MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
CAMBRIDGE, MASSACHUSETTS 02139
2.29 NUMERICAL FLUID MECHANICS — SPRING 2015

QUIZ 2   Wednesday, April 15, 2015

The goals of quiz 2 are to: (i) ask some general higher-level questions to ensure you understand broad concepts and are able to discuss such concepts and issues with others familiar with numerical fluid mechanics; (ii) show that you understand methods and schemes that you learned and that you can apply them in idealized computational problems; and (iii), evaluate if you can read numerical codes and recognize what they accomplish. Partial credit will be given to partial answers (e.g. accurate descriptions in words of computations required but not carried out).

A. Shorter Concept Questions

Problem I (24 points)

*Briefly answer **6** of the following 7 questions* (a few words to a few sentences is enough, with or without equations depending on the question). If you answer 7 questions, we will give a bonus.

a) Provide one reason why polynomial approximations are often used to derive finite-difference/volume schemes. Provide another type of approximation that might be more adequate in specific cases and briefly state why.

b) Using a Taylor series expansion, briefly explain when and why a non-uniform grid can be advantageous.

c) You derive a discretization on a non-uniform grid and find that the truncation error is of a lower order than that on a uniform grid. What would be the advantage of selecting a non-uniform grid progression such that when you refine the non-uniform grid, the grid tends towards a uniform one?

d) Two of your friends compare their spatial FD discretization for the same PDE. Each tells you that her/his scheme is stable and of second order accuracy. What are two analyses that you can suggest to your friends to determine which of the two schemes is better?

e) The von Neumann analysis tests the stability of a spatial discretization using Fourier series. Provide two reasons for this.

f) Finite-volume schemes clearly involve integrals but why do they also involve interpolations?

g) For three-dimensional problems in space, provide a major reason why it is challenging to obtain and utilize finite-volume methods of high-order (higher than 2)?

B. Computational Schemes and Idealized Computational Problems

Problem II: *Stability Analysis for a Parabolic Diffusion Equation in n-dimensions* **(14 points)**

Consider the *n*-dimensional heat conduction equation with a uniform thermal diffusivity,

$$\frac{\partial T}{\partial t} = \alpha \sum_{\ell=1}^{n} \frac{\partial^2 T}{\partial x_\ell^2}.$$

We employ the following numerical discretization using a uniform grid spacing in the *n*-dimensional space, i.e. $\Delta x_\ell = \Delta x \ \forall \ell$,

$$\frac{T_j^{k+1} - T_j^k}{\Delta t} = \alpha \sum_{\ell=1}^{n} \frac{T_{j+e_\ell}^k - 2T_j^k + T_{j-e_\ell}^k}{\Delta x_\ell^2} \quad \Rightarrow \quad T_j^{k+1} - T_j^k = r \sum_{\ell=1}^{n} \left(T_{j+e_\ell}^k - 2T_j^k + T_{j-e_\ell}^k \right),$$

where: $r = \alpha \dfrac{\Delta t}{\Delta x^2}$, the vector index *j* denotes a point in the n-dimensional space (e.g., *j* is a set of 3 indices in 3-d), and $e_\ell = [0, \cdots, 1, \cdots, 0]$ is a unit vector of indices (all zero except 1 in position ℓ).

a) Using a von Neumann stability analysis, assuming a uniform modal decomposition in space (i.e. the variable β is chosen uniform and independent of ℓ), determine the stability criterion for the above discretization.

b) Discuss your result in a) in terms of *n*. In particular, consider the cases $n = 2$ and $n = 3$.

c) Briefly suggest how you could modify the given numerical discretization so as to relax the stiff stability condition obtained but without substantially increasing the computational cost.

Problem III: *Analysis of Projection Methods in the Pressure-Correction Form* (22 points)

Consider the first two projection methods presented in class (see eqs. sheet), the so-called non-incremental and incremental projection methods. For each of these schemes, the result of the first PDE solve, the predictor velocity u_i^*, satisfies the boundary conditions (BCs) for velocity. For example, in the case of a no-slip Dirichlet condition at a wall, $\left. \left(\rho u_i^*\right)^{n+1}\right|_{\partial D} = \rho v_{wall}$.

First, consider the simplest non-incremental projection method:

  a) Explain how its third algebraic equation, the velocity correction, can be utilized to obtain the BC on the new pressure p^{n+1}, i.e. $\left.\dfrac{\delta p^{n+1}}{\delta n}\right|_{\partial D} = 0$.

  b) Is this BC on p^{n+1} physical? Briefly discuss (*Hint*: consider for example a viscous flow impinging on a wall).


Next, consider the incremental projection method:

  c) Its BCs on the pressure correction, $\left.\dfrac{\delta\left(p^{n+1} - p^n\right)}{\delta n}\right|_{\partial D} = 0$, can be obtained as in part a) (no need to show this). What does this BC imply for the normal pressure gradient at the wall? Is it physical?

  d) Using the third equation of the incremental projection method, obtain and discuss the projection of the corrected velocity $\left(\rho u_i\right)^{n+1}$ along the wall, i.e. the tangential velocity component.

  e) Obtain a quick estimate of the order of "accuracy" of the difference $\left(\rho u_i\right)^{n+1} - \left(\rho u_i^*\right)^{n+1}$, a so-called splitting error.

We now look into the cost of the linear system solves for projection methods.

  f) Consider any of the three projection schemes, what are the linear systems that it solves? You can define a matrix **A** and right-hand-side **b** for each system.

  g) Consider now a full implicit system solve for the Navier-Stokes eqs. (e.g. the Backward-Euler Implicit in Time scheme, see eqs. sheet) or the same but with an explicit advection term. Compare the cost of solving either of these full systems to the cost of solving the systems in part f). (*Hint*: Drawing a block-matrix to represent a full system should help).

Problem IV: *Compact Scheme for Finite Volume Methods* (20 points)

In finite volume methods, one step is to express flux variables at CV edges in terms of CV-averaged values through interpolation. To obtain such an expression, we can use a parametric shape function approach, but we can also employ a Taylor expansion approach, as for finite differences. In this problem, we will use this Taylor expansion approach to obtain a compact system for the values of the field u at CV surfaces (cell vertex in 1D, edges in 2D, faces in 3D).
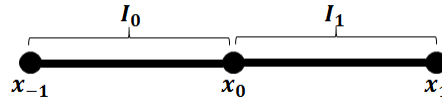
Consider a uniform 1D mesh with spacing Δx as sketched above. We set the origin of the local x axis at x_0. Therefore, the coordinate of a cell vertex is $x_k = k\Delta x$. For a field u, we denote its cell-averaged value over cell I_k by $\bar{u}_k$ and its cell-vertex value at x_k by u_k. To express the latter values in terms of the former averages, we employ the following compact scheme:

$$au_{-1} + u_0 + bu_1 = c\bar{u}_0 + d\bar{u}_1 \qquad (1)$$

where a, b, c and d are coefficients to be determined by the Taylor expansion approach.

a) The Taylor expansion at $x_0 = 0$ of a *cell-averaged value $\bar{u}_k$* is

$$\bar{u}_k = \sum_{i=0}^{\infty} \alpha_i u_0^{(i)} (\Delta x)^i \, , \text{ where } \alpha_i = \frac{k^{i+1} - (k-1)^{i+1}}{(i+1)!} \, ,$$

and $u_0^{(i)}$ is the i-th derivative of u at x_0 (No need to derive this, but if you do, we will give a bonus). Using this result, fill out the following Taylor table for the compact scheme (1) (*either here or in your answer book*):

	$u_0^{(0)}$	$u_0^{(1)}(\Delta x)$	$u_0^{(2)}(\Delta x)^2$	$u_0^{(3)}(\Delta x)^3$	$u_0^{(4)}(\Delta x)^4$
au_{-1}					
u_0					
bu_1					
$-c\bar{u}_0$					
$-d\bar{u}_1$					

b) By setting to zero as many leading order terms of $au_{-1} + u_0 + bu_1 - c\bar{u}_{-1} - d\bar{u}_1$ as possible, write down the linear system for the undetermined coefficients a, b, c and d. The solution to this linear system is

$$[a,b,c,d] = [\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}] \, .$$

Obtain the final compact scheme as well as the leading order term of the truncation error. Briefly explain how you would use this scheme to compute all cell-vertex values (fluxes) from all cell-average values.

c) For an equation with a diffusion term $\partial^2 u / \partial x^2$, to obtain a compact scheme, it may seem that we would need to utilize a Padé scheme to express the diffusive flux $\partial u / \partial x$ at cell vertices in terms of cell-averaged values $\bar{u}_k$. However, with the compact scheme we just derived, we can actually approximate $\partial u / \partial x$ at cell vertices by $\bar{u}_k$ without using a new Padé scheme. Could you briefly explain in words how this could be done?

C. Numerical Code Evaluations

Problem V: *Finite Difference Solution using MCVNCD* (20 points)
Read the following MATLAB function and answer the questions that follow.

```
a = 1; L = 1; T = 1;

dt = 5e-4; dx = 1e-3;
A1 = 0.5*(1-a*dt/dx);
A2 = 0.5*(1+a*dt/dx);

u = sin(pi*(0:dx:L-dx)');

for i = 1 : round(T/dt)
    u = [[u(2:end);u(1)],[u(end);u(1:end-1)]]*[A1;A2];
    if mod(i,10)==0, plot(0:dx:L-dx, u); pause(1e-9); end;
end;
```

a) Write down the PDE that is solved and the finite difference scheme that is used.

b) What is the leading order term of the truncation error of this scheme?

c) In the following sketch, the circles represent the nodes of the space-time grid. Please identify the numerical domain of dependence by filling the circles at which these node values can influence the value at the solid circle. What is the CFL criterion for this scheme? Please also sketch the physical domain of dependence for the solid circle when the CFL criterion is met. (*Please sketch either here or on the answer book*)
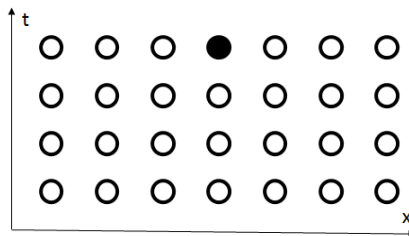
d) Under what condition does this scheme satisfy boundedness and why? Boundedness means that the numerical solution at the next time step will never exceed the maximum (nor go below the minimum) of the solution at previous time steps.

2.29 Numerical Fluid Mechanics
Spring 2015