GEORGE BARBASTATHIS: OK, so let's get started. I'm just trying to wire myself up. OK. So before we get into the PowerPoint here. So basically, what I'm doing is I'm picking up from the point where we left last time, where we're discussing waves. And what I would like to do now is derive a wave equation basically, for a mathematical form that should allow us to predict different kinds of waves. Of waves, not the waves.

All right, so what I start with here. I start with a general function, psi, of an argument, eta. So for those of you who are not fluent in Greek, this is psi, and this is eta. OK. At the moment, these don't necessarily have a physical meaning. But in order to produce a wave out of this function, what I can do is I can write it as psi, or f of z comma t, equals psi of z minus ct.

OK, now why is that a wave? First of all, z now has become a spacial coordinate. t, as usual, is time. So if you actually hide time. So now time is zero. Time is a constant or something like that. Then you actually have basically the same argument. But if you allow time to increase, then you can see that the time increase will have the effect of sliding the same shape along the spatial axis.

So for example, let's take this as our reference point here. The first peak after you advance time a little further than this point has moved fast, has moved to the right. And how much is this distance? Well, anybody? This is easy. I'm just trying to see if the people in Boston have woken up. Yes? Nobody's awake in Boston. So this distance equals ct. Because if I set the argument of this function to zero, then basically I recover my original function. So basically as I advance time, the fact is that this waveform moves to the right.

And at the moment, there's no obvious motivation for why I'm going to do what I'm about to do. But please take my word for it for a while. I'm going to take two partial derivatives. I'm going to take df dz, and df dt. OK, df dz equals basically c prime, where c prime-- I use c prime simply to denote the derivative of c with respect to its mysterious argument. So that's very easy. What about the affinity?

That's right. I have to apply the chain rule, because there is a multiplicative argument inside. So this is going to be minus c times c prime. So because c prime appears in both places over here, I can combine these equations and say, df over dz equals minus 1 over c, df dt.

So that is a partial differential equation. And you might say, well, what I did is a bit stupid, right? Because I had a relatively simple explanation for the wave. I actually had the wave just in front of me. And what I did is I wrote a partial differential equation which is not very intuitive. But the benefit of doing that is, of course, that the wave that I wrote here is a very special, very limited kind of wave that propagates without changing shape, without anything fancy happening.

We know from experience, for example, in real life, if you have ever seen water waves. Water waves, they don't quite propagate. They change shape in all kinds of complicated ways. So even though I derive this equation from that simple shape. What we'll see a little bit later that this equation actually can describe much more general waves. We're not quite done yet, because the question that I derived here applies for a wave moving to the right.

I can equally well contribute a wave that is moving to the left. And in this case, what I would have to do is I would have to write f of z comma t equals psi or z plus ct. A little bit of thought will convince you now that if you advance time, then the wave will have actually to move to the left, towards negative z in order to maintain its argument. And if I do the same game now. Again, I can write df dz equals c prime. That did not change.

This time, df dt equals plus c prime. Now, I have another partial differential equation, df dz equals 1 over c, df dt. OK, let me put back the equation. So this is a wave going to the left, backwards. And I had another equation, df dz equals minus 1 over c, df dt. And that is a way of going to the right.

OK. You can see very easily that if you take a second derivative of these equations. Let's go back to the first one. If you do the second derivative, d squared f dz square. You will pick up an extra minus sign from the same equation. And you will find that it is 1 over c square, d square f, dt square. You can do the same in this equation.

There's no sign here to begin with. But there is one over here, c square, d squared f, dt squared. So basically, this single equation. This single equation can describe both forward and backward propagating waves. And this equation. Actually, it's a three dimensional version. Not this one, but it's three dimensional version. It has the name Helmholtz wave equation. And because of Helmholtz, it's a little bit difficult to pronounce. Actually, that's not the real one. Anyway, it's a very commonly encountered wave equation. So typically we simply omit Helmholtz, and we say wave equation.

Now, why you might want to distinguish this as opposed to are there other wave equations? Yes, there is the Schrodinger equation. There is the Klein-Gordon equation. For those of you who do fluid mechanics, there is a Korteweg-de Vries equation, also known as KDV. There's a bunch of different wave equations. So we need some way to discriminate. Nevertheless, the other wave equations, they are referred with their name. So when you want to refer to Schrodinger's equation, you say Schrodinger's equation. By convention, when you say wave equation, you mean this one, so the Helmholtz. OK.

So this is all very nice, but it's actually down on your slide. So there's no reason to agonize over what I did over here. What I would like to remind you before I move on is for the simple case of a-- first of all, you can imagine that this is a very general kind of solution. Because what is c? After all, I put up this strange shape over here. I didn't say where it came from. So for that matter, if someone gives you the wave question, how can you go ahead and solve it?

Well, we're not going to do this in great detail here. This is a little bit beside the scope of the class. But generally, if you need to solve a partial differential like this one, you need additional information. You need the initial conditions. How did the wave start at that time equals zero? And you also need the boundary conditions. So what happens to the wave as it expands?

Sometimes, the boundary condition is at infinity. Maybe the wave is not bounded. It can propagate outwards all the way. And that's a good example of what happens if you are flying with a helicopter on top of the ocean. And you drop a-- I don't know. You drop a stone into the surface of the ocean. Then the wave that you generate, the wave that you generate. To a good approximation, you can use infinite boundary conditions. Because the ocean is not quite infinite, but very large.

If you generate in a wave in a small channel, then you might have to include boundary conditions. Because the wave will be confined by the walls of the channel. And we will see later that this is actually quite commonly the case. In optics, we don't call them channels. We'll call them wave guides. But the boundary conditions can actually significantly alter the shape of the wave.

So what I did here is just to give you an idea, I quote unquote, "solved." Nothing interesting happens here, but this is actually solution of the wave equation with a particular initial condition, where time equals zero. The wave is a sinusoid. I said that this is my equation. And I say that f of z at t equals 0, equals some sinusoid amplitude a, and wave number k. If that is the case, then by inspection. I don't have to do anything. All very important.

And what I did not write down, but it is implied, is that the boundary conditions are free. So this wave is free to go as far as it pleases without any special constraints. So if that is the case, then you can write the solution almost by inspection. Half of z comma t equals a cosine kz minus ct. And you can see by inspection here that because of the way I wrote it, this actually satisfies both the partial differential equation, the wave equation, and the boundary condition.

So therefore, this is a proper solution of the wave equation. In fact, I only wrote half of it. The wave, remember, it can go forwards, and can go backwards. To cover both cases, I actually write the solution this way. With the data that I have here, actually, I don't have enough to specify. Or actually, in principle, the wave can go both ways. I would have to apply to give you additional information to know which actually happened.

The other thing that I wanted to point out, which is not really difficult, but it takes some getting used to, is the various terms that appear inside this equation over here. So the term that we have been very familiar with is the wavelength. There's another of those crazy Greek symbols. It's called lambda. And the wavelength is related to the quantity that we have over here, as 2 pi over lambda equals k. So lambda is the wavelength, as I said.

K is known as the wave number. So the relationship that they have is if you think of the wavelength as a period-- which it is, the period of the wave in the space domain-- then k would actually be the angular frequency. But we'll call it wave number to keep them separated. OK.

Equivalently, you can define the actual angular frequency of the wave. Usually this goes by the symbol omega. And from this equation over here, very simply you can see that if you wanted to write it in the form a cosine kz minus omega t, then clearly, omega is the angular frequency. And when comparing the two equations, you can see that omega equals kc. So the angular frequency, then, is related to the spatial frequency, the wave number, and the speed of the wave.

And of course, this is nothing new. We've seen this equation before, but we saw it in disguise. And that's why I'm going through this. Because all the symbols, they can be played around with quite a bit. So this equation, I can rewrite. And the [INAUDIBLE] is like this. So omega is the angular frequency. I also have the plane, so to speak. Frequency nu, which is related as omega over 2 pi. One of them goes [INAUDIBLE] radians. I'm sorry, it's measured in full radians per second. This is omega. Nu is measured in hertz, simply inverse seconds. Bless you.

So I can substitute in this equation. Omega is 2 pi times nu. k, according to my equation over here, is 2 pi up on lambda. And c is c. So the two pis cancel here. The two pis cancel here, and I end up with equation c equals lambda nu, which I think we've already seen a couple of times before. Who call it the dispersion relation of the wave.

So this is how all this came about. I haven't said anything new about this. I'm just trying to point out the different ways that this equation can manifest itself. And the other two terms that we encountered last time, and I wanted to remind you. One is the amplitude of the wave. This is this constant a over here. For some reason, in the slide, they decided to call it a0. Anyway, it is the same thing.

And also, so we said last time that in general, the wave can also have a phase delay. So in here, I can not cite the constant plus fee, which does not really do anything by itself. All it does if you wish equivalently, it shifts the time origin. So that's a relatively trivial thing because after all, I am free to start my clock whenever I want. If I have a wave that never changes, I can start my clock whenever I want. So it doesn't really make a big difference.

But we saw last time that if you have two waves that are, as we said, interference-- or they're simultaneously happening, so to speak-- then the relative phase delay between the two can actually be quite significant. And I will play this movie again. In this case, the two waves are actually-- they have the same phase between them. In other words, if you look at the center, they're both simultaneously bright or simultaneously dark.

Whereas in the other case, it's the other way around. They're off by pi. So when one is maximum, the other is minimum. I think it is worth playing this one. This also has the interesting effect that this can hypnotize you. So for those of you who are in Boston and not fully awake yet. Yes. Oh, so when you ask a question, you need to push the button.

AUDIENCE: So for them for now are two waves travel in the opposite direction. They have same amplitude, about perfect cancel each other. So in this case, where does the energy go?

GEORGE BARBASTATHIS: They don't quite cancel each other, right? So as we will see I will actually do this in a second. It's called a standing wave. So you will see that they don't quite cancel. They just produce a, well, what is called a standing wave. So I will get to your question. I think it's actually coming up next. Yeah, so give me a second and I will get to your question.

So that is a standing wave. So like your colleague asked over here. Suppose that you have a superposition of two waves. Now, both are solutions to the wave equation. Now, by truce. I don't want to do the proof here. It takes three lines. But by its looks, this is a linear differential equation. So basically, it means that if you know one solution, you know another solution. And then you add these two solutions after multiplying by arbitrary constants. The result from the addition is still a solution to this equation. A very fundamental property that we use again and again later in the class.

So that said, suppose that I pick two solutions that I already saw. One of them is a forward propagating wave, the other is a backward propagating wave. So here they are. The one with the minus is propagating forward. The one with the plus is propagating backward. So your colleague here asked, what will happen if I superimpose them? So one of them is going to the right. One is to the left. Are they going to cancel? What are they going to do?

OK, so this is a computational. So perhaps you should not trust it. After all, what is this? But anyway, this is what I got in Matlab when I simulated this. OK, let me play this again. So this is the result of the two waves. OK. I pushed the button. I did not play the movie again. I revealed the derivation, but that's quite all right.

Actually, I meant to do this on the free hand here. OK, never mind. Ignore what you see on the slide, and just watch the whiteboard. I guess I can call it whiteboard, can't I? I mean, it is a whiteboard. OK, so here are the two waves. So I actually omitted. I should have said that not only are they counterpropagating, but also, I pick them both to be harmonic waves. So that we can do the math easily.

OK, so let's add this out. So we picked the identical amplitude. So one of them will be written as-- did I use a, or was it-- a, yeah, OK. OK, now, for generality. I don't really have to, but for generality, I also added the phase, a phase delay field to the first wave. So this one is forward. And then I have the other one, which looks very similar, except for this devious minus sign that goes backward. OK. And the superposition principle says that I can simply add the two waves, and the summation is still a solution to the wave equation, therefore, still a valid wave.

OK, so what is the result? So now, I don't know how to tell you to remember this. Most people look it up in books of trigonometry. I have a second mnemonic for it, but I will not tell you. So there's a mnemonic that says if you have the sum of two cosines equals-- now, this is a mathematical property, so there's no intuition to be derived here. This is just obtained by trigonometric manipulation. Equals twice cosine, a plus b over 2, cosine a minus b over 2.

So I never remember this one. I will tell you actually my mnemonic. It's nothing strange. All I remember is that if you have cosine of two things, it equals-- let me write it cleanly on a clean piece of paper. So cosine of two things. It has a minus sign here. And sine of two things.

OK, this, you have memorized. And you can see very easily that from these two, you can obtain the equation we said before. Because, for example, you can write cosine alpha plus beta, plus minus beta equals cosine alpha cosine beta minus plus sine alpha sine beta. OK, so if you add the term with a plus, and the term with a minus. The sine terms evaporate, and you end up with the identity that they had before. So this is my mnemonic for remembering this one.

We will not have to do this too often because last time, if you remember, we introduced phasors. We introduced a complex notation for waves, so we don't have to deal with this trigonometry very often. In this case, there's no way out. Well, actually, there's a way out, but it is simpler to do it this way.

So now, I have basically a equals kz minus omega t plus phi. b equals plus omega t plus phi. So if you apply my formula here, you will find that the wave actually-- OK, the amplitude is the same. I pick up a factor of two. So it is two. And then I have the cosine of the sum. So if I sum the two, the time term will disappear. So we'll have cosine. Well, you get twice of it, but there's a factor of two here, so they cancel. It'll be cosine kz plus phi. And if I subtract the two, then naturally, the spatial terms and the phase will disappear. And I will get the cosine omega t.

OK, now do I believe Matlab? I think I should believe Matlab, because let's see what Matlab saw. Here, time is frozen. So you'll see basically the same term where this has been replaced by constant, because I froze time. So in there, this looks like a sinusoid of the same frequency as we had before. And actually, of course, you have no way of knowing that. But the waves that I entered had an amplitude that was half of this. So basically, the factor of two was also reproduced by my simulation.

OK, and if I play the movie again, it is basically like unfreezing time. OK, so here, I have unfrozen time. And you see that I get a sinusoidal variation in the time domain. So therefore, that's my term cosine omega t here. So basically, the Matlab simulation is correct. So we're very happy. OK.

So now, let me come to your question. What happens to the wave when it vanishes? First of all, it doesn't really vanish. I don't know how to go back. So it does look like at some point the wave vanishes, and then it reappears. OK, it is a little bit annoying that it vanishes. But in electromagnetic waves, and usually in other kinds of waves, we don't worry about instantaneous energy. We worry about the average energy.

So what you can see here is that the average energy is actually conserved. Because as the wave oscillates from a maximum to a minimum, it's average energy remains the same. Now, you might ask, well, what happens in between? I mean, what happens when this wave has gone to zero? Again, I will [INAUDIBLE] kinds of electromagnetic waves. What you have done here is you have basically launched two waves that are moving in opposite directions. So who is providing the energy? Well, if you are providing the energy on one end, and someone else is providing the energy on the other end.

What happens when the two waves cancel is that the energy is actually stored in another place, which does not show up in this equation. That's one way to think about it. So if you remember when I saw-- actually, you don't remember, because you were not in the class one week ago. But I showed an example of a simple harmonic oscillator.

So simple harmonic oscillator goes from zero velocity maximum potential energy to the edge, where the velocity now is-- I'm sorry, to the rest position, where the velocity's maximum and the potential energy has become zero at the rest position. So it's the same kind of thing. The kinetic energy disappears. Well, what happens to it? It has become potential. So it is very similar to this. The energy has become potential energy, and you don't see it here.

AUDIENCE: OK, maybe a follow up on the question. So now, let's assume the two wave are traveling along the same direction, and they are out of phase by pi. In this case, they'd be a perfect cancelling each other. So where does the energy go? Just for your information, the previous answer, again. Such thing will not happen in the first place, but just not comfortable with this answer. So I would like to know your comments.

GEORGE BARBASTATHIS: [INAUDIBLE] the wave start at the pi phase shift. It's equivalent to not having launched any wave at all. Whoever launched the two waves basically canceled his own action. So if you think about it physically, what happened? First of all, have the two waves been launched at infinity, or at the finite time?

AUDIENCE: So for the [INAUDIBLE] have, and as that equal zero. And this wave traveling to the positive direction and continuously. Now, I'm not sure another wave added them [INAUDIBLE] that equals minus 10. But this show is that we've only lasted for three period for the input. So as we are traveling into a positive direction, it is going to cancel three periods out the previous wave, right?

GEORGE BARBASTATHIS: Now you're making it a little bit more complicated. Because what happens to the edges of the wave? Are you multiplying by a boxcar function? What happens later? You're not given enough information to answer the question. Because then we had three periods cancel, but then what happens later? Did you just stop the wave?

AUDIENCE: I'm still not so sure, yeah. That's why I check. Yeah, thanks.

GEORGE BARBASTATHIS: The question's a bit ill defined, I guess. So I can not answer it. But I can tell you this. Suppose that I launch a wave at some time, and then a certain time later, I launch the same wave with biphasic trait. So there's no doubt that the two waves at this point will cancel.

So what happened to the energy? Well, it is equivalent to thinking that I did not really launch a second wave with a pi phase shift. But I actually turned off the swings that was producing my first wave in the first place. There might be a tangent or something like that, but in fact, I've just canceled the wave. So simply stop the flow of energy. Everybody happy with this answer, or should we discuss this?

And I think to realize that actually this is interesting, that we brought up energy. In the standing wave, the energy is actually not going anywhere. Well, that's one way to think about it. It's not going anywhere. Or you can think about it in the way that the energy is going back and forth. I did not realize, actually. I did not the phasors last time, did I? Did I? I guess I didn't.

So I guess I pulled a fast one. A few minutes ago, I was talking about the difficulty of trigonometry. And I said that's why I introduced phasors last time. I was mistaken. I had not introduced any phasors last time. I'm introducing phasors now. And the reason, of course, that we're introducing phasors is because we don't want to have to remember complicated identities like this one.

By the way, another point that I would like to bring up before I move on to phasors is that the wave that I'm showing here is actually very different than the traveling wave that led us to the wave equation. You'll agree with that. That looks very different, both in the way it behaves with time and space, and the way it looks like an equation. Yet, it still originated from the same wave equation.

So this is actually the power of partial differential equations and superposition. That I took two solutions with my partial differential equation that were not particularly interesting, I superimposed them, and then all of a sudden, I created another solution, which is still a very valid wave, but we have totally different. It is a stationary wave. It has this oscillation, and so on and so forth. So that is why we bother to write this partial differential equation, because it allows us to generate a much richer set of waves that are, of course, very important in practice. Standing waves happen all the time.

OK, so now, let's go to phasors. OK, so just like before, I wrote two waves that were propagating in opposite directions. Now, what I'm going to do is I'm going to write two waves that are phase shifted by pi over two. So one of them is cosine kz minus omega t plus phi. And the other is phase shifted by pi over 2, which means instead of a cosine, it is a sine.

OK, both are valid solutions to the wave equation with different initial conditions, but that doesn't bother us. They're still valid solutions. Therefore, their superposition is also a solution. And I said before that when you do a linear superposition, you can also multiply the waves by arbitrary constants. I said this a little bit fast, but I hope that remember that I said it. In fact, in this class, we have evidence of what I said and what I did not say. Then go back to the video and see if I said it, and I did.

So I will pick a very particular constant, which may sound like a crazy choice, but I will pick it. And that is the imaginary constant, i, also known as one of the two square root of minus 1. OK, what is this now? So the first confession that I will make is that it is not physical anymore. There's no such thing as a complex wave. The waves that we see around us are real. They can be measured by instruments, such as oscilloscope, scanners, I don't know what else. They're real. As far as I know, nobody has ever measured anything complex.

However, what I'm doing here is mathematically correct, even if it has no physical meaning. And as it turns out, it provides a huge mathematical convenience. So I'm justified in doing it as long as I remember what is the physics that is behind. And the physics that is behind it is that every time I see a wave like this, I have to remember that the physical quantity is the real part.

So the physical wave-- the one that I can actually measure, observe, generate with natural means-- is the real part. The imaginary part is something that I added for mathematical convenience. And it isn't reality because I forget whose formula. It is called the De Moivre. Anyway, one of those. D-E? Moivre, yeah.

AUDIENCE: De Moivre.

GEORGE BARBASTATHIS: De Moivre, yeah. OK, I got it right. So De Moivre, then, if I remember correctly, that's how you spell it. De Moivre says that this thing here is a complex exponential. So that's very nice because for example, if you have to multiply two cosines, it's a big pain. You have to remember the trig to multiply two exponential. It is trivial. You just add their phases, I mean, their exponents. So a lot of inconveniences to be to be expected.

So this representation, it comes with different names, which are sometimes used confusingly in an interchangeable way. So some people call this a phasor notation. I prefer to call this the complex representation of the wave. It becomes a phasor when we drop-- I'm going to do something nasty here.

OK, remember that I can write this as a project. So I can write this as e to the ikz, e to the minus i omega t, e to the i phi. OK. If you recall during the beginnings of the class, we saw that the wavelength in the middle changes. But there is a property of the wave that does not change. And that property, if you recall, is the temporal frequency, omega, or nu, which are related by two pi.

That is true in linear media. Of course, there is a class of optics called non-linear, where actually, the frequency can change. It can double, or it can half, or [INAUDIBLE] of things can happen to it. But in this class, we don't deal with these complicated things. We just deal with linear optics. So in linear optics, the temporal frequency of the light, no matter what happens to the light. If it goes through glass and reflects, and all kinds of things. The temporal frequency remains the same.

So when we deal with optical waves of a single frequency-- that is the quality's fixed and the medium is linear-- then this term is actually superfluous. We don't have to remind ourselves that the light goes like e to the minus omega t. It does, but we don't have to write it. Because if we keep writing it, it clutters our equations. So we simply drop it. We drop it.

Very important to remember. We are allowed to drop it if we know that we have a single color in our system, a single frequency. If there is multiple frequencies, we can not do that anymore. Because then these frequencies have to adapt in a certain way, and you have to keep track of their additive phases. So we can only drop this term if we-- two conditions. If we know that we have a linear medium, and a single frequency light propagating in that medium.

So in my terminology, at least, after we drop this term, then it becomes a phasor. So what is left? e to the ikz plus 5. That, I call a phasor. And you notice what happened there is I only left the spatial dependency of the wave plus the phase delay. The temporal variation is-- it's hidden. It is not gone. It is just hidden.

OK, now let's see this in action. So what I'm going to do is I'm going to rederive the standing wave, but using the phasor. So you can see how nice and simple it comes out, and we don't have to agonize over it. But before I do it, I actually have to solve a little problem. And the problem is that I did this very nice derivation for the phasor of the forward propagating wave. What about the phasor for the backward propagating wave?

The backward propagating wave will be something of this sort. So this will always forward, right? Let me write the backward. I think I have enough space here, so I can do it. OK, so the backward would be something of the sort of the form of cosine, az plus omega t plus phi. And it is very inconvenient. Because the way I got the phasor is I got rid of a term of the forward to the minus omega t.

If I add the imaginary part here, I don't have a term e to the minus omega t, so I'm stuck. But I can do some mathematical trickery. Now, this is pure trickery, but it works. The trick, it is the following. We know that the cosine is an even function. So this is actually identical. I can replace it, in other words, with the cosine of the negative element. No harm done. It is still a backward propagating wave. Nothing has changed. It is the same equation.

Now, according to my good principle here, I add. That is not the same as before. Because the sign is an odd function. But this is a non-physical term, anyway. I just put it there for mathematical convenience, so I don't particularly care. I'm very happy, because the real part is the same as I started with. What the imaginary part does-- as long as it's mathematically convenient, I couldn't care less.

OK, so that gives me the license to write this. Now, this, of course, equals e to the minus ikz minus omega t minus phi. Now very happy, because I did get a term e to the minus i omega t in the phase of this exponential. And therefore, I can drop it to produce my phasor. So the last one, then, is that the phasor for the backward wave is e to the minus ikz minus i phi. In other words, if I give you a forward propagating wave, and you compute its phasor. And then I asked you, well, what is the phasor for the backward propagating wave? Well, all you do is you flip the sign and the exponent.

Whoa. Where did this come from? I think this is an old paper from some previous lecture. But anyway, ignore this. I have to remember to ask for a new paper next time. OK, so what I will do now. Let me reproduce the standing wave. So the standing wave is using the phasors now. It's e to the ikz plus phi. That is the forward wave, plus e to the minus ikz minus i phi. That is the backward wave.

Can I deal with that? Well, yes, actually. Because, again, De Moivre says that if you add two exponential of this form-- notice they have the same phase within a minus sign-- I will simply get two i's cosine az plus phi. So I've got my thing here. Not quite, though, because the standing wave. It had the cosine omega t term. What happened to the cosine omega t?

Oh, come on. OK. What happened to the cosine omega t? Well, remember, we neglected an e to the minus omega t. We didn't really neglect it. We hit it, because we didn't want to carry it when we write the equation. First of all, if I don't care about the temporal variation, I don't really have to do anything. I'm done. But if I want to put a temporal variation back in, what I have to do is I have to put back what I took when it did not belong to me. And what did not belong to me is this e to the minus i omega t. OK, I put it back.

Now, that is still not a physical wave because it is complex. How can I find the actual, physical wave? Well, I have to take its real part. And the real part is easy here, because the only complex thing is this exponential. So if I take the real part of this, of all of this now. It would simply be 2 cosine kz plus phi cosine omega t. And I'm done. I've actually produced my standing wave.

Any questions? We're almost done here. So I have about three minutes. So I think I'll use them to finish up this lecture, so we can move on to bigger and better things. The question that I wrote before is the one dimensional wave equation. Because the waves that I've written. They can only go down one spatial axis, z. The more generalized, of course, situation is three dimensional waves. And what you see here is the generalized three dimensional wave equation, which, of course, I will not derive. But it can be pretty easily derived using the same arguments that we did before, except you have to generalize in the third dimension.

And this produces things that we've seen before in geometrical optics, for example, the plain wave. Except now, you see the wave description, it is you have alternating positive and negative-- how do you call those-- peaks and troughs, which are propagating along an arbitrary angle. Now, they can go in any angle in 3D space. So you just see a cross-section here.

Or you can have a spherical wave, which we saw it before. It starts at a given point, and then radiates outwards. So that's a spherical wave. Well, we're not quite finished, yet. But I think we're almost out of time. So I think I'll stop here, and we can still continue next time. Are there any questions? There's no homework due on Wednesday, so what I would like to-- you have a question? Yeah, there's a question. Go ahead.

AUDIENCE: How is the wave equation intuitively different from simple harmonic oscillator? The variation in space depends on the relation in time. So that's coupling?

GEORGE BARBASTATHIS: Yeah, the mathematical answer, of course, is very easy. They're simply harmonic oscillators in an ordinary differential equation. This is a partial differential equation. But let me think how to formulate a physical answer. OK, so the physical answer is that the harmonic oscillator is a singly particle whose position you track as a function of time.

A wave is a distributed physical system. So you can think of a wave-- in this case, for example. You can interpret the physical meaning of these black and white stripes as particles which are moving. For example, the particle that is on the white stripe is moving up. The particle in the black stripe is moving down.

If I play the wave again, what you will see that the particles are kind of executing a coordinated motion. Because you have particles that were down, then they go up, and then they go down again. But they go out in a coordinated motion throughout the entire plane. How do they know? Well, there is a physical system associated with this. For example, this could be a membrane with the sound wave propagating on it. And now the reason that particles are connected, that they execute a coordinated motion is because of the interatomic forces between the membrane particles.

If it is an electromagnetic wave, then it is a field that is distributed. And, of course, it knows to be connected because of Maxim's equations. There's things like charge conservation and so on that force it to act this way. So a wave is actually a distributed system, where the physics force an extended systems-- particles, or fields, or whatever the case may be-- to execute a coordinated motion. That's the difference between that simple harmonic oscillator where you just have one particle.

Now, you can go from one to the other. Imagine you take a single harmonic oscillator. That's a good one. So let me draw it here. Wow, we're actually going backwards now in the knowledge that they used in the past. So you take a simple harmonic oscillator. Here it is, a pendulum. This is a simple harmonic oscillator. Then you couple it with a separate pendulum. And how do you couple-- they're not coupled at the moment, but you can couple them, for example, by connecting the spring in between.

But that's not quite a wave. But you can see now that because I coupled them, if I kick one of these, it will also cause the other one to move, so they become coordinated. Well, now, let me generalize. Imagine that I have a bunch of harmonic oscillators. And, in fact, I can make them infinite, and I couple them. Now, in fact, the system will become wave like, in the sense that if you kick it at one end. Let's say not at infinity. Let's say it starts actually here, and then it goes to infinity.

So if you kick it at one end, then the disturbance will actually propagate very similar to a wave. It's a discrete wave, but it's still a wave. So you can go from one to the other by introducing kind of a coordination mechanism. So it is the coupling here that does the coordination. Any other questions? We can stay here, actually. Nobody claims the classrooms either here or there. So if you have more questions, I'll be happy to stay and answer them.

But about the phasors, is it clear? When I was your age, when I was a student, I was also very upset by phasors. Because they look very much like mathematical trickery. I mean, I'm trying to find my-- I guess that's one benefit of this over the whiteboard, that I have a history of what I did. But yeah.

But the phasors, they look very much like mathematical trickery. I mean, this looks like someone pulled it out of a hat. But the only reason to justify it in your mind is that it provides huge mathematical convenience. Now, this looks like a trivial thing. I mean, the derivation of the standing wave was maybe four lines with trigonometry, two lines with the complex exponential. So OK, why mess with complex numbers?

Well, try superposition. If you have several waves superimposed, or if you have an integral, a continuum of superimposed waves, which we will see very soon. Like Fourier transforms and so on. Fourier transforms are extremely inconvenient if you write them as cosine transforms. Big mess. If you write them as complex exponentials, it's actually very simple to do the math. It is a very, very time saving tool, basically. Strangely enough, it leads to some insights that real numbers do not give you. So that's an additional convenience. We will appreciate that later. For now, it is simply the mathematical convenience that drives this mathematical wizardry.

AUDIENCE: Yeah. You're saying [INAUDIBLE] angular frequency and the temporal frequency are different, or it's the same? Because you initially defined omega as angular frequency.

GEORGE BARBASTATHIS: I confuse those myself, usually, the terms. Usually nu. Usually nu is called the frequency. And it is measured in hertz. 2 pi nu is measured in radians per second. Angular frequency, isn't it? Yeah. As opposed to what else? I guess I don't understand the question. Yeah, what I mean is that, for example, take the propagating wave.

The propagating wave is e to the i-- let's say forward-- kz minus omega t. The standing wave is cosine kz e to the minus i omega t. I'll step ahead a little bit, and write something awful. This is actually a wave. What we'll see is called a diffraction integral. And you can see it is nothing-- even though it looks really complicated, it is just a superposition. Because the integral is a summation.

And then what I have here is a phasor, and I put a lot of these phasors together. What they all share is this term, e to the i minus I omega t. Actually, to be strictly correct, I should really carry this term. I'm not really allowed to drop it. But because it is the same always, I might as well just as well not write it. So it basically saves me writing. And also confusion, because as you know, the more things you write, the more likely you are to make mistakes.

It is typical in these derivations. If you don't need something, you drop it. Because if you start carrying it around, it involves mistakes. So that's why I would drop this term. So I can just as well say that if I have a propagating wave, I can just write it as e to the ikz. That's a propagating wave. I have a standing wave. I can just as well write it cosine kz. OK, the e to the i minus i omega t is implicit, right? If I have this diffraction integral, I can just write it without this term.

But again, the term is implicit. If I have to figure out what is the temporal variation of the wave, I have to put it back in. And again, I cannot do it if I have two waves of different frequencies. If I have e to the i omega. Let me write another superposition. e to the i k1 z minus omega one t, plus e to the i k2 z minus omega 2t. That's still a valid superposition, right? Provided that the ratios of these, k1 over omega 1 equals c. k2 over omega 2 equal c. Still a valid superposition. I'm not violating anything.

But I cannot do the phasor trick anymore. I have to keep the e to the i omega times, or I will make a mistake. There's one more case when if you produce any kind of nonlinearity, where well we'll see later. And again, we're going a little bit ahead. I'm in trouble. These are from the quiz. We'll a little bit later. When we try to compute the flux of electromagnetic energy, there's something called the pointing vector, which is defined as the electric field cross with the magnetic field.

If you tried to write the expression for the pointing vector, you cannot just multiply the phasors of the electric and the magnetic field, because it's a product. So you can use phasors only when you have linear operations, such as addition or integrals. In this case, you can't. We will see what you do in this case. You do a temporal average. There's a few cases where one has to be careful with this phasor.