MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.71 Optics **Spring '09**

Solutions to Problem Set #3

Due Wednesday, March 4, 2009

Problem 1: Wanda's world

a) The geometry for this problem is shown in Figure 1. For part (a), the object (Wanda) is located inside the bowl and we are interested to find where the image is formed. We start by using the matrix formulation to analyze the given system,

$$\begin{aligned}\begin{bmatrix}\alpha_i\\ x_i\end{bmatrix} &= \begin{bmatrix}1 & 0\\ s & 1\end{bmatrix}\begin{bmatrix}1 & -\frac{(1-n)}{-R}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ \frac{R}{n} & 1\end{bmatrix}\begin{bmatrix}n\alpha_o\\ x_o\end{bmatrix} \\ &= \begin{bmatrix}\frac{1}{n} & \frac{(1-n)}{R}\\ \frac{R}{n}+\frac{s}{n} & 1+\frac{s(1-n)}{R}\end{bmatrix}\begin{bmatrix}n\alpha_o\\ x_o\end{bmatrix}.\end{aligned} \tag{1}$$

To find the imaging condition, we note that all the rays, regardless of their departure angle α_o, arrive at the image point x_i (i.e. $\partial x_i/\partial\alpha_o = 0$),

$$\begin{aligned}\frac{R}{n}+\frac{s}{n} &= 0 \Rightarrow \\ s &= -R.\end{aligned} \tag{2}$$

We see that the image is formed at the center of the bowl and is *virtual*. Using this result in equation 1,

$$\begin{bmatrix}\alpha_i\\ x_i\end{bmatrix} = \begin{bmatrix}\frac{1}{n} & \frac{(1-n)}{R}\\ 0 & n\end{bmatrix}\begin{bmatrix}n\alpha_o\\ x_o\end{bmatrix}. \tag{3}$$

The lateral magnification is,

$$M_L = \frac{x_i}{x_o} = n, \tag{4}$$

and therefore, the image is *erect*.

b) For this part, the object (Olive) is located outside the bowl and we are interested to find where the image is formed. Again, we solve this part using the matrix formulation,

$$\begin{aligned}\begin{bmatrix}\alpha_i\\ x_i\end{bmatrix} &= \begin{bmatrix}1 & 0\\ \frac{s'}{n} & 1\end{bmatrix}\begin{bmatrix}1 & -\frac{(n-1)}{R}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ s & 1\end{bmatrix}\begin{bmatrix}n\alpha_o\\ x_o\end{bmatrix} \\ &= \begin{bmatrix}1-\frac{s(n-1)}{R} & -\frac{(n-1)}{R}\\ \frac{s'}{n}+s-\frac{ss'(n-1)}{nR} & 1-\frac{s'(n-1)}{nR}\end{bmatrix}\begin{bmatrix}n\alpha_o\\ x_o\end{bmatrix}.\end{aligned} \tag{5}$$

From the imaging condition ($\partial x_i/\partial \alpha_o = 0$), we solve for s',

$$
\begin{aligned}
\frac{s'}{n} + s - \frac{ss'\,(n-1)}{nR} &= 0 \Rightarrow \\
s' &= \frac{snR}{s\,(n-1) - R}.
\end{aligned} \tag{6}
$$

The lateral magnification is (for an on-axis ray, $\alpha_o = 0$),

$$
\begin{aligned}
M_L &= \frac{x_i}{x_o} = 1 - \frac{s'\,(n-1)}{nR} \\
&= 1 - \frac{snR\,(n-1)}{(s\,(n-1) - R)\,nR} = -\frac{R}{s\,(n-1) - R}.
\end{aligned} \tag{7}
$$

From equations 6 and 7, the following cases arise:
1. If $R > s(n-1) \rightarrow s' < 0$, the image is *virtual, erect* and is located at a distance $|s'|$ outside the bowl.
2. If $R > s(n-1) \rightarrow s' > 0$, the image is *real, inverted* and is located at a distance $|s'|$ inside the bowl.

c) If we were to consider the glass container of thickness t as well as the inner, R_1, and outer, R_2, radii, the matrix formulation becomes,

$$
\begin{aligned}
\begin{bmatrix} \alpha_i \\ x_i \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ \frac{s'}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{(n-n_g)}{R_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{t}{n_g} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{(n_g-1)}{R_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} n\alpha_o \\ x_o \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ \frac{s'}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 - \frac{t(n-n_g)}{R_1 n_g} & P \\ \frac{t}{n_g} & 1 - \frac{t(n_g-1)}{R_2 n_g} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} n\alpha_o \\ x_o \end{bmatrix} \\
&= \begin{bmatrix} 1 + Ps - \frac{t(n-n_g)}{R_1 n_g} & P \\ \frac{s'}{n}\left(1 - \frac{t(n-n_g)}{R_1 n_g}\right) + \frac{t}{n_g} + s\left(\frac{s'P}{n} + 1 - \frac{t(n_g-1)}{R_2 n_g}\right) & 1 + \frac{s'P}{n} - \frac{t(n_g-1)}{R_2 n_g} \end{bmatrix} \begin{bmatrix} n\alpha_o \\ x_o \end{bmatrix},
\end{aligned} \tag{8}
$$

where,

$$
P = -\left[\frac{(n-n_g)}{R_1} + \frac{(n_g-1)}{R_2} - \frac{t}{n_g R_1 R_2}\,(n-n_g)\,(n_g-1)\right]. \tag{9}
$$

For the case of uniform glass: $R_1 = R_2 = R$. The imaging condition is,

$$
\frac{s'}{n} + s - \frac{ss'\,(n-1)}{nR} - \frac{s'\,t\,(n-n_g)}{Rn_g n} + \frac{t}{n_g} - \frac{st\,(n_g-1)}{Rn_g} + \frac{ss'}{nR}\left(\frac{t\,(n-n_g)\,(n_g-1)}{Rn_g}\right) = 0
$$

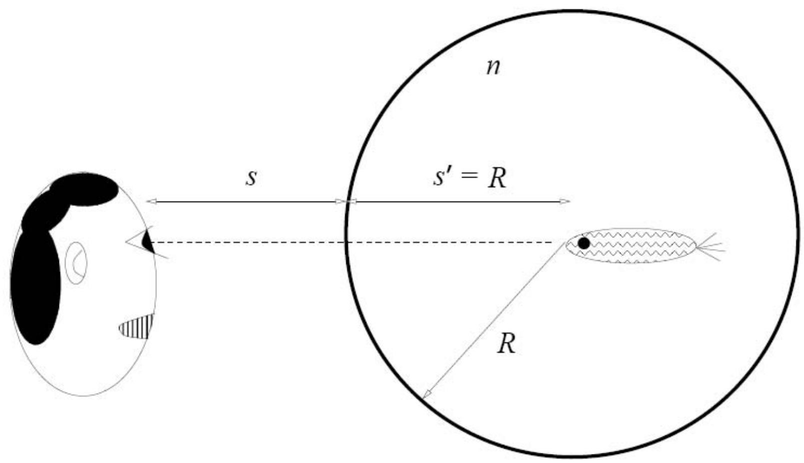

Figure 1: Wanda's world problem

$$\frac{s'}{n} + s - \frac{ss'(n-1)}{nR} + \delta_g = 0, \tag{10}$$

where,

$$\delta_g = -\frac{s' t(n-n_g)}{Rn_g n} + \frac{t}{n_g} - \frac{st(n_g-1)}{Rn_g} + \frac{ss'}{nR}\left(\frac{t(n-n_g)(n_g-1)}{Rn_g}\right). \tag{11}$$

Comparing equations 10 and 6, we see that in order to neglect the aquarium walls we require that $\delta_g << 1$,

$$t\left[-\frac{s'(n-n_g)}{Rn_g n} + \frac{1}{n_g} - \frac{s(n_g-1)}{Rn_g} + \frac{ss'}{nR}\left(\frac{(n-n_g)(n_g-1)}{Rn_g}\right)\right] << 1 \Rightarrow$$

$$t << \frac{R^2 n_g n}{R^2 n + ss'(n-n_g)(n_g-1) - s'R(n-n_g) - sRn(n_g-1)}. \tag{12}$$

Problem 2: Ball lens magnifier

a) In class we derived the composite matrix for a thick lens surrounded by air and is given by,

$$M_{thick} = \begin{bmatrix} 1 + \frac{d}{R_2}\left(\frac{n-1}{n}\right) & -\left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{d(n-1)^2}{nR_1R_2}\right] \\ \frac{d}{n} & 1 - \frac{d}{R_1}\left(\frac{n-1}{n}\right) \end{bmatrix}. \tag{13}$$

For a ball lens magnifier, the lens thickness and radii are: $d = 2R$, and $R_1 = -R_2 = R$. Using these values on equation 13 we get,

$$M_{ball} = \begin{bmatrix} 1-2\left(\frac{n-1}{n}\right) & -\frac{2}{R}\frac{(n-1)}{n} \\ \frac{2R}{n} & 1-2\left(\frac{n-1}{n}\right) \end{bmatrix}, \tag{14}$$

so the EFL is,

$$EFL = -\frac{1}{P} = \frac{Rn}{2\,(n-1)}. \tag{15}$$

b) To calculate the BFL we consider an on-axis object source at infinity (i.e. $s_o = \infty$, and $\alpha_o = 0$) which focuses by the ball lens at the optical axis ($x_i = 0$). Using the matrix formulation,

$$\begin{aligned} \begin{bmatrix} \alpha_i \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ BFL & 1 \end{bmatrix} \begin{bmatrix} 1-2\left(\frac{n-1}{n}\right) & -\frac{2}{R}\frac{(n-1)}{n} \\ \frac{2R}{n} & 1-2\left(\frac{n-1}{n}\right) \end{bmatrix} \begin{bmatrix} 0 \\ x_o \end{bmatrix} \\ &= \begin{bmatrix} 1-2\left(\frac{n-1}{n}\right) & -\frac{1}{EFL} \\ BFL\left(1-2\left(\frac{n-1}{n}\right)\right)+\frac{2R}{n} & -\frac{BFL}{EFL}+1-2\left(\frac{n-1}{n}\right) \end{bmatrix} \begin{bmatrix} 0 \\ x_o \end{bmatrix} \Rightarrow \end{aligned} \tag{16}$$

$$\begin{aligned} -\frac{BFL}{EFL}+1-2\left(\frac{n-1}{n}\right) &= 0 \Rightarrow \\ BFL &= \frac{R(n-2)}{2(1-n)}. \end{aligned} \tag{17}$$

By symmetry, the $FFL = BFL$. Since $1 < n < 4/3$, $BFL > 0$. The location of the 2nd principal plane respect to the back of the lens is,

$$x_{2PP} = BFL - EFL = -R. \tag{18}$$

Again, by symmetry the location of the 1st principal plane is also at the center of the sphere as shown in Figure 2.

c) As shown in Figure 3, an object is located at a distance d to the left of the back surface of the ball lens, where

$$d = R\frac{4-3n}{4(n-1)}. \tag{19}$$

The distance of the object to the 1st principal plane is,

$$s_o = d + (EFL - FFL) = \frac{1}{2}\frac{Rn}{2(n-1)} = \frac{1}{2}EFL. \tag{20}$$

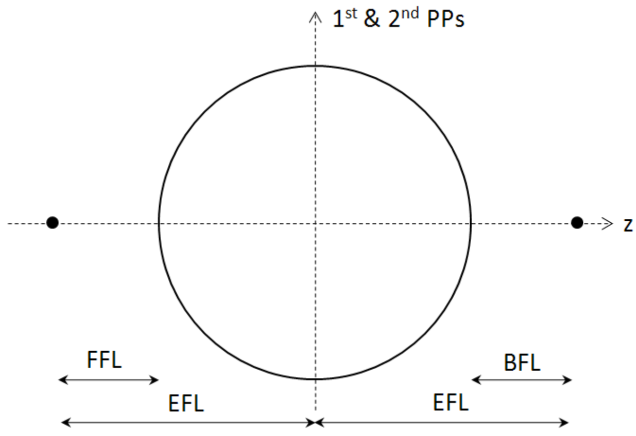

Figure 2: Location of principal planes: ball lens magnifier.

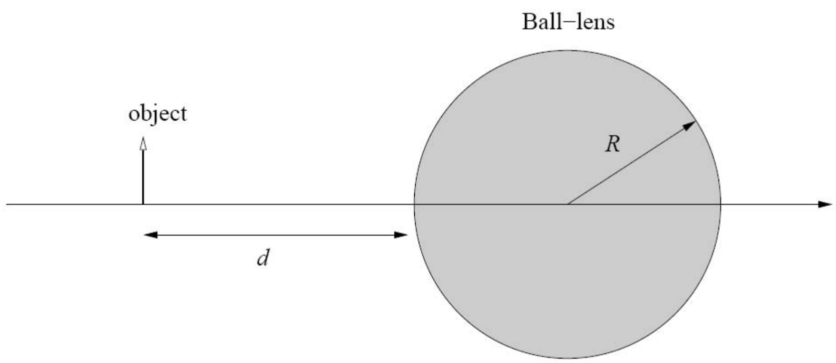

Figure 3: Imaging with a ball lens magnifier.

Since the object is located between the principal plane and the focal point, as indicated by equation 20, the image formed is *virtual*. To find the image position we use the imaging condition,

$$\begin{aligned} \frac{1}{s_o} + \frac{1}{s_i} &= \frac{1}{EFL} \Rightarrow \\ s_i &= -EFL, \end{aligned} \tag{21}$$

so the image is formed to the left of the 2nd principal plane (i.e. to the left of the center of the ball lens).

d) The lateral magnification is,

$$M_L = -\frac{s_i}{s_o} = 2, \tag{22}$$

and therefore, the image is *virtual* and *erect*.

e) The aperture stop (AS) is given by the finite size of the ball lens and thus is located at the center of the lens. The numerical aperture, NA, is,

$$NA = n_{air} \sin\alpha = \alpha = \tan\alpha,$$

where we have used the *paraxial approximation* and α is the angle formed by an on-axis point object and the edge of the AS,

$$NA = \frac{R}{d+R} = 4\left(1 - \frac{1}{n}\right). \tag{23}$$

f) By observing the object through the lens, our eye's lens is effectively imaging the image generated by the sphere. So, if our eye is located at a distance L behind the lens, the new object distance is $s_o = L + R + EFL$. The crystalline lens of the eye accomodates by changing the eye's focal length f in order to still satisfy a positive image distance of $s_i \tilde{=} 5$cm. From the imaging condition, we find that the resulting focal length after accomodation is,

$$\frac{1}{f} = \frac{2(n-1)}{2(L+R)(n-1) + Rn} + \frac{1}{s_i}. \tag{24}$$

To be able to see an object at a relaxed state (i.e. without accomodation of the eye), the image generated by the spherical lens should be at infinity, so the object should be placed at a distance equal to EFL to the left of the 1st principal plane.

Problem 3: Telephoto lens design

a) For this problem, we are given two requirements of a telephoto lens system: i) for an object at infinity, $\alpha = 10^{-2}$ radians and $h = 5 \times 10^{-2} f$ (cm); ii) The location of the real image produced by the system. The telephoto lens system is shown in Figure 4. By inspection, we see that the first lens, L_1, focuses the object at infinity to a point of height $x_i = \alpha f$, at a distance of f to its right. This image point now acts as the object source for the second lens, L_2. The object and image distances are,

$$\begin{aligned} s_o &= d - f, \text{ and} \\ s_i &= 3f - d. \end{aligned} \tag{25}$$

We use the equation for the lateral magnification to find the separation distance d,

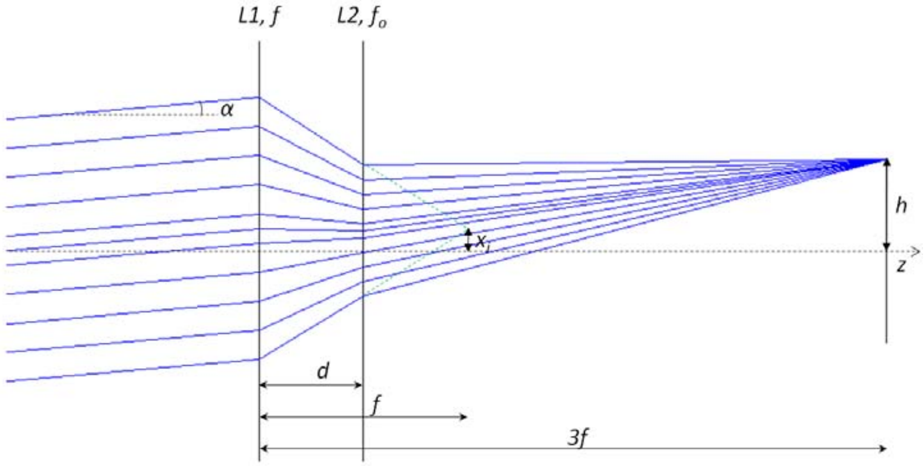

Figure 4: Telephoto lens system.

$$
\begin{aligned}
M_L &= \frac{h}{x_i} = \frac{5\times 10^{-2} f}{10^{-2} f} = 5 \\
&= -\frac{s_i}{s_o} = -\frac{3f-d}{d-f} \Rightarrow
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\frac{3f-d}{f-d} &= 5 \Rightarrow \\
d &= \frac{f}{2} = 5\text{cm}.
\end{aligned}
$$

We use the imaging condition to find the focal length of $L2$,

$$
\begin{aligned}
\frac{1}{s_o} + \frac{1}{s_i} &= \frac{1}{f_o} \Rightarrow \\
\frac{1}{\left(-\frac{f}{2}\right)} + \frac{1}{\left(\frac{5}{2}f\right)} &= \frac{1}{f_o} \Rightarrow \\
f_o &= -\frac{5}{8} f = -6.25\text{cm}.
\end{aligned} \tag{27}
$$

b) Referring to Figure 5, we see that,

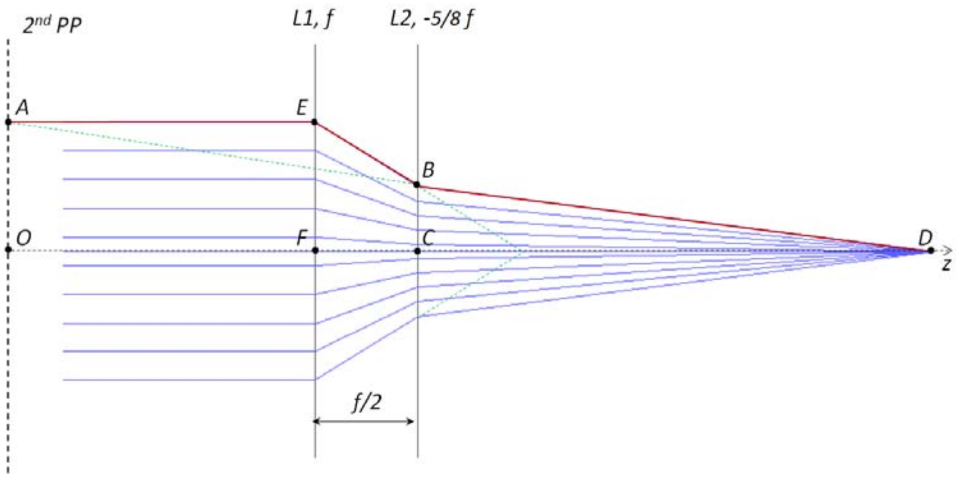

Figure 5: Locating 2nd principal plane.

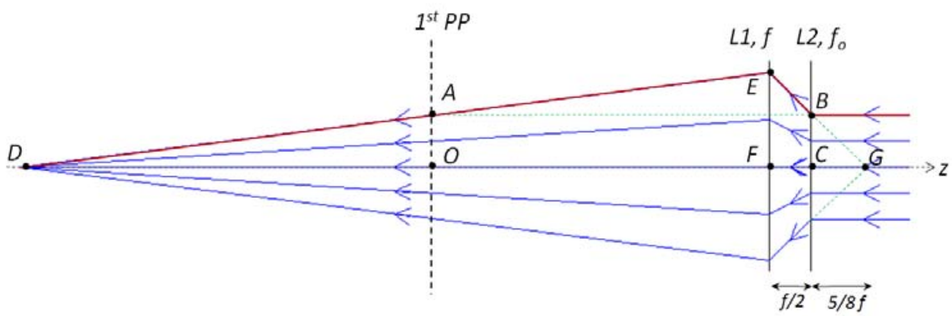

Figure 6: Locating 1st principal plane.

$$
\begin{aligned}
\overline{BC} &= \frac{1}{2}\overline{EF} = \frac{1}{2}\overline{AO} \Rightarrow \\
\overline{OD} &= 2\overline{CD} = 2\left(3f - \frac{f}{2}\right) = 5f \Rightarrow \\
\overline{OF} &= 5f - 3f = 2f.
\end{aligned}
\tag{28}
$$

So the 2nd principal plane is located to the left of $L1$ at a distance of $2f = 20$cm.

To find the 1st principal plane, we reverse the propagation direction of the rays and consider a point at infinity coming from the right as shown in Figure 6. The parallel rays will have a virtual image at point G produced by $L2$. This point image then acts as a point object for $L1$ and we use the imaging condition to find the location of the image (point D),

$$
\begin{aligned}
\frac{1}{s_o}+\frac{1}{s_i} &= \frac{1}{f} \Rightarrow \\
\frac{1}{\left(\frac{9}{8}f\right)}+\frac{1}{s_i} &= \frac{1}{f} \Rightarrow \\
s_i &= 9f.
\end{aligned}
\tag{29}
$$

From Figure 6 we see that,

$$
\begin{aligned}
\frac{\overline{BC}}{\overline{EF}} &= \frac{\frac{5}{8}f}{\frac{9}{8}f}=\frac{5}{9}=\frac{\overline{AO}}{\overline{EF}} \Rightarrow \\
\frac{\overline{DO}}{\overline{DF}} &= \frac{5}{9} \Rightarrow \overline{DO}=5f \Rightarrow \\
\overline{OF} &= 4f.
\end{aligned}
\tag{30}
$$

So the 1st principal plane is located to the left of $L1$ at a distance of $4f = 40$cm.

c) From part (b) we know that,

$$
EFL = \overline{DO} = 5f = 50\text{cm}. \tag{31}
$$

Or even from the problem we can know the EFL without any calculations: because the parallel ray bundle with angle α passing through the system will focus at the focal plane with height h, so the effective focal length is $EFL = h/\alpha = 5f$.

d) This part can be answered by using only the principal planes and applying the imaging condition,

$$
\begin{aligned}
s_o &= 240-40=200\text{cm} \Rightarrow \\
\frac{1}{s_o}+\frac{1}{s_i} &= \frac{1}{50} \Rightarrow \\
s_i &= 66.67 \Rightarrow \\
M_L &= -\frac{1}{3}.
\end{aligned}
\tag{32}
$$

The image plane is located to the right of $L1$ at a distance of $20/3 \cdot f - 2f = 14/3 \cdot f = 46.67$cm.

An alternative way to solve this problem is by using the matrix formulation,

$$\begin{bmatrix} \alpha_{out} \\ x_{out} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3f-d & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f_o} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{in} \\ x_{in} \end{bmatrix} \tag{33}$$

$$= \begin{bmatrix} 1-\frac{d}{f_o} & -\left[\frac{1}{f}+\frac{1}{f_o}-\frac{d}{ff_o}\right] \\ d+(3f-d)\left(1-\frac{d}{f_o}\right) & \left(1-\frac{d}{f}\right)-(3f-d)\left(\frac{1}{f}+\frac{1}{f_o}-\frac{d}{ff_o}\right) \end{bmatrix} \begin{bmatrix} \alpha_{in} \\ x_{in} \end{bmatrix},$$

$$x_{out} = \left[d+(3f-d)\left(1-\frac{d}{f_o}\right)\right]\alpha_{in} + \left[\left(1-\frac{d}{f}\right)-(3f-d)\left(\frac{1}{f}+\frac{1}{f_o}-\frac{d}{ff_o}\right)\right]x_{in}. \tag{34}$$

Because all the parallel rays with angle α will focus at the point at the image plane with $h = 5\alpha f$, x_{out} is independent of x_{in},

$$d+(3f-d)\left(1-\frac{d}{f_o}\right) = 5f, \tag{35}$$

$$\left(1-\frac{d}{f}\right)-(3f-d)\left(\frac{1}{f}+\frac{1}{f_o}-\frac{d}{ff_o}\right) = 0. \tag{36}$$

From equations 35 and 36 we have,

$$\frac{1}{f_o} = \left(1-\frac{5f-d}{3f-d}\right)\frac{1}{d} \Rightarrow \tag{37}$$

$$\begin{aligned} \left(1-\frac{d}{f}\right)-(3f-d)\left(\frac{1}{f}+\left(1-\frac{5f-d}{3f-d}\right)\frac{1}{d}\left(1-\frac{d}{f}\right)\right) &= 0 \Rightarrow \\ d &= \frac{f}{2}, \end{aligned}$$

which is consistent with the result of equation 26.

Problem 4: Microscope design

a) The geometry of this problem is shown in Figure 7. To let the human's unaccommodated eye to focus the image on the observer's retina, $L3$ should form an image at infinity (i.e. for a given point in the object the corresponding output is an off-axis parallel ray bundle). The first lens, $L1$, forms an intermediate image at the $S2$ plane,

$$\begin{aligned} \frac{1}{s_o}+\frac{1}{170} &= \frac{1}{10} \Rightarrow \\ s_o &= 10.625\text{mm}. \end{aligned} \tag{38}$$

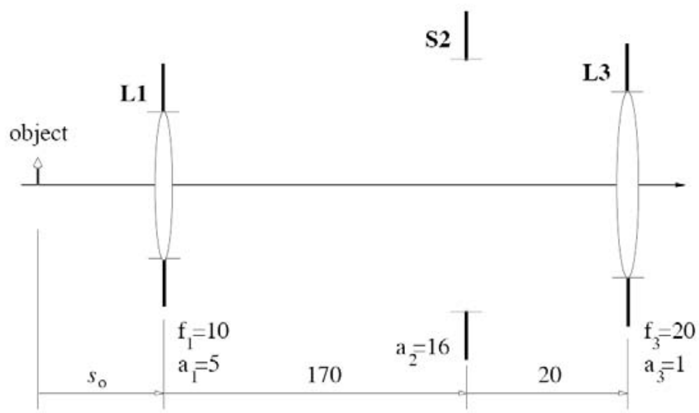

Figure 7: Microscope design problem.

b) If we place a small object at s_o in front of $L1$, the instrument acts as a microscope. The magnifying power, MP, is the ratio of the image size formed on the human's retina when using the instrument and the image size when viewed with the naked eye,

$$\begin{aligned} MP &= M_{L_1} M_{A_3} \\ &= \left(\frac{170}{10.625}\right)\left(\frac{254}{20}\right) \\ &= 203\text{X}, \end{aligned} \tag{39}$$

where, M_{L_1} is the lateral magnification of the objective ($L1$) and M_{A_3} is the angular magnification of the eyepiece ($L3$).