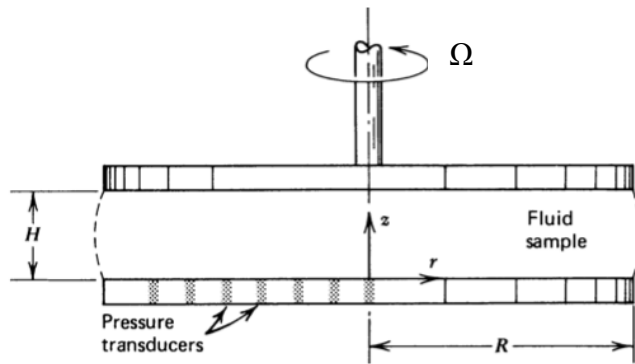**Problem 1: Measuring Viscometric Functions in the Parallel Plate Rheometer** [25 points]

We have mentioned in class briefly the use of a parallel plate fixture for rheometric measurements. Such a fixture (as shown opposite) consists of a rotating cylindrical fixture of radius R mounted coaxially above another plate of same radius (or just a large flat plate). Here we consider the steady flow of a non-Newtonian fluid in such a device; in which the upper plate is rotated at a constant rotation rate Ω and the total torque $\mathcal{T}$ is measured.

=>Use cylindrical polar coordinates $\{r,\theta,z\}$ in the following analysis, and assume that $H \ll R$.

a) [4 points] Assume a purely axisymmetric and steady flow with velocity field of separable form $v_\theta(r) = rw(z)$, where $w(z)$ is an (as yet unknown) function. Evaluate the velocity gradient tensor ∇v, the deformation rate tensor $\dot{\gamma}$ and all three invariants *I, II, III* of the deformation rate tensor for this flow field. Is this a *simple shear flow*?

b) [3 points] Give a dimensionless criterion under which inertia in the flow is negligible. In the inertialess limit show that the theta component of the equations of motion tells us that the shear stress (and therefore the corresponding component of the shear rate tensor) is independent of the axial position z. Use this fact to integrate the equation and find the full velocity field $v_\theta(r)$ which satisfies appropriate boundary conditions on the upper and lower plates. Show that the local shear rate in the device $\dot{\gamma}_{\theta z}(r)$ varies <u>linearly</u> across the device; i.e. $\dot{\gamma}(r) = \dot{\gamma}_R\, r/R$. Give an explicit expression for the *rim shear rate* $\dot{\gamma}_R$ at $r = R$. We shall use this identity extensively below.

c) [4 points] Show that the torque acting on the rotating plate is given by $\mathcal{T} = 2\pi \int_0^R \eta(\dot{\gamma})\dot{\gamma} r^2\, dr$ and explain why this function can't readily be simplified further.

   Using the fact that the shear rate varies linearly across the device to change variables in this integral using the identity $\dot{\gamma} = \dot{\gamma}_R r/R$ and arrive at a new integral of the form $\mathcal{T} = (?) \int_0^{\dot{\gamma}_R} (?)\, d\dot{\gamma}$ in which the argument inside the integrand is only a function of the local shear rate. {you need to specify the terms in the parentheses}

d) [4 points] To simplify this result, we will use the Liebniz formula for differentiating an integral (see footnote at end if you don't recall what this is exactly). To start with rearrange your result from part (c) into a form that looks like the following:

$$\frac{d\left(T'\dot{\gamma}_R^{3}/2\pi R^{3}\right)}{d\dot{\gamma}_R} = \text{RHS?}$$

And then use the Leibniz rule to show that:

$$\eta(\dot{\gamma}_R) = \frac{(T'/A)}{\dot{\gamma}_R}\left[B + \frac{d\ln(T'/A)}{d\ln\dot{\gamma}_R}\right] \tag{1}$$

where A and B are constants. <u>Give values for these constants.</u>

e) [4 points] This expression is nice as it gives us a way to evaluate the viscosity for *any fluid* in a parallel plate rheometer from the measured torque. Using the results in (c) show how your result in (d) simplifies for two special cases: *(i)* a Newtonian fluid with $\eta(\dot{\gamma}) = \mu$, and *(ii)* a power law fluid with $\eta(\dot{\gamma}) = K\dot{\gamma}^{n-1}$.

f) [2 points] Once the viscosity is measured using eq. (1) it can be compared with different models, such as the Carreau-Yasuda equation (discussed in class) and given by:

$$\frac{\eta(\dot{\gamma}) - \eta_s}{\eta_0 - \eta_s} = \left[1 + (\lambda\dot{\gamma})^2\right]^{(n-1)/2}$$
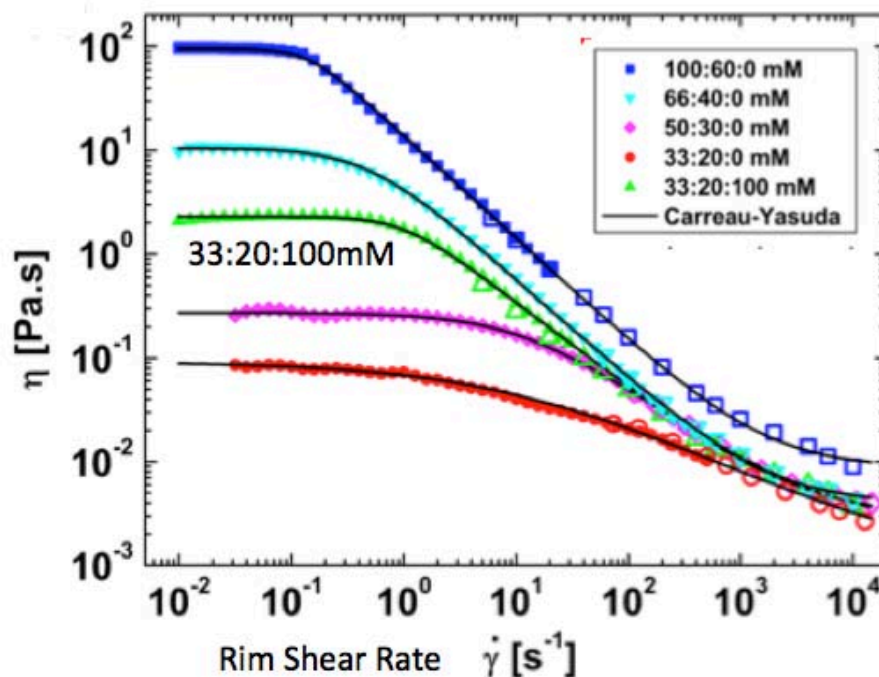
In the figure above you can see data for a range of different wormlike micellar fluids (with different chemical compositions). For the 33:20:100 mM fluid (green triangles) *carefully estimate* (to within ±50% is fine) appropriate values of the <u>four parameters</u> in the Carreau-Yasuda model (which as you can see describes the data very well). List the four values clearly in a table.

This parallel plate device can also be used to measure the two normal stress coefficients $\Psi_1(\dot{\gamma}) = (\tau_{11} - \tau_{22})/\dot{\gamma}^2$ and $\Psi_2(\dot{\gamma}) = (\tau_{22} - \tau_{33})/\dot{\gamma}^2$. After some more algebra (in curvilinear coordinates) it can be shown that the stress tensor for steady shear flow in cylindrical coordinates $\{r,\theta,z\}$ can be written in most general form as:

$$\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\Psi_1 + \Psi_2)\dot{\gamma}^2 & \eta\dot{\gamma} \\ 0 & \eta\dot{\gamma} & \Psi_2\dot{\gamma}^2 \end{pmatrix} \qquad (2)$$

g) [2 points] Substitute these components into the radial component of the equation of motion and change the integration variable once again to show that the pressure field in the sample varies as

$$p(r) - p_a = \int_{\dot{\gamma}_r}^{\dot{\gamma}_R} (\ )\dot{\gamma}\,d\dot{\gamma}$$

where again you need to specify the argument inside the parentheses.

h) [2 points] Recognizing that the local measured axial thrust acting at any point on the plate at $z = 0$ is not just the local pressure $p(r)$ but in fact π_{zz} find an expression (in terms of an integral) for the local thrust at the middle of the plate $r = 0$. Is your result finite? or infinite? (like we saw in the cone-and-plate analysis).

In principle, this pressure/thrust can be measured as the rim shear rate is changed. However, the result is still not that useful as it involves an integral moment of the material functions. With some more algebra one can obtain a more useful expression similar to that in eq. (1) above, but this involves some more effort, and I imagine you would rather go home ☺. [See the solution for details after Spring Break].

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the "Leibniz formula" for differentiating an integral:

$$\frac{d}{dt}\int_{a_1(t)}^{a_2(t)} f(x, t)\,dx = \int_{a_1(t)}^{a_2(t)} \frac{\partial f}{\partial t}\,dx + f(a_2, t)\frac{da_2}{dt} - f(a_1, t)\frac{da_1}{dt}$$

The Equations of motion in cylindrical coordinates are :

Cylindrical Coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}\right] - \frac{\partial p}{\partial r} + \rho g_r \quad (\text{B.1-4})$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right] - \frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta$$

$$(\text{B.1-5})$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z} + \frac{\partial}{\partial z}\tau_{zz}\right] - \frac{\partial p}{\partial z} + \rho g_z \quad (\text{B.1-6})$$

** In case you need the components of ∇v see overleaf:

In cylindrical coordinates the components of ∇v :

$$\{\nabla v\}_{rr} = \frac{\partial v_r}{\partial r} \tag{S}$$

$$\{\nabla v\}_{r\theta} = \frac{\partial v_\theta}{\partial r} \tag{T}$$

$$\{\nabla v\}_{rz} = \frac{\partial v_z}{\partial r} \tag{U}$$

$$\{\nabla v\}_{\theta r} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \tag{V}$$

$$\{\nabla v\}_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \tag{W}$$

$$\{\nabla v\}_{\theta z} = \frac{1}{r}\frac{\partial v_z}{\partial \theta} \tag{X}$$

$$\{\nabla v\}_{zr} = \frac{\partial v_r}{\partial z} \tag{Y}$$

$$\{\nabla v\}_{z\theta} = \frac{\partial v_\theta}{\partial z} \tag{Z}$$

$$\{\nabla v\}_{zz} = \frac{\partial v_z}{\partial z} \tag{AA}$$

2.341J / 10.531J Macromolecular Hydrodynamics
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