7B.1 NORMAL STRESSES IN SMALL-AMPLITUDE OSCILLATORY SHEARING OF THE CONVECTED JEFREY'S MODEL [GHM]

a) The xx-component of time-dependent shearing flow for this model is given as eq 7.2-3

$$\left(1 + \lambda_1 \frac{d}{dt}\right)\tau_{xx} - 2\tau_{yx}\lambda_1\dot{\gamma}_{yx}(t) = 2\eta_0\lambda_2\dot{\gamma}_{yx}^2(t) \qquad (7.2\text{-}3)$$

where $\dot{\gamma}_{yx}(t) = \dot{\gamma}_0 \cos\omega t$ and $\gamma_0 = \dot{\gamma}_0/\omega$

The expression for the shear stress $\tau_{yx}(t)$ is given in equations 7.2-9, 10, 11 as

$$\tau_{yx} = A\cos\omega t + B\sin\omega t \qquad (7.2\text{-}9)$$

where
$$A = -\eta_0\left(\frac{1+\lambda_1\lambda_2\omega^2}{1+\lambda_1^2\omega^2}\right)\gamma_0\omega \qquad (7.2\text{-}10)$$

$$B = -\frac{\eta_0(\lambda_1-\lambda_2)\gamma_0\omega^2}{(1+\lambda_1^2\omega^2)} \qquad (7.2\text{-}11)$$

Substituting (7.2-9) into (7.2-3) gives

$$\left(1+\lambda_1\frac{d}{dt}\right)\tau_{xx} = 2\lambda_1\gamma_0\omega\cos\omega t\left(A\cos\omega t + B\sin\omega t\right) + 2\eta_0\lambda_2\gamma_0^2\omega^2\cos^2\omega t$$

$$= \lambda_1\omega\gamma_0 B\sin 2\omega t + \left(2\lambda_1\gamma_0\omega A + \eta_0\lambda_2\gamma_0^2\omega^2\right)\left\{\tfrac{1}{2} + \tfrac{1}{2}\cos 2\omega t\right\}$$

Identify $T = \gamma_0 B = -\eta_0(\lambda_1-\lambda_2)\gamma_0^2\omega^2/(1+\lambda_1^2\omega^2)$ (7B.1-2)

and substitute for A in (----) term gives

$$\left(1+\lambda_1\frac{d}{dt}\right)\tau_{xx} = T\left(1 + \cos 2\omega t + \lambda_1\omega\sin 2\omega t\right) \qquad (7B.1\text{-}1)$$

Since we expect a solution periodic in time we guess form

$$\tau_{xx}/T = A^* \cos 2\omega t + B^* \sin 2\omega t + C^*$$

Substituting in (7B.1-1) and comparing terms

$$\left.\begin{array}{lll} \text{cos term} & A^* + 2\lambda_1\omega B^* & = 1 \\ \text{sin term} & B^* - 2\lambda_1\omega A^* & = \lambda_1\omega \\ \text{constant term} & C^* + 0 & = 1 \end{array}\right\} \Rightarrow \begin{array}{l} A^* = \dfrac{1 - 2\lambda_1^2\omega^2}{1 + (2\lambda_1\omega)^2} \\ B^* = \dfrac{3\lambda_1\omega}{(1 + (2\lambda_1\omega)^2} \\ C^* = 1 \end{array}$$

* Note that τ_{xx} has a nonzero mean $\overline{\tau_{xx}} = T$ and oscillates with frequency 2ω.

c)

$$\lim_{\gamma_0 \to 0} \frac{|\tau_{xx}|}{|\tau_{yx}|} = \lim_{\gamma_0 \to 0} \frac{|T|\,|A^* \cos 2\omega t + B^* \sin 2\omega t + 1|}{|A \cos \omega t + B \sin \omega t|}$$

Note that A^*, B^* are independent of γ_0 and thus

$$\cong k \lim_{\gamma_0 \to 0} \frac{|T|}{|A \cos \omega t + B \sin \omega t| + 1)}$$

Dividing by γ_0 {or using L'Hôpitals rule} gives

$$\cong k \lim \frac{|T/\gamma_0|}{|\underbrace{A/\gamma_0 \cos \omega t + B/\gamma_0 \sin \omega t}_{\text{independent of } \gamma_0}|}$$

$$\therefore \lim_{\gamma_0 \to 0} \frac{|\tau_{xx}|}{|\tau_{yx}|} = \frac{A^* \cos 2\omega t + B^* \sin 2\omega t + 1}{A/\gamma_0 \cos \omega t + B/\gamma_0 \sin \omega t} \lim_{\gamma_0 \to 0} \left\{ \frac{\eta_0 (\lambda_1 - \lambda_2)\omega^2}{(1 + \lambda_1^2 \omega^2)} \gamma_0 \right\}$$

$\rightarrow 0$ for any $\omega, \lambda_1, \omega$

K. C. Armstrong

7B.2 Normal Stresses in Small-Amplitude Oscillatory Shearing of the Convected Jeffreys Model [RBB]

(a) From Eq. 7.2-3 and $\dot{\gamma}_{yx}(t) = \gamma_0 \omega \, \mathcal{R}e\{e^{i\omega t}\}$:

$$\left(1+\lambda_1 \frac{d}{dt}\right)\tau_{xx} - 2\tau_{yx}\lambda_1 \gamma_0 \omega \, \mathcal{R}e\{e^{i\omega t}\}$$

$$= 2\eta_0 \lambda_2 \gamma_0^2 \omega^2 \left[\mathcal{R}e\{e^{i\omega t}\}\right]^2$$

But $\quad \tau_{yx} = \mathcal{R}e\{\tau_{yx}^0 e^{i\omega t}\} = -\mathcal{R}e\{\eta^* e^{i\omega t}\}\gamma_0 \omega$

Therefore, using fn. 11 on p. 188:

$$\left(1+\lambda_1 \frac{d}{dt}\right)\tau_{xx} =$$

$$-2\lambda_1 \gamma_0^2 \omega^2 \cdot \frac{1}{2}\left[\mathcal{R}e\{\eta^* e^{2i\omega t}\} + \mathcal{R}e\{\eta^*\}\right]$$
$$+ 2\eta_0 \lambda_2 \gamma_0^2 \omega^2 \cdot \frac{1}{2}\left[\mathcal{R}e\{e^{2i\omega t}\} + \mathcal{R}e\{1\}\right]$$

$$= -\lambda_1 \gamma_0^2 \omega^2 \left[\eta' \cos 2\omega t + \eta'' \sin \omega t + \eta'\right]$$
$$+ \eta_0 \lambda_2 \gamma_0^2 \omega^2 \left[\cos 2\omega t + 1\right]$$

Substitution of $\eta'(\omega)$ and $\eta''(\omega)$ from Eqs. 7.2-10 and 11 into these expressions gives Eqs. 7B.1-1 and 2.

(b) The form of the differential equation for τ_{xx} suggests that at the "sinusoidal steady state," τ_{xx} should be of the form Eq. 7B.1-3. Alternatively one can seek a solution of the form

$$\tau_{xx} = C + \mathcal{R}e\left\{\overset{\circ}{\tau}_{xx} e^{2i\omega t}\right\}$$

in which C and $\overset{\circ}{\tau}_{xx}$ are functions of ω. Substitution of this postulated form into Eq. 7B.1-1 (or the $e^{i\omega t}$ equivalent in (a)) gives

$$C + \mathcal{R}e\left\{\overset{\circ}{\tau}_{xx} e^{2i\omega t}\right\} + \lambda_1 \mathcal{R}e\left\{\overset{\circ}{\tau}_{xx}\, 2i\omega\, e^{2i\omega t}\right\}$$

$$= -\lambda_1 \gamma_0^2 \omega^2 \left[\mathcal{R}e\left\{\eta^* e^{2i\omega t}\right\} + \eta'\right]$$

$$+ \eta_0 \lambda_2 \gamma_0^2 \omega^2 \left[\mathcal{R}e\left\{e^{2i\omega t}\right\} + 1\right]$$

Now equating all terms independent of t, we get

$$C = \gamma_0^2\omega^2\left(\eta_0\lambda_2 - \lambda_1\eta'\right) = \gamma_0^2\omega^2\left[-\eta_0 \frac{(\lambda_1-\lambda_2)}{1+\lambda_1^2\omega^2}\right]$$

Equating the terms containing a t-dependence, we get:

$$\overset{\circ}{\tau}_{xx}\left(1 + 2i\lambda_1\omega\right) = \gamma_0^2\omega^2\left(\eta_0\lambda_2 - \lambda_1\eta^*\right)$$

or $$\overset{\circ}{\tau}_{xx} = \gamma_0^2\omega^2 \frac{\eta_0\lambda_2 - \lambda_1\eta^*}{1 + 2i\lambda_1\omega}$$

$$= -\eta_0\gamma_0^2\omega^2 \frac{(\lambda_1 - \lambda_2)}{(1+2i\lambda_1\omega)(1+i\lambda_1\omega)}$$

(c) The amplitude of the stress τ_{yx} is given by:

$$\text{amp}(\tau_{yx}) = \sqrt{\tau^{\circ}_{yx}\,\bar{\tau}^{\circ}_{yx}} = \gamma_0\omega|\eta^*|$$

and is thus proportional to γ_0. The normal stress displacement of the normal stress, C, is

$$C = -\gamma_0^2\omega\eta'$$

which is proportional to γ_0^2. The amplitude of the oscillation of τ_{xx} about its nonzero mean value is

$$\text{amp}(\tau_{xx}) = \sqrt{\tau^{\circ}_{xx}\,\bar{\tau}^{\circ}_{xx}}$$

and from (b) it is clear that this is $\gamma_0^2 \times$ (function of ω). Therefore the ratio of the magnitude of τ_{xx} to that of τ_{yx} is proportional to γ_0. As $\gamma_0 \to 0$, then, the ratio of magnitudes goes to zero for all values of ω. We thus conclude that the Oldroyd-B model (or "convected Jeffreys model) give identical results in sinusoidal shear flow in the limit of vanishingly small displacement gradients.

Example 7.2-1 (Part(c)) ALTERNATIVE SOLUTION

Given: $\gamma_{yx}(0,t) = \int_0^t \dot{\gamma}_0 \cos \omega t'\, dt' = \gamma_0 \sin \omega t \qquad [\gamma_0 = \dot{\gamma}_0/\omega]$

This may also be written

$$\gamma_{yx}(0,t) = \int_0^t \dot{\gamma}_0\, \mathcal{R}e\{e^{i\omega t'}\}\, dt'$$

Then $\dot{\gamma}_{yx}(t) = \dot{\gamma}_0\, \mathcal{R}e\{e^{i\omega t}\} = \gamma_0 \omega\, \mathcal{R}e\{e^{i\omega t}\}$

Thus, in lieu of Eq 7.2-8 we have

$$\frac{d}{dt}\tau_{yx} + \frac{1}{\lambda_1}\tau_{yx} = -\frac{\eta_0}{\lambda_1}\mathcal{R}e\left\{\gamma_0 \omega e^{i\omega t} + \gamma_0 \omega \lambda_2 i\omega e^{i\omega t}\right\},$$

Now postulate a solution of the form $\tau_{yx} = \mathcal{R}e\{\tau_{yx}^0 e^{i\omega t}\}$.

Then substitution into the above equation gives:

$$\left(i\omega + \frac{1}{\lambda_1}\right)\tau_{yx}^0 = -\frac{\eta_0 \gamma_0 \omega}{\lambda_1}\left(1 + \lambda_2 i\omega\right)$$

and

$$\tau_{yx}^0 = -\eta_0 \gamma_0 \omega \left(\frac{1 + i\lambda_2\omega}{1 + i\lambda_1\omega}\right) \equiv -\eta^* \dot{\gamma}_0$$

From this we get:

$$\begin{cases} \eta'(\omega) = \eta_0 \left(\dfrac{1 + \lambda_1\lambda_2\omega^2}{1 + \lambda_1^2\omega^2}\right) \\ \eta''(\omega) = \eta_0 \left(\dfrac{\lambda_1 - \lambda_2}{1 + \lambda_1^2\omega^2}\right)\omega \end{cases}$$

This agrees with Eqs 7.2-10 and 7.2-11.

7B.2 Shearfree Flow for the White-Metzner Model [RBB]

a. From Eq. 4.1-8 $\dot{\gamma} = \sqrt{\frac{1}{2}(\dot{\underline{\underline{\gamma}}} : \dot{\underline{\underline{\gamma}}})}$. For shearfree flows we get from Table C.1, p. 622:

$$\dot{\underline{\underline{\gamma}}} = \begin{pmatrix} -(1+b) & 0 & 0 \\ 0 & -(1-b) & 0 \\ 0 & 0 & +2 \end{pmatrix} \dot{\epsilon}(t)$$

And for steady shearfree flow with $\dot{\epsilon}(t) = \dot{\epsilon}_0$, a constant, we get:

$$\{\dot{\underline{\underline{\gamma}}} \cdot \dot{\underline{\underline{\gamma}}}\} = \begin{pmatrix} (1+b)^2 & 0 & 0 \\ 0 & (1-b)^2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \dot{\epsilon}_0^2$$

Note that $\dot{\epsilon}_0$ may be either positive or negative.

Then

$$\frac{1}{2}(\dot{\underline{\underline{\gamma}}} : \dot{\underline{\underline{\gamma}}}) = \frac{1}{2} \operatorname{tr}\{\dot{\underline{\underline{\gamma}}} \cdot \dot{\underline{\underline{\gamma}}}\} = \frac{1}{2}(6 + 2b^2)\dot{\epsilon}_0^2$$

So that

$$\dot{\gamma} = \sqrt{\frac{1}{2}(\dot{\underline{\underline{\gamma}}} : \dot{\underline{\underline{\gamma}}})} = \sqrt{3+b^2}\,|\dot{\epsilon}_0|$$

b. Recall from Eqs. 3.5-1 and 2

$$\tau_{zz} - \tau_{xx} = -\bar{\eta}_1 \dot{\epsilon}$$

$$\tau_{yy} - \tau_{xx} = -\bar{\eta}_2 \dot{\epsilon}$$

From Table C.1 and Eq. 7.3-1 we get:

$$\begin{pmatrix} \tau_{xx} & 0 & 0 \\ 0 & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix} - \frac{\eta(\dot{\gamma})}{G}\begin{pmatrix} -(1+b)\tau_{xx} & 0 & 0 \\ 0 & -(1-b)\tau_{yy} & 0 \\ 0 & 0 & 2\tau_{zz} \end{pmatrix}\dot{\epsilon}_0$$

$$= -\eta(\dot{\gamma})\begin{pmatrix} -(1+b) & 0 & 0 \\ 0 & -(1-b) & 0 \\ 0 & 0 & 2 \end{pmatrix}\dot{\epsilon}_0$$

From this matrix equation we get:

$$\begin{cases} \tau_{xx}\left[1+(1+b)(\eta/G)\dot{\epsilon}_0\right] = +(1+b)\,\eta\,\dot{\epsilon}_0 \\ \tau_{yy}\left[1+(1-b)(\eta/G)\dot{\epsilon}_0\right] = +(1-b)\,\eta\,\dot{\epsilon}_0 \\ \tau_{zz}\left[1-2(\eta/G)\dot{\epsilon}_0\right] = -2\eta\,\dot{\epsilon}_0 \end{cases}$$

From these we get

$$\tau_{zz}-\tau_{xx} = -\frac{(3+b)\,\eta(\dot{\gamma})\,\dot{\epsilon}_0}{\left[1+(1+b)(\eta/G)\dot{\epsilon}_0\right]\left[1-2(\eta/G)\dot{\epsilon}_0\right]}$$

$$\tau_{yy}-\tau_{xx} = -\frac{2b\,\eta(\dot{\gamma})\,\dot{\epsilon}_0}{\left[1+(1+b)(\eta/G)\dot{\epsilon}_0\right]\left[1+(1-b)(\eta/G)\dot{\epsilon}_0\right]}$$

These results, along with the definitions in Eqs. 3.5-1 and 2, give Eqs. (C) in Table 7.3-1 on p. 351.

c. If we presume that $\bar{\eta}_1$ going to ∞ is physically unrealistic, then, for $\dot{\epsilon}_0$ positive:

For convected Maxwell model: $\dot{\epsilon}_0 < \dfrac{1}{2\lambda_1}$

For White-Metzner model:

$$\dot{\epsilon}_0 < \frac{1}{2\eta/G} = \frac{G}{2m\dot{\gamma}^{n-1}}$$

$$= \frac{G}{2m\left[\sqrt{3+b^2}\,\dot{\epsilon}_0\right]^{n-1}}$$

whence:

$$\dot{\epsilon}^{\,n}_{0,max} = \frac{G}{2m\,(3+b^2)^{\frac{n-1}{2}}}$$

or

$$\dot{\epsilon}_{0,max} = \frac{(G/2m)^{1/n}}{(3+b^2)^{\frac{n-1}{2n}}}$$

There is really no way to decide whether the White-Metzner model is an improvement over the convected Maxwell model in elongational flow, <u>unless</u> some statement is made about the relations between the model parameters. For example, if we agree to equate $(G/2m)^{1/n}$ with $(1/2\lambda_1)$, then

$$\frac{(\dot{\epsilon}_{0,max})_{W.M.}}{(\dot{\epsilon}_{0,max})_{C.M.}} = (3+b^2)^{(1-n)/2n}$$

which suggests that the W.M. model postpones the onset of $\bar{\eta}_1 = \infty$ to a slightly higher value of $\dot{\epsilon}_0$.

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7B.10 Radial Flow between Two Lubricated Disks [RBB]

a. Continuity equation: $\frac{1}{r}\frac{d}{dr}(r v_r) = 0$

Integration gives $\quad v_r = -\frac{C}{r}$

where C is a constant of integration. Since the volume flow rate is Q, we have

$$Q = \int_0^{2\pi}\int_0^B (-v_r)\, dz\; r\,d\theta = C \cdot 2\pi B$$

so that $C = Q/2\pi B$.

b. For the velocity distribution being studied here, we have:

$$\nabla \underline{v} = \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{C}{r^2}; \qquad \underline{\underline{\gamma}}_{(1)} = \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{2C}{r^2}$$

$$\{\underline{v} \cdot \nabla \underline{\underline{\tau}}\} = \begin{pmatrix} (-C/r)(\partial \tau_{rr}/\partial r) & 0 & 0 \\ 0 & (-C/r)(\partial \tau_{rr}/\partial r) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\{\underline{\underline{\tau}} \cdot \nabla \underline{v}\} = \begin{pmatrix} \tau_{rr} & 0 & 0 \\ 0 & \tau_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(-\frac{C}{r^2}\right) = \begin{pmatrix} \tau_{rr} & 0 & 0 \\ 0 & -\tau_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{C}{r^2}$$

Then the convected Maxwell $\underline{\underline{\tau}} + \lambda_1 \underline{\underline{\tau}}_{(1)} = -\eta_0 \underline{\underline{\gamma}}_{(1)}$

becomes:

$$\begin{cases} \tau_{rr} - \lambda_1 \left[\frac{C}{r}\frac{\partial \tau_{rr}}{\partial r} + 2\frac{C}{r^2}\tau_{rr} \right] = -2\frac{C}{r^2}\eta_0 \\ \tau_{\theta\theta} - \lambda_1 \left[\frac{C}{r}\frac{\partial \tau_{\theta\theta}}{\partial r} - 2\frac{C}{r^2}\tau_{\theta\theta} \right] = +2\frac{C}{r^2}\eta_0 \end{cases}$$

or

$$\begin{cases} r\dfrac{d\tau_{rr}}{dr} + \left(-\dfrac{r^2}{C\lambda_1} + 2\right)\tau_{rr} = \dfrac{2\eta_0}{\lambda_1} \\ r\dfrac{d\tau_{\theta\theta}}{dr} + \left(-\dfrac{r^2}{C\lambda_1} - 2\right)\tau_{\theta\theta} = -\dfrac{2\eta_0}{\lambda_1} \end{cases}$$

Make the change of variables suggested in the text to get

$$\begin{cases} \dfrac{dT_{rr}}{dx} + \left(-1 + \dfrac{1}{x}\right) T_{rr} = \dfrac{1}{x} \\ \dfrac{dT_{\theta\theta}}{dx} + \left(-1 - \dfrac{1}{x}\right) T_{\theta\theta} = -\dfrac{1}{x} \end{cases}$$

c. First Solve the T_{rr} equation:

$$T_{rr} = e^{\int(1-\frac{1}{x})\,dx}\left[\int \frac{1}{x} e^{-\int(1-\frac{1}{x})dx}\,dx + C_1\right]$$

$$= e^x \frac{1}{x}\left[\int_0^x e^{-x}dx + C_1\right] = -\frac{1}{x} + C_1\frac{1}{x}e^x$$

$$T_{rr,0} = -\frac{1}{x_0} + C_1\frac{1}{x_0}e^{x_0} \longrightarrow \quad C_1 = x_0 e^{-x_0}\left[T_{rr,0} + \frac{1}{x_0}\right]$$

Next look at the $T_{\theta\theta}$-equation:

$$T_{\theta\theta} = e^{\int(1+\frac{1}{x})\,dx}\left[-\int \frac{1}{x} e^{-\int(1+\frac{1}{x})}\,dx + C_2\right]$$

$$= -e^x x\left[\int_\infty^x e^{-x}\frac{1}{x^2}\,dx - C_2\right]$$

$$= -e^x x\left[-\frac{e^{-x}}{x} - \int_\infty^x \frac{e^{-x}}{x}\,dx - C_2\right]$$

$$= 1 - xe^x\int_x^\infty e^{-x}\frac{1}{x}\,dx + C_2 xe^x$$

$$= 1 - xe^x E_1(x) + C_2 xe^x$$

$$T_{\theta\theta,0} = 1 - x_0e^{x_0}E_1(x_0) + C_2x_0e^{x_0} \quad \text{or}$$

$$C_2 = x_0^{-1}e^{-x_0}\left[T_{\theta\theta,0} - 1 + x_0e^{x_0}E_1(x_0)\right]$$

d. Next use the equation of motion

$$0 = -\frac{dp}{dr} - \frac{1}{r}\frac{d}{dr}(r\tau_{rr}) + \frac{\tau_{\theta\theta}}{r}$$

or
$$\frac{d}{dr}(p+\tau_{rr}) + \frac{\tau_{rr}-\tau_{\theta\theta}}{r} = 0$$

Let $P = p\lambda_1/\eta_0$ and rewrite the differential equation as:

$$2x\frac{d}{dx}(P+T_{rr}) = r\frac{d}{dr}(P+T_{rr}) = -(T_{rr}-T_{\theta\theta})$$

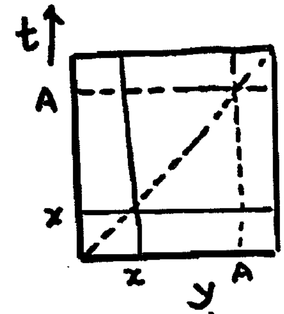
Whence

$$P + T_{rr} = -\int \frac{T_{rr}-T_{\theta\theta}}{2x}\,dx + C_3$$

$$= \frac{1}{2}\int\left\{\left[\frac{1}{x^2} - C_1\frac{1}{x^2}e^x\right] + \left[\frac{1}{x} - e^x E_1(x) + C_2 e^x\right]\right\}dx + C_3$$

$$= \frac{1}{2}\left[-\frac{1}{x} - C_1\left\{-\frac{e^x}{x} + \int_{-\infty}^{x}\frac{e^x}{x}dx\right\} + \ln x - \int e^x E_1(x)\,dx + C_2 e^x\right] + C_3$$

$$= \frac{1}{2}\left[-\frac{1}{x} - C_1\left\{-\frac{e^x}{x} + Ei(x)\right\} + \ln x - \ln x - e^x E_1(x) + C_2 e^x\right] + C_3$$



Note on evaluation of $-\int e^x E_1(x)\,dx$:

$$-\int_A^x e^x E_1(x)\,dx = +\int_x^A e^x E_1(x)\,dx$$

$$= \int_x^A e^y \int_y^\infty \frac{e^{-t}}{t}\,dt\,dy$$

$$= \int_x^\infty \frac{e^{-t}}{t}\left[\int_x^t e^y dy\right]dt - \int_A^\infty \frac{e^{-t}}{t}\left[\int_A^t e^y dy\right]dt$$

$$= \int_x^\infty \frac{e^{-t}}{t}(e^t - e^x)dt - \int_A^\infty \frac{e^{-t}}{t}(e^t - e^A)dt$$

$$= \ln t\Big|_x^\infty - e^x E_1(x) - \ln t\Big|_A^\infty + e^A E_1(A)$$

$$= -\ln x - e^x E_1(x) + \text{consts. to be absorbed into } C_3.$$

Then P is given by:

$$P = \frac{1}{x} - C_1\frac{1}{x}e^x + \frac{1}{2}\left[-\frac{1}{x} - C_1\left\{-\frac{e^x}{x} + Ei(x)\right\}\right.$$
$$\left. - e^x E_1(x) + C_2 e^x\right] + C_3$$

or

$$P = \frac{1}{2x} - \frac{C_1}{2x}e^x - \frac{C_1}{2}Ei(x) - \frac{1}{2}e^x E_1(x) + \frac{C_2}{2}e^x + C_3$$

Then

$$C_3 = P_0 - \frac{1}{x_0} + \frac{C_1}{2}\left[\frac{1}{x_0} + Ei(x_0)\right] + \frac{1}{2}e^{x_0}E_1(x_0) - \frac{C_2}{2}e^{x_0}$$

Substitution of the expressions for C_1 and C_2 and simplifying gives:

$$C_3 = P_0 + \frac{1}{2}T_{rr,0}\left(1 + x_0 e^{-x_0}Ei(x_0)\right)$$
$$+ \frac{1}{2}e^{-x_0}Ei(x_0) - \frac{1}{2x_0}T_{\theta\theta,0} + \frac{1}{2x_0}$$

Selecting $T_{rr,0} = 0$ and $T_{\theta\theta,0} = 0$ simplifies equations :

$$T_{rr} \cong -\frac{1}{x} + \frac{1}{x} e^{x - x_0}$$

$$T_{\theta\theta} \cong 1 - x e^{x} E_1(x) + \left\{ E_1(x_0) - \frac{1}{x_0} e^{-x_0} \right\} x e^{x}$$

The Pressure drop required is given in terms of $(P + T_{rr})$ as

$$P + T_{rr} \cong \frac{1}{2} \Big\{ -\frac{1}{x} + e^{-x_0} \left[\frac{1}{x} e^{x} - Ei(x) \right] - e^{x} E_1(x) +$$

$$\left(E_1(x_0) - \frac{1}{x_0} e^{-x_0} \right) e^{x} + e^{-x_0} Ei(x_0) + \frac{1}{x_0} \Big\} +$$

The Pressure drop is thus

$$\Delta P = (P + T_{rr,0}) - (P + T_{rr})_i \simeq P_0 - (P + T_{rr})_{x = x_i}$$

$$= -\frac{1}{2} \Big\{ -\frac{1}{x_i} + e^{-x_0} \left[\frac{1}{x_i} e^{x_i} - Ei(x_i) \right] - e^{x_i} E_1(x_i) +$$

$$\left(E_1(x_0) - \frac{1}{x_0} e^{-x_0} \right) e^{x_i} + e^{-x_0} Ei(x_0) + \frac{1}{x_0} \Big\}$$

$$\Delta P = \frac{1}{2} \left\{ \left(\frac{1}{x_i} - \frac{1}{x_0} \right) \left(1 - e^{(x_i - x_0)} \right) + \left[Ei(x_i) - Ei(x_0) \right] e^{-x_0} + \left[E_1(x_i) - E_1(x_0) \right] e^{x} \right.$$

The functions $Ei(x)$ and $E_1(x)$ cannot simplify by combined any further....