

Lecture #10: Background

When my two daughters were younger, I tutored them in chemistry, which they cordially hated (at first). Invariably during a session, I would hear the phrase “don’t tell me what it means, just show me how to get the answer fast!” If my daughters had taken an ocean acoustics course, they’d have been big fans of the parabolic equation (PE) method. It (by itself) gives you no physical insight, but it produces an answer – fast!

In the notes, there are an initial few lines on how second order PDE’s are categorized by their discriminants (gotten from the coefficients). The only thing to note here is that the Helmholtz equation is elliptic, and has slow, clunky numerical solutions. But parabolic equations can be solved far more quickly and efficiently. So the question is: can one approximate the elliptic Helmholtz equation by a parabolic form? It turns out yes – otherwise why would we be chatting about this? ☺

Another small aside – in teaching about PE since 1984 (almost 30 years), the method has advanced from a small angle, lower frequency, deep water tool to a very general, fully 3-D, all angle code that works well even up to a kilohertz and a bit beyond. The “basic” small angle derivation I present first is from a 1978 “SACLANT Center” report, and is very standard. However, that derivation is now pure pedagogy, and the “state of the art” (or close) is the material that follows it. Again, COA is a great reference for the newer numerics. (Note: There is still some good development in PE going on even today, specifically for 3-D and shear effects!).

The “old school,” basic PE derivation starts out with the Helmholtz equation (as it must), but immediately puts the $k(r,z) = \omega/c(p,z)$ into the form of a reference (background) wavenumber k_0 and a relative index of refraction, $n(r,z)$. This is not an unusual thing to do, and it turns out to be very useful. Next, the PE derivation invokes partial separation of variables, as we did for range dependent mode theory. However, the form of the solution is now $\psi(r,z)S(r)$; the $\psi(r,z)$ is just a plain vanilla, slowly varying function here, and not a sum over local modes. So, already there are two differences. As usual, we plunk the partially separable solution back into the PDE, and here we come up with two “separate” equations. The first one, for $S(r)$ is just Bessels equation, and as before we can break J_0 into outgoing and ingoing parts and just retain the outgoing. This $S(r)$ can now be put back in the second equation (which had both ψ and S in it), leaving a much simpler equation with second order partial derivatives of ψ in r and z in it. This still isn’t parabolic. The big trick now is, if we insist that $\psi(r,z)$ to be slowly varying in r , we can kick out the $\partial^2\psi/\partial r^2$ term, and only keep first derivatives in r . When we do this, we do indeed have a parabolic form! This equation can be solved rather neatly, as we discuss next!

This “marching solution” to the simple P.E. is actually rather trivial to show formally (though its numerical implementation is a bit more work, and is left for a term project!). The course notes are pretty detailed, so all we will say is that we get $\psi(r + \Delta r, z)$ from $\psi(r, z)$ via two simple Fourier transforms and two phase factor multiplications. These are very fast, numerically.

To examine the error in our solution, we put the marching solution back into the “exact” PE, and see what terms show up that do not solve the PE. This is a good way to examine error in a numerical solution.

Another way to examine error in our simple PE solution is to contrast the P.E. solution to a normal mode solution for a range independent waveguide where the mode solution is exact, but the P.E. is not. This is shown in section 7.4 of B&L, who also provide another simple PE derivation. The material on the “extended” P.E. in B&L is dated now, and can be ignored.

We return to COA now, and to “modern era” PE forms. In section 6.2.2 of COA, a “generalized PE derivation” is presented which factors the elliptic wave equation into operators, the most important of which is the “square root operator” shown in Eq. 6.14. Getting a good approximation to the square root operator, which extends the (vertical) angular range of the PE, has been the focus of much research over the past three decades. (The PE is so important, this level of effort was quite justified!). In section 6.2.3, this effort is discussed in detail, following the historical development right up to the present front-runner, the Pade series expansion of the square root operator.

Next we look briefly at the “starting field” for the PE at range r_0 . We just look at the model starter (COA sections 6.4.1 and 6.4.1.1), as it is the most common.

In 6.6., 6.6.1 and 6.6.2, the very popular “implicit finite difference (IFD)” form of PE solution is discussed. While there is a bit of algebra in this section, it is pretty straightforward, and the range step “answer” in Eq. (6.18) has a rather simple tridiagonal matrix form, which can be solved stably and accurately. It also admits wide angle solutions.

There is much more to the P.E. that we have omitted (e.g., shear, 3-D, etc.), but that is for Professor Schmidt’s course.

Also, at the end of the class notes in Appendix 1, which is the “recipe for a simple PE code.” As mentioned, this is part of the numerical methods term project.

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