Problem 9.04
This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin
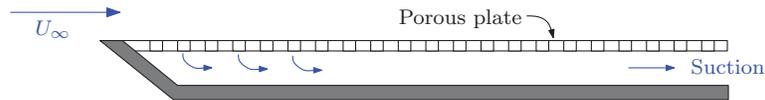
Consider a flat porous surface with no lengthwise pressure gradient to which suction is applied for the purpose of sucking off the laminar boundary layer. Let us denote by v_0 the downward component of velocity at the surface of the plate ($y = 0$). Then it may be shown that for very small values of v_0/U_∞ compared with unity, the laminar boundary layer becomes constant in both thickness and velocity at large distances from the leading edge, provided that v_0 is constant.

For this case of small v_0/U_∞ and large distance from the leading edge, find:

(a) the velocity profile, by relating u/U_∞ to $v_0 y/\nu$

(b) the displacement thickness Reynolds number, $\delta^* v_0/\nu$.

(c) the skin-friction coefficient, $2\tau_0/\rho U_\infty{}^2$, in terms of v_0, U_∞, and ν.

(d) develop an estimate for the distance (in x) at which the boundary layer thickness approaches the asymptotic constant value found in Parts (a)-(c).

Hint: Start with the differential equations of the laminar boundary layer.

Solution:

Assumptions:

1. $\frac{dp}{dx}$ from outside the boundary layer $= 0$

2. $v_y = -v_0$ at $y = 0$ where $v_0 = $ constant.

3. For $\frac{v_0}{U_\infty} \ll 1$ and sufficiently large x, boundary layer is constant <u>thickness</u> and <u>velocity</u>

Consider <u>reduced</u> Navier-Stokes in x-direction (derived in lecture):

$$\underbrace{\frac{\partial v_x}{\partial t}}_{\text{assume steady}} + \underbrace{v_x \frac{\partial v_x}{\partial x}}_{\text{assumption 3}} + v_y \frac{\partial v_x}{\partial y} = \underbrace{-\frac{1}{\rho} \frac{dp}{dx}}_{\text{assumption 1}} + \nu \frac{\partial^2 v_x}{\partial y^2}$$

From continuity

$$\underbrace{\frac{\partial v_x}{\partial x}}_{\text{assumption 3}} + \frac{\partial v_y}{\partial y} = 0 \qquad \Rightarrow \qquad v_y = \text{constant} = -v_0$$

$$\Rightarrow \boxed{0 = \nu \frac{\partial^2 v_x}{\partial y^2} + v_0 \frac{\partial v_x}{\partial y}} \tag{9.04a}$$

$$\text{B.C.} \begin{cases} v_x = 0 & \text{at } y = 0 \quad \forall x \\ v_x = U_\infty & \text{as } y \to \infty \quad \forall x \end{cases}$$

(a) Solve for v_x by assuming $v_x = A \exp(By) + C$ and substitute that into Eq. (9.04a):

$$0 = \nu A B^2 \exp(By) + v_0 A B \exp(By) \qquad \Rightarrow \qquad \begin{cases} B = -\dfrac{v_0}{\nu} \\ v_x = A \exp\left(\dfrac{-v_0 y}{\nu}\right) + C \end{cases}$$

Find A & C by applying boundary conditions:

$$v_x(y = 0) = 0 = A + C$$
$$\Rightarrow A = -C$$
$$v_x(y \to \infty) = U_\infty = C$$
$$\Rightarrow C = U_\infty$$
$$A = -U_\infty$$

$$\Rightarrow \boxed{v_x = U_\infty \left[1 - \exp\left(\frac{-v_0 y}{\nu}\right)\right]} \tag{9.04b}$$

(b) Based on Eq. (9.04b), find δ^*:

$$\delta^* = \int_0^\infty \left(1 - \frac{v_x}{U_\infty}\right) dy = \int_0^\infty \exp\left(\frac{v_x}{U_\infty}\right) dy = \frac{\nu}{v_0}$$

$$\Rightarrow \boxed{\frac{\delta^* v_0}{\nu} = 1} \tag{9.04c}$$

(c)

$$\tau_0 = \mu \frac{\partial u}{\partial y}\bigg|_{y=0}$$

$$\text{OR } \tau_0 \sim \mu \frac{U_\infty}{\delta^*} = \frac{\mu U_\infty v_0}{\nu} = \rho U_\infty v_0$$

$$\Rightarrow C_f = \frac{2\tau}{\rho U_\infty^2} = \frac{2\cancel{\rho} U_\infty v_0}{\cancel{\rho} U_\infty^2}$$

$$\boxed{C_f = \frac{2v_0}{U_\infty}}$$

(9.04d)

(d) We could think of the problem in terms of the Rayleigh problem, then

$$\delta \sim \sqrt{\nu t^*},$$

(9.04e)

but what is t^*? t^* is the characteristic time for the fluid particles to reach the point x, then $t^* = \frac{x}{U_\infty}$. Then,

$$\delta \sim \sqrt{\frac{\nu x}{U_\infty}},$$

(9.04f)

which is the growth of the BL for the normal Blasius Solution. Now, when is the characteristic vertical length equal to δ? Let's assume that the characteristic vertical length is δ, i.e. the BL reaches its steady state, then from mass conservation,

$$\frac{U_\infty}{x} \text{ " + " } \frac{v_0}{\delta} \text{ " = " } 0,$$

(9.04g)

then, $v_0 \sim \frac{\delta}{x}U_\infty$. Now, elevating to the second power this expression, and replacing the order of magnitude of δ obtained before,

$$v_0^2 \sim \frac{\nu x}{U_\infty} \frac{U_\infty^2}{x^2},$$

(9.04h)

simplifying,

$$x \sim \frac{\nu U_\infty}{v_0^2},$$

(9.04i)

then, for $\boxed{x \gg \frac{\nu U_\infty}{v_0^2}}$ the BL reaches its maximum value.

Note: Using suction to enhance the B.L adhesion is an idea that has been investigated since 1940, one of the last research airplanes that incorporated this technology is the NASA experimental version of the F-16XL. It has laser-cut holes to allow suction and enhance the attachment of the boundary layer.

http://grin.hq.nasa.gov/ABSTRACTS/GPN-2000-001935.html

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Problem Solution by SL, MC (Updated), Fall 2008

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