GEORGE BARBASTATHIS: So you guys walked in just in time. So today's topic is slightly different, but before we start, you asked a question last time, right?

AUDIENCE: Yes.

GEORGE BARBASTATHIS: And what's your name again?

AUDIENCE: [INAUDIBLE]

GEORGE BARBASTATHIS: [INAUDIBLE]

AUDIENCE: [INAUDIBLE]

GEORGE BARBASTATHIS: [? Kalpesh, ?] OK, again, so [? Kalpesh, ?] my friend here, asked me a question. And I think you asked it in the lecture before the quiz. So I answered it.

I guess you were not convinced. So he asked me again just before the quiz started. I told him, well, wait until actually we, the class, have convened, so we can all discuss it together. So I was thinking how to answer his question.

And then I decided it was a really interesting question. So I made some additional slides to discuss the topic that he asked. And these slides are not on the website, so don't look for them. You cannot find them, even though I will probably post them anyway. But for now, you just have to watch and participate in the discussion, and see what happens.

So [? Kalpesh ?] asked the following, when I started, actually, the class, I claimed that optical rays have to satisfy Fermat's principle. So therefore, they must always go where their path is minimal. So in the case of a GRIN, I use the notation here where the gray area in the slide corresponds to high index of refraction.

The sort of whitest area is low in index. So this is like a typical GRIN. You can imagine this profile may be quadratic or something like that.

And what we said in the class, and you all believed me except for [? Kalpesh, ?] I guess. What I said in the class is that if you have a ray that enters this medium sort of off axis, like shown here, the ray will actually bend inwards. So all of the rays will come to a focus somewhere around there.

But [? Kalpesh ?] asked, why doesn't it do this? I mean, the rays here seems to actually-- it seems to do the wrong thing, doesn't it? It seems to go towards the high-index area, whereas the minimum path principle would suggest that the ray should go outwards like this, towards the low-index area. So the question is, what will the ray do? Will it do this or that?

We know from experience-- I mean, we know from experience, from experiment, I should say-- that gain optics do focus. So we know that physics actually behaves like this. But the question is, how is that consistent with our description of the physical system? What is going on here?

So that is [? Kalpesh's ?] question. And what I said when he asked it in class, more or less-- I cannot remember exactly what I said-- but I said to properly contemplate the answer, we must consider not just this ray. If we just have this ray, we don't really know whether the ray will go.

But we also have another ray, the one that is on axis. Now these two rays, they actually originated at the same point. Where did they originate? At infinity, somewhere far out to the left over here.

And what I said is that basically these rays, because this ray goes through a longer optical path, because it is entered in a high-index medium, this has too much, its optical path. So this ray actually enters a lower index. So therefore, it starts with a sort of predicament that points to a smaller optical path.

So in order to catch up, in order to match the longer optical path of this ray, this ray, the sort of off-axis ray, must follow along at a geometrical distance so that the longer geometrical distance times the lower index kind of will balance, will compensate. So this is where we left it. But then [? Kalpesh ?] came back to me at the quiz, and he told me, I don't believe it. You want to rephrase your question?

AUDIENCE: No, [INAUDIBLE].

GEORGE BARBASTATHIS: OK, so I presented your objection well. So first of all, does anybody else from here or from Boston have some additional insight about this? I can go back to the original. We'll call it the [? "Kalpesh ?] Question." Yes.

AUDIENCE: [INAUDIBLE]

AUDIENCE: We didn't hear the question.

GEORGE BARBASTATHIS: Is your button pushed? Yes. You have to make sure that the red light is on.

AUDIENCE: OK. Yeah, in the standard case of an interface, we have a ray propagating from left to right and we have an interface, which is top to bottom. But in this case, the interface could be considered to be left to right, couldn't we? Because the refractive index change is from top to bottom. So the interface-- I'm just thinking that if we consider the interface to be from left to right, then the normal will be from top to bottom. And still, the Snell's law will lead us to the correct answer, correct [INAUDIBLE].

GEORGE BARBASTATHIS: Yes. You can actually-- by the time you reach here whether the ray has already been, you can make that argument. Because now the ray is really--

[INTERPOSING VOICES]

AUDIENCE: Yeah.

GEORGE BARBASTATHIS: But the question is, why should the ray start bending inwards to begin with? Why should it not bend this way?

AUDIENCE: Yeah.

GEORGE BARBASTATHIS: And if it starts bending this way, then Snell's law also said that it should go like this. So there's something amiss here.

AUDIENCE: Yeah.

GEORGE BARBASTATHIS: OK. So there's no doubt-- as I said, there's no doubt that the correct physics is this one. The question is, is our model correct? Is our claim, all this stuff we developed about optical paths and so on, does it really agree with the physics?

So I don't hear any-- any other comments or objections about this? Or-- so let's go with a little bit more care through this argument. What is happening. Let's take out the medium, so there's nothing now, just free space.

So the rays started at infinity. They're heading towards infinity. Everybody's happy with a straight path, because clearly, the straight line in uniform space is the shortest between two points. And clearly, these two rays-- they never really meet, but we say they meet at infinity at the left and they also meet again at infinity to the right. And they will have traversed the same optical path by the same-- by the time they meet again at infinity.

Now, you might ask, well, what does it really mean? I mean, after all, the path itself is infinite. What does it mean they travel equal infinite paths?

Well, I can get, actually, around this problem pretty easily because I can actually put some boundaries to the left and to the right. And I can keep these boundaries at a finite distance. Then you can definitely measure the optical path.

Then you can start moving the boundaries farther away to the left and farther away to the right. The optical paths remain the same as I move the boundaries. And I can continue this process until the boundaries are really, really very far. This is mathematically really what we mean by "infinity."

So I can get around this difficulty. I can certainly talk about equal paths, even if the paths are actually infinite. This is sort of a technicality, but nevertheless an important one so we know that we're on solid ground. So I hope I have convinced you that Fermat is happy in this case of free space, because the rays indeed propagate through the minimum path, and indeed, they're all equal in terms of path.

Now, this is really not anything new. This is what I said before. And basically, this says that the longer geometrical path actually gets to be multiplied by a lower index, so therefore, by the time the rays meet again here, they still have equal optical paths.

Because the path to the left is equal-- they both came from infinity-- then the two trajectories here, if you sort integrate the optical path all along, you would get equal answer. Are you happy? Maybe push the button in.

AUDIENCE: [INAUDIBLE] Instead of a wave coming towards a whole cross-section of the GRIN lens, suppose if you have a small spot coming only towards the top. So will it go towards up in that case?

GEORGE BARBASTATHIS: So the question is, suppose I take out this ray. Will the ray-- will this thing also focus? What do you think?

AUDIENCE: I think we have only a single ray towards the top, which is coming towards the lower refractive index, then it should go towards the lower refractive index because it doesn't need to match with the same phase as the center wave.

GEORGE BARBASTATHIS: OK, let's not quite argue about a single ray, because there's no such thing as a single ray. If you illuminate a very, real infinitesimal optical spot over here, as we will learn in wave optics, what you will get is not a sort of straight line of light, but you will get a very strong diffraction.

So what will happen if you illuminate a single spot here is the light will spread violently into the GRIN. And then what will happen is a kind of a complicated issue. So the way, essentially by claiming the fraction, I got around the problem that you're posing, because as long as I have a finite pencil of rays coming in here, then the same thing will happen, because I can always compare two rays that enter at slightly different indices.

Then the same argument follows. So this pencil will also kind of focus in the same-- so basically, I can remove parts of that ray diagram and the same thing will still happen according to geometrical optics. Diffraction says differently, but according to geometrical optics, I can remove portions of these rays and still have focusing.

There's another objection that I haven't had yet. I'm not finished. I am not satisfied that I answered [? Kalpesh's ?] question yet.

And I see that [? Kalpesh ?] is also skeptical, and I hope someone-- I have another slide to bring up the objection, but I'm wondering if someone will bring it up, or maybe another objection. At least there's one objection that I thought of, but there may be others.

AUDIENCE: Question.

GEORGE BARBASTATHIS: Yeah.

AUDIENCE: Why is it that the ray on top bends to slow down to meet the ray that's traveling through the denser medium versus both of them traveling to the less dense optical medium, because that could also--

[INTERPOSING VOICES]

GEORGE BARBASTATHIS: That's the objection. That's exactly the objection. So the objection is who says that the central ray should go straight? I sort of silently took it for granted that the central ray here goes straight, and therefore, this other ray has to race in order to meet it.

Why don't the two rays go like this? Is this your question? I mean, yes. Why don't the two rays-- why don't they both do like this? They seem to be doing the correct thing. They seem to be both going toward the lower index.

We know again-- we can argue that this would happen, but we know from physics that this does not happen. So what is the physical--

AUDIENCE: That can't possibly happen by symmetry.

GEORGE BARBASTATHIS: OK, thank you. Accurately, it is not a question of symmetry because indeed, in this case, I assume that the refractive index is symmetric. However, now let me see if I can do this without revealing my next slide.

However, I can make an asymmetric index profile. So this is a simulation of the GRIN with a symmetric quadratic index. Here, I made the profile asymmetric. So basically, this is the equation here.

I should simply draw it, except I don't have a marker. Anyway, basically here, the index of refraction is a parabola that has a different slope above and different slope below. So you see that even though this is asymmetric, stubbornly, the central ray still goes straight. Therefore, it is not symmetry which makes the previous case impossible.

AUDIENCE: But, well, I said "symmetry," but if you go to your next one, that ray that's going straight along the center-- is it gone? Yeah, that ray goes straight, doesn't it?

GEORGE BARBASTATHIS: Yep.

AUDIENCE: And that's because there's no transverse gradient in the refractive index where it's propagating, I guess.

GEORGE BARBASTATHIS: OK, yes. That is true. There's no movement to push it. Yeah, and that, I agree with.

However, that really still does not answer his question, because the reason momentum works, the reason we apply Snell's Law, for example, is because we claim path minimization. So I suppose you could argue this way. I have another argument, actually, why this ray must go straight. Actually, I don't know if it must go straight, but then another argument why basically this is impossible, why this is physically incorrect.

AUDIENCE: If you do that, isn't it true that the bottom ray's path is still longer than the top ray's path because of the index of refraction is low, and then it's just low all the way, and-- yeah.

GEORGE BARBASTATHIS: Exactly. If you do that, there is no way ever that this ray will catch up. Because this ray now, not only is it going through a higher index of refraction, it is also following a longer geometrical trajectory. So therefore, if these two rays ever meet again-- they might meet at a finite distance. The way I drew them here, at some point they will meet again, or they might meet at infinity. They might somehow become parallel and meet again at infinity.

But no matter what, this ray over here has traveled the longer optical path. And now that is a clear violation of Fermat. So we basically arrived at an impossibility. So this is physically, then, incorrect.

That is the reason why the rays have to bend inwards. And so the proper interpretation of Fermat's principle-- and actually, the book has an interesting discussion about this-- we can claim, yes, that Fermat minimizes the optical path, but perhaps, at least linguistically a more precise definition is that Fermat requires that the path of the rays is stationary.

Now, "stationary" in mathematics-- what it means is that if you perturb it in some way, you will get a bigger answer-- if you perturb it in anyway. So remember-- I wish I had the marker. I think I can just use a pen, though. Let's try with a pen, and you guys tell me in Boston if you can see it.

Remember with Fermat, we did this analogy. We said-- you can see it, right-- we said-- I'm sorry, I should have said remember with Snell's Law. Snell's Law says that the ray must do something like this in order to connect this point to this point.

So mathematically, what we derive is that any other possibility-- this, this, this, this, and also, why not this? These are all different possible ray trajectories. There's a mathematical tool. It's called "calculus of variation," which does a miraculous thing.

It somehow compares all of these possible trajectories, and then it arrives at the minimum trajectory. And it tells us, no, no. No matter what-- these are infinite possibilities. That you can imagine.

But this tool, calculus of variations, actually tells us eventually that all of these give us a longer total optical path. And the only true one, if you wish, the one that minimizes, the stationary one, is the one that I drew kind of bolder here. So this is the proper way to interpret Fermat.

It is not that a single ray will actually look for the lower index of refraction. It is actually that when you have a bundle of rays that share a common beginning and a common end, then the trajectory that these rays will follow will be such that they're all equal, and that this trajectory now is stationary, in the sense that if you were to perturb it in any way-- if you were to move some of these rays up, or down, or bend them, or whatever-- you would end up getting a longer trajectory. That's the proper interpretation of Fermat.

So this is actually a very interesting topic. And I confess that when I was a student, I was also bothered by this. I mean, the rays seems to be doing the wrong thing, after all. It seems to be going really the way it shouldn't go. Why? The reason is as we just described-- because the path has to be stationary.

The bottom line out of this, which will actually be useful in our ensuing discussion of Hamiltonian optics, is that the index of refraction is acting a little bit like gravity. Actually, it's acting like gravity more than you would think. The index of refraction seems to be attracting the rays, which is, again, a little bit counterintuitive if you interpret Fermat as a minimum path.

But after we went through this explanation, it makes sense. The index of refraction pretty much collects rays near it. Yes, there's a question.

AUDIENCE: Yes. George, I think if you consider the interfaces left to right, the geometric works out. And so if we assume any ray coming like this, going from left bottom to top right, then it will-- it is going from a higher index to lower index, which means it will tend to increase its angle with respect to normal. So it will keep bending like this. And at some point, there will be total internal reflection, and it will bend towards the--

GEORGE BARBASTATHIS: That's right, yeah. Doesn't happen here, for example. If this ray continues--

AUDIENCE: And then again, it will--

GEORGE BARBASTATHIS: It will kind of bend.

AUDIENCE: Yeah.

GEORGE BARBASTATHIS: Yeah.

AUDIENCE: So maybe-- I mean, that's just another way of looking at the problem, that if you consider the interface to be normal to the change in the refractive index, then-- yeah.

[INTERPOSING VOICES]

GEORGE BARBASTATHIS: Actually, that's true. Snell's Law also justifies this attraction. I mean, the point that is really hard to swallow is why this ray starts bending inwards at this interface. After it has started, then it has to fall, because Snell says that it should fall. So that is consistent.

So really, this acts-- you can think of a mechanical analogy, which would be like imagine that you take a tube and you cut it in half. So now you have these hollow sphere. And then you launch a billiard ball into that surface.

What will the billiard ball do? It will spin. It will go down, and then up again, and then down again, and spin, and spin, and spin. So basically, the surface in this sort of mechanical analogy, the equivalent of the surface shape is the index of refraction here.

And of course, the billiard ball is subject to its own gravity. And the surface-- how do you call it, the [INAUDIBLE] contact from the surface, and so on, and so forth. But the bottom line is that this is a mechanical analog. I'm sorry, this is-- whatever, the mechanical analogy for this is gravity, attraction.

AUDIENCE: [INAUDIBLE]

GEORGE BARBASTATHIS: Yes, provided you define the gravitational potential appropriately in terms of the initial refraction. And this will actually come up near the end of the Hamiltonian lecture. So I will show you exactly the formula. Did everybody hear his question?

AUDIENCE: No.

GEORGE BARBASTATHIS: OK, could you repeat it?

AUDIENCE: Yeah, I was asking, is the analogy between the gravitational force and the reflective [INAUDIBLE] complete?

GEORGE BARBASTATHIS: Yeah, and you have an answer. OK, now that I have a marker, let me also answer-- so in this case, what I did is on the left-hand side, we have the familiar quadratic index profile. So this is n0 minus the some coefficient times x squared. On the right-hand side, what I did is I chose an index that [INAUDIBLE] like this.

I'm exaggerating, but basically, it is still a parabola on the left and the right, but has a different curvature on the left and the right. And the point here is that the central ray still goes straight. Of course, this is a terrible lens. It does something really horrible back here, does not focus properly, and so on and so forth.

But the only reason I did this is to show that the ray here goes straight not because the index of refraction is symmetrical, even though Colin's point is well taken. I mean, the reason it goes straight here-- again, you can think of it in terms of Snell's Law. It does not want to go this way, because it would bend to a longer optical path then the ray next to it. It does not want to go down, because again, it would end up with a longer optical path with respect to the ray below it. So therefore, it has to go straight.

Can you imagine a situation-- is it always straight? Can I actually bend this ray? What would I have to do to make the central ray bend? Actually, Colin answered it already. Yeah, they cannot see you. Can you--

AUDIENCE: Curve the whole profile.

GEORGE BARBASTATHIS: So you have to put a gradient along z in the index of refraction. Then if the ray sees a tilted interface, which basically means a gradient of the index of refraction here, it will bend. Because again, Snell tells it to bend. But since there is no-- since here, the index has no z dependency, then the ray must go straight.

So first of all, thanks to [? Kalpesh ?] for asking this question, because it turned out to be a really, really helpful one. And in addition to that, today we have kind of two more things to do. So I need to really finish the Hamiltonian optics discussion, and then we'll start talking about wave optics. So we'll see how far we go. And I have to make some announcements near the end. So I will try to save about five minutes near the end for my announcements.

So last time, we were discussing gradient index optics. And just to briefly remind you, this is actually what prompted the [? Kalpesh ?] question. We went through this derivation where we had the gradient index element. I will stop saying gradient index it's difficult for me to pronounce-- a GRIN light.

So we had the GRIN element, and we did this simple derivation which basically said that this GRIN element is acting like a lens. And again, with our newfound wisdom, it is acting like a lens by virtue of the attraction of the rays by the origin of high index near the center of this element. And we also went through this other case of axial GRIN. And we also said that this is-- now that's interesting.

If you only have z gradient, then it is still not a lens. I'll let you think about this by yourself. In order to make it a lens, you have to have an x gradient, the index of refraction.

So this case is somewhere in between, where you have a z gradient-- so the index is higher in the front and lower in the back-- but in order to get optical power, that is to get focusing power, in this case you actually grind. In addition, you grind it to a sphere. And we said that the reason you do this axial GRIN here is because you're going to correct for spherical aberration.

So if you compare the paths now of this ray to this ray, in this case, this ray goes through a longer optical path, because the index is uniformly high here. Therefore, it bends faster. In this case, by lowering the index of refraction that this ray goes, you actually force it to go to a longer geometrical trajectory. Therefore, you're pushing it further to the back.

This is another way to interpret the way this element corrects. You can interpret it with Snell's law. This is what we did last time.

But now, with our newfound wisdom on optical paths and Fermat, we can also explain it this way. I lowered the optical path by virtue of lowering the index here. Therefore, the ray has to go outwards in order to increase its travel distance, and still maintain the same path as the central ray. So the net effect is that you're correcting the bad quality focus here that was caused by the spherical aberration.

And I believe where we left it last time is we asked this question, what if I have an arbitrary index of refraction as a function of position? What if this is not quadratic, this is not axial, this is some general function of r, the Cartesian coordinate? What then? What is the-- I lost my train of thought. What is the path of the-- what is the trajectory of the rays that go through this variable index medium.

And I don't remember how far I went here, but we basically took a turn here, and we started talking about mechanics probably to some of your surprise. Can you remind me, how far did I go? Did I actually finish this example or not?

AUDIENCE: Yeah, I think so. Yeah, we covered this.

GEORGE BARBASTATHIS: So I don't feel like going into it again, anyway, because this is sort of good, old physics 101 kind of thing. What might be a little bit unfamiliar is this development here, which some of you may have seen, some of you may not have seen. So this is really not doing anything spectacular. This is a simple mathematical manipulation.

So we start with the equation of motion for the mechanical oscillator. So the equation of motion for the mechanical-- I'm sorry not the equation of motion, but the energy for the mechanical oscillator. So the energy consists, of course, of two terms. One of them is, if you wish, the elastic energy, the potential that is due to the restoring force of the spring. And the other is the kinetic energy.

And v is the velocity, so v is basically x dot, the first time derivative of the position. So this is one thing that we know. This is the energy of the system.

And since in our model, there is no dissipation, the energy must be conserved. I can also write an equation of motion for this system. And since it is a simple spring force kind of system, I expect my equation of motion to be second order, so there would be something of this sort-- m x double dot plus kx equals the force, the excitation.

So let's say that this system is free. There is no force. So if it is moving, it is moving because of a prior excitation, which if I start it [INAUDIBLE] it will go on forever, because there is no dissipation. There is no place for the energy to go.

If you get this thing moving, it will go on forever. But now, there's no force. So basically, I can write down the equation of motion like this, as a homogeneous equation.

So there's is a trick that those of you who use Matlab to solve differential equations are very familiar.

And the trick is the following-- you break this higher order differential equation. You break it down into as many as you need first order differential equations. Since this is a second order equation, you only need two first order equations.

And the way you do it is basically, you write down an equation that looks kind of stupid. The equation says that x dot equals v. This is one equation.

And then the other equation comes from this one-- since x dot equals v, it means that x double dot equals v dot. This is the acceleration. So the second equation now comes from this, that is, v dot equals minus whatever k over m times x.

Now, I can define the vector x v. And what this set of equations here has done, it has actually connected the first derivative of this vector with the vector itself via matrix. And here is the vector in its first derivative. Here is the vector itself.

And what I have to put here in order to make this look correct is I have to put 1 here, and then minus k over m here, and then 0s, and then I'm done. So again, if you ever used Matlab to solve an ordinary differential equation, you're familiar with this. Matlab actually forces you to rewrite your equation this way.

But it turns out that this kind of method actually preceded Matlab by about a century, because this is a standard way of writing the equation in-- actually, I believe it was Poincare who actually first came up with this. It's the standard way to derive the dynamics of a system. And this is not-- I don't want to spend too much time with this.

But what this slide basically points out to you is the following-- that if you take this expression here, which I will call the energy, but now we'll call it-- of course, you cannot see it because it's hidden. Now you can see it.

So this expression is to call the energy but to that I will call it the Hamiltonian and it will become apparent in a second why, but if you take this expression, this is now a function of the two dynamical variables, the same two variables that appeared in this expression over here. They actually get into this equation, x comma v.

It's a bit of a funny way to think about it, but certainly you see x here and you see v here. And actually, I don't like v very much, because if I keep using v, I will not come up with the correct set of equations. I will replace v with the momentum. So that's a minor cheating. Instead of v, I will replace it with p.

So p, the momentum, is simply m times v. So I haven't cheated in any major way. I just did the simple multiplication. So if you replace p here, then the equation will look like this. Nothing changes here. This part will look like 1/2 p squared over m. Oh, sorry. OK.

So nothing major here. The magic now-- it's not really magic. There's a good reason for it, but it looks like magic is that if I do the following, if I take the two derivatives of this expression with respect to x and p-- so if I take dh dx and if I take dh dp. Can you still see? What do I get? If I get dh dx, what is that? It is kx. And if I take dh dp, I will get p over m.

So if I look carefully here, this is p over m. That's the velocity. And this is minus kp, isn't it? Wait a minute-- of course, I know why. I have v dot here.

Instead of v dot, let me write it in terms of p dot. If I do in terms of p dot, I multiply by m both sides of the equation, I will get minus kx. I don't have to do anything else. So what do I get?

I get that dh dp, which equals p over m, is this term in this equation. And if I get dh dx, which is kx, it's actually this term in this equation. So basically, by taking the partial derivatives of this expression, the Hamiltonian, with respect to my two dynamical variables' position and momentum, I came up with the equation of motion decomposed in the dynamical variables, Matlab style, or actually Poincare's probably shifting in his grave-- Poincare style, in the way [INAUDIBLE] dynamical systems.

So this is really what I say in the top of the slide. Of course, I say it in a slightly more formal way. And then the rest of what I do is I simply solve this equation, but I will not do it here. This is a simple harmonic oscillator.

The mechanical system, as I said before, is a car that will move forever, eternally in a sinusoidal fashion. And it's kind of interesting, I guess, that the energy alternates between 100% kinetic, which happens when the cart, the particle moves through the rest position, because then it is the maximum velocity and has no potential. It is the rest position of the spring.

And it converts to completely elastic when it goes to the edge of the range of motion, because over there, it has to reverse. Therefore, it will go velocity zero momentarily, and the kinetic energy has vanished at this point. It has become all potential. This is probably boring, because this stuff is very well known.

What I really want to do is I want to establish a connection between this very nice, very well-known mechanical stuff with optics. So basically, what I will do now is I will derive-- I will not really derive, but I will show you a set of Hamiltonian equations that actually describe ray trajectories. And what is really spectacular is that, as you pointed out earlier, they will turn out to have an exact mechanical analogy if you use a certain equation for the index of refraction. So we'll get to that.

So there's two ways to derive the Hamiltonian equations. Does anybody know, in the field of mechanics? Does anybody know how we get the Hamiltonian equations, what is the process?

Well, I sort of just described it over here, but I guess what I'm trying to get to is, how did this come about? Of course, it came about from my physical intuition, because I know that to have a kinetic-- someone told me 1,000 years ago when I was a student in school that this is the case. But--

AUDIENCE: It's the invariant property of the system. It's the total energy we put in at the start.

GEORGE BARBASTATHIS: Yes, so it is the invariance of the system, and also, mechanical systems, they also have a Fermat principle of their own. If you recall, mechanical systems also try to minimize their potential. The trajectory that they follow tries to minimize their potential.

So a good example of that is actually, forgetting that this is optics, imagine that what you see here in this sort of fancy-looking diagram, this is not the index of refraction, but this is a star. Imagine that the center of this [INAUDIBLE] massive object, a massive gravitational attraction, like actually the sun. And imagine that it's not a ray anymore, but imagine that that trajectory is the path of the satellite.

This is convincing, isn't it? If you allow the grayscale to denote gravity now, not index of refraction but potential due to the gravitational field exerted by the planet, then if I launch a spacecraft, or a satellite, or whatever in the vicinity of this gravitational field, as you know if you are a sort of amateur astronomer or aspiring astronaut-- if you are, then you know that depending on how you launch your spacecraft, sometimes, it will actually sort of circle once and then escape.

That's very bad, because it means that say, if you're an astronaut, you're going to space forever. But also you know that if you launch it properly, then you can actually get into orbit. And orbit means that the dynamics here is such that you will actually start circling this object. In fact, we never launch spacecraft to stars. Typically, we launch them to planets nearby, but then--

So it turns out that the trajectories that these spacecraft follow, actually they also minimize their "optical path," where there's no index of refraction then. It is just gravity. But they actually do follow the minimum path around the star. And in fact, this has been known for more than a century. I mean, Copernicus even hinted at that in his work, when he computed the orbits of the planets in the solar system. So these are very well known stuff.

So what I will do now, then, is I will actually drive you, without really proving anything, but that will drive you to a set of Hamiltonian equations that describe the same analogy, but for optics now. So in order to derive our Hamiltonian equations, we actually need two postulates. Now what does it mean, "postulate?" Do you know that word? Yes, [INAUDIBLE].

AUDIENCE: [INAUDIBLE] I have a doubt regarding definition of Hamiltonian. Why can't it be some other invariant quantity like angular momentum or linear momentum, and energy in this case?

GEORGE BARBASTATHIS: So you're actually there. You're forcing me to say what I didn't say before. How did this come about?

This comes about-- and I have to apologize now, because I will throw a lot of jargon at you. This comes about from something called the Lagrangian principle. So the Lagrangian is very-- actually, if you write the Lagrangian, it looks very similar to this one. Looks similar to this.

The question you're asking the mechanical system is, what is the trajectory that the system will follow? So we have a particle or a set of particles, and you ask what is the trajectory. There Lagrangian actually looks very similar to this, but with a minus sign. And I never quite remember if you put the minus sign here or here, but anyway, one of those has a minus sign.

And so what you do then is you put this Lagrangian into this integral. And you say that the trajectory that the particle will follow actually minimizes this integral. Now, this has a very good physical justification, because this is actually Newton's Law. The way you get this Lagrangian is you start with good, old Newton, which says that--

Then you multiply. I haven't cheated. I've multiplied it with a quantity that is hopefully non-zero. And then you can write it down as 1 and-- actually not quite 0 any more, but constant. Now confuse my mind with science somehow. Well, anyway.

AUDIENCE: [INAUDIBLE]

GEORGE BARBASTATHIS: That's a good plan, because this is a restoring force. But anyway, how do we actually derive the Hamiltonian? So I don't know what has happened, but if you think about it, this is now the Hamiltonian, isn't it?

So by a simple trick on Newton's Law, it derive-- that's not quite what I was expecting, but it happened anyway. The answer to the question is simpler than I was expecting it to be. So there is the Hamiltonian. now why that and why not the momentum? That really has to do with the Lagrangian, but let's--

It's clear what I did here, right? This is just a simple manipulation that I learned from Professor [? Jerry ?] [? Wickham ?] at Caltech, actually.

I think I skipped a slide. Here we go. So we were at this word, "postulate." What does "postulate" mean?

AUDIENCE: We accept something without-- we say something and we accept something.

GEORGE BARBASTATHIS: That's right, yeah. So the postulate is a good way to develop a scientific theory in the following sense-- first of all, postulates cannot be very complicated. So it is something simple that we accept with two caveats-- one is that we accept it because it has justified the observation.

So postulate had better be something that has never been violated to our knowledge. For example, the sun rises every day. As far as humanity has recorded history, this has happened every day, except, of course, in the north above the arctic, where some days the sun does not rise.

So that's one. And the second is that the theory that follows from the postulate has to be self consistent. So it has to be mathematically consistent. So typically the postulates are things that I don't quite feel like proving, but they're good assumptions, like assumptions that cannot be very easily challenged.

So the assumptions here, one of them is that rays are continuous. That really cannot be very easily challenged, because if a ray were to jump, that would be really strange. And I can quote a number of physical reasons why this would be really strange.

For one is that relativity-- it means that somehow, a photon, if we believe that photons follow the ray trajectories, it means that somehow light within zero time, actually traverses a finite distance. That is impossible, so that is one good reason. So if you accept, then, that the rays must be continuous, actually, it is only continuity.

I am not saying that the rays are differentiable. So the rays can bend like this. So here, actually, obviously, it is non-differentiable, but it's still continuous.

So the postulate actually talks about continuity, not differentiability. Piece-wise differentiable, yes, it is, because it is certainly piece-wise differentiable here, piece-wise differentiable there. This is actually a technicality.

Functions that are not piece-wise differentiable are really nasty beasts, like Weierstrass functions and stuff like that, which normally we don't deal with in engineering. So basically, the second condition here is a mathematical technicality. So we don't have to dwell on that.

Now here's something that probably as engineers, we've got turned off by when we learned it in calculus. But it's actually a useful thing that is called the "mean value theorem." And the mean value theorem says the following-- says that if you are given a continuous piece-wise differentiable function-- so here it is-- so this is x. This is f of-- so this curve here is f of x. This is some function of x1, x2, f of x1, f of x2.

The mean value theorem says that according to the conditions I just stated, there is a point somewhere in this interval satisfying the following property, that f of x2 minus x1 over x2 minus x1 equals f prime of x. So basically, this ratio, I can pick a point xi somewhere in this interval such that the value of the derivative of a function at this point equals this ratio. And of course, you can realize immediately that if you make the interval very, very tiny, then of course, you get the definition of the derivative itself.

But the mean value theorem is a little bit more general. And it is the mean value theorem that I use here in order to write this equation. This is basically the mean value theorem, where the quantity of interest that I applied is the ray trajectory.

Let me back up for a second. I should have started with something else. What am I really plotting here?

So the red curve is the ray trajectory. So I describe the ray trajectory as a vector, Q. So the vector is a position vector. This is the origin of the coordinates and Q is the position vector. And then I parameterize it with a variable s.

So basically, s is an index. As I increase, as I step through s, I basically walk along this trajectory. So you can think of s as time, as time goes forward, the light particles, if you wish, are moving along this curve.

And at each point on the curve, since it is continuous and piece-wise differentiable, at each point I can define a tangent-- except, of course, the point where it is not differentiable, but let's skip those for a moment. In the point that is differentiable, we can define a tangent. And this tangent is the vector P. So this vector P is actually the momentum of the ray.

So since the tangent is associated with a derivative-- in fact, in high-dimensional space, it is even more obvious because the tangent really points towards the-- it is the tangent. Then this really is an expression of the mean value theorem for the continuous and piece-wise differentiable ray. And of course, I can write it down as a differential equation by taking the limit.

And there is one element that is sort of missing here. That's the normalization. Why did I pick to normalize by the value of the tangent? Let's leave that alone for now. Let's assume that it is correct, and see if it will lead us to an acceptable physical result.

So what I'm trying to say is that the denominator here is arbitrary. I don't really need it. I can throw it out. It is a normalization parameter, and I put it there because with hindsight, it would lead me to a physically acceptable result.

The second postulate is a little bit easier to swallow, thanks to [? Kalpesh's ?] question, because the second postulate says that the ray, very similar to a spacecraft approaching a planet, a ray actually is attracted by a high index of refraction. And the way we quantify this assumption is we say that if the ray enters a region where the index of refraction is variable, that this has a gradient. So that's the gradient, in case you have forgotten-- I hope not.

So if there's a gradient in index of refraction, the ray will receive a kick. And the kick of the ray-- what the kick really means is that its momentum will change. Therefore, the trajectory will bend and the kick that is a change in momentum will be proportional to the gradient.

That is the postulate. And I will not do it here, but if you-- it is a little bit of a long derivation, I have to warn you. I went through it and I decided-- actually, I went through it and then I decided not to put it in the notes, because it is a little bit involved. But this actually turns out to be equivalent to Snell's Law.

So if you want, I can give you a copy of my notes where I derived it, but if you don't feel like going through two pages of calculus of variations, then please take my word for it. This is actually Snell's Law, but written in a funny way. So that's, then, the sort of variational version. And we can write it down as a differential equation, and we get this.

So now, we're kind of going backwards. Now, we have two differential equations. Yes.

AUDIENCE: Why the position variables are different on both sides, s and s dash?

GEORGE BARBASTATHIS: You mean the s prime?

AUDIENCE: Yeah, s prime.

GEORGE BARBASTATHIS: Oh, the s prime is actually the xi from the mean value theorem.

AUDIENCE: Oh, OK.

GEORGE BARBASTATHIS: When you write it in the variational form, s prime can you be anywhere in here.

AUDIENCE: Right, right.

GEORGE BARBASTATHIS: I don't know exactly where it is. And very similar to here, actually again here I invoke the mean value theorem. I didn't say that then.

AUDIENCE: Yeah. Yeah.

GEORGE BARBASTATHIS: But of course, when you do the differential form, it vanishes, because the interval collapses. So now what I have is actually two differential equations. One of them contains a trajectory, the other contains a momentum.

So this is basically the two-dimensional analog to the dynamical system that I wrote here. I have an equation for the position. This is it. And I have a differential equation for the momentum. This is it.

This is the equation for the position. This is the equation for the momentum. I used q for the position here, because that's a convention in geometrical optics. People use q, not x, but q is the position vector.

Well, that's fine. The question is, is there anything that is conserved here? Did I derive any sort of Hamiltonian?

It turns out-- and again, this requires a bit of hindsight, but I can pick the Hamiltonian to be of this form, to be equal to the magnitude of the momentum minus the index of refraction. Let's write down the two equations again. So this is the equations.

And the Hamiltonian that I picked is H equals p minus n of q. You can see now that if I do the gradient-- now in the high-dimensional case, instead of the derivatives of the Hamiltonian, I have to take the gradient. So if I do the gradient of the Hamiltonian with respect to-- let's do the first one-- with respect to q, this is really what I'm going to get.

And if I do the Hamiltonian with respect to p-- what is p? So for example, if you do dH dp x, you will get p x over the square root. And then you can follow dH dp y and so on. So therefore, the gradient, which is defined as a vector--

So you can basically see from this simple derivation here that by this choice of a Hamiltonian, the Hamiltonian that I wrote here, I can actually derive consistently the set of Hamiltonian equations. So I basically find that the dq ds equals the derivative or the gradient of the Hamiltonian with respect to the conjugate variable, the momentum, and the other way around for dp ds.

So this equation, then, is the Hamiltonian. It is not the only one, by the way. Hamiltonians, they're not unique. Clearly for example, I can add an arbitrary constant here. I can get another Hamiltonian. And we know that for energy also. Energy is not unique. I can add constants to it, and it is still remains energy. So you need the reference basically to decide which way to go. But anyway, this is one possible Hamiltonian.

How do we explain it physically? Well, the momentum is a vector. And what the Hamiltonian says, that the difference between the magnitude of the vector at the given position of the index of refraction, the difference has to be constant. I can pick this constant to be 0. Since it is arbitrary, let me pick it to be 0.

If that is the case, what I'm really saying is that the momentum, the vector of the momentum is constrained to lie on a sphere. And that actually is known as Descartes' sphere, but it is not really-- Descartes was not the first one to make this observation. It was an Arab scientist whose name is, I believe, al-Haytham, who actually made this observation for the first time. And actually, al-Haytham also derived Snell's Law for the first time.

So basically what the Hamiltonian that we derived says is that the momentum of the array is constrained to lie on a sphere. Now I did not go through the slides when we did Snell's law, but it is in your notes. And you may have noticed it if you were diligent when you were studying the notes.

Because I actually showed-- this slide is another way to derive Descartes' Law. So what is happening here is I have the familiar situation of Snell's Law. I have a medium of index n to the left and the medium of index n prime to the right.

And in each medium, I associate a sphere with a radius equal to what Descartes predicts. So the sphere has radius n on the left and another sphere with radius n prime on the right. In this case, n prime is bigger than n, so the sphere is bigger on the right-hand side.

So Snell's Law here is derived by applied momentum observation, which actually, Professor Sheppard mentioned in the beginning when we were discussing [? Kalpesh's ?] question. And momentum observation says the following-- that in the vertical direction, up and down, clearly, this geometry is invariant. There's no change.

So therefore, the momentum must be conserved. The momentum on the left-hand side, which we denote as p vertical, and the momentum on the right-hand side, p prime vertical, must be equal. And since the momenta controls Descartes' calculation equal to this, then basically, these vertical components are obtained by projecting the radius of the sphere onto the vertical axis.

And of course, if you do this now, you immediately get Snell's Law, because this angle is theta. So therefore, this length equals n sine theta on the left-hand side, and n prime sine theta prime on the right-hand side. So that is yet another way to obtain Snell's Law.

And of course, you might wonder what happens to the momentum on the [? back, ?] on the longitudinal dimension. That's OK. You can have a change in momentum there because there is a change in media.

So the change in media actually corresponds to a change in momentum. And momentum is conserved, with actually means that the medium here, if you actually use the proper electrodynamics, there is actually a force applied in this medium here due to the change in momentum of the light. But this is beyond the scope of the class, so we'll not discuss it in any greater detail.

So this kind of picture here, with the Descartes sphere, it also applies to a GRIN medium. So here is a medium whose index of refraction is variable. So you go from low to high. Again, the convention is gray means higher index.

And you can break it down. Like we did, again, in the beginning of the class, we can break it down into slices of progressively higher index. And you can apply Snell's Law, using either Snell itself or the succession of Descartes sphere. You can apply it as the ray goes down. And then either because it has too much momentum or because it has to satisfy Snell's Law, these two statements are exactly equivalent, the ray must bend its path.

So this is yet another way to justify why the ray is bending. It is bending because it must conserve the value of this Hamiltonian. It must lie on the Descartes sphere. But of course, in the GRIN medium, we have to be a little bit careful because this sphere has a variable radius, as you can see in this diagram over here.

Now let's come to the other question that was asked. Is there an analogy between mechanics and optics? And because we're running a little bit late, I don't want to spend too long with this analogy. It's a very interesting one. I'll let you study it. And if you'd like, we can also discuss it later.

But basically, the analogy comes from this-- we go back and rewrite the Hamiltonian equation of the mechanical system as a sort of unassailable kinetic energy term. The first term over here is clearly the kinetic energy, the term that looks like momentum squared over 2m. Then the potential-- we wrote the potential before.

In the case of the elastic spring, the elastic potential was v of q equals 1/2 k q [INAUDIBLE]. That's one special case of a restoring force, of an elastic potential. But it could be more general. So we write it as v of q. And then the rest of the definitions are basically straightforward. The momentum equals m times the time derivative of the trajectory.

Anyway, like I said, I don't want to go through this in great detail. But basically, if as a starting force we use the index of refraction squared divided by 2m, where m is like a fictitious mass that we have to assign to the photon, then we would basically get the one-to-one analogy. This is quite interesting, because it is really acting like a spring or like gravity, but gravity is not exactly the same, because gravity is like 1 over r.

So you have to make an approximation and blah, blah, blah. But if you think of it as a spring, then you get basically a one-to-one analogy. So it's kind of interesting.

And the last thing I want to say, because it has a practical significance, is that once we write down the Hamiltonian equations, then it is very convenient. In addition to all this nice physics, and intuition, and so on, it is very convenient, because we can just plug them in Matlab and solve them. So if someone is giving us a graded index medium-- now whether it is quadratic, or cubic, or whatever, 27th polynomial, or anything for that matter, we can just solve it using this set of equations.

And here, what I did is I did an example of a quadratic index medium on-axis, off-axis. You can see that on-axis actually pretty good. It is not completely free of spherical, but pretty close to free.

You can see it does come to a very nice geometrical focus. Of course, off-axis, it has a terrible aberration that we've called what? How do we call this particular aberration that we see here?

AUDIENCE: Coma.

GEORGE BARBASTATHIS: Coma. Yes.

AUDIENCE: Can you use the index of refraction to correct for coma?

GEORGE BARBASTATHIS: Yes, of course. In fact, believe it or not, this is a research project in my group, which says that suppose you wanted to define an arbitrary index distribution here in order to correct for a given set of aberrations. So someone is giving you the specifications. They say, I want to eliminate the coma for this angle of incidence. I want to eliminate other aberrations, for example astigmatism and so on, then in principle, you can come up with some index of refraction, n of r, which if properly chosen, will eliminate this specified aberration.

Now of course, that's pretty easy, actually. That doesn't need research. It's kind of like a computational problem.

But how do you implement an arbitrary index of refraction? That's difficult. So we're actually working on that in my group. But yeah, in principle, you can. In practice, with commercial GRINs, it a bit difficult, actually. You would probably have to use something more than a GRIN, as far as I know. Yeah.

AUDIENCE: On the previous slide, you said n has to be-- [INAUDIBLE]

AUDIENCE: Can you press the button, please? Can you press the button?

AUDIENCE: He is pressing, but it's broken. [INAUDIBLE]

AUDIENCE: On the previous slide, you said something about physical-- n has to be greater than 0.

GEORGE BARBASTATHIS: You caught me. Yes. Yes, I said that n has to be bigger than 1,

AUDIENCE: [INAUDIBLE]

GEORGE BARBASTATHIS: Better than 1. The question is, can n be less than 1 and less than 0? From the point of view of geometrical optics, n is the speed of light in a bulk medium. And in ordinary bulk media that we use in geometrical optics, light is always slower than in vacuum, so n is always bigger than 1.

In a few lectures, when we do proper electromagnetics-- also, let me say that so far, n has been a phenomenological quantity. That's another funny word like "postulate." What does it mean, "phenomenological."

It means a quantity that I have not justified its physical origin. I just pulled it out of a hat. And I said, hey, you know what, the speed of light in the medium is less than the speed in a vacuum. So therefore, it is bigger than 1.

Now, when we do the proper physical origins of n, then we will see that indeed, n can be less than 1. It can even be negative, and that's another hot topic in research optics nowadays. So it is not necessary. That's true.

But if you take it into the mechanical analogy, interestingly enough, if you allow n to go into these sort of strange modern sort of regions, where it is less than 1 and even negative, then in the mechanical analogy you would end up with a negative energy, which is kind of interesting. It doesn't mean anything. This is just an analogy. It doesn't invalidate anything, but it is kind of interesting.

And let me say upfront, n becoming smaller than 1 does not mean that I violate relativity. It does not mean that something moves faster than the speed of light. And as we will see, just today, actually, n is what we call the "phase velocity."

And the phase velocity of wave can be much greater than the speed of light, because the phase carries no information. That's sort of the classical way to justify it. So I can allow, actually, n to be less than 1, and am perfectly fine.

So it is not necessary, but I should put a qualification here. Not physically allowable, but how you say-- every day intuition allowable, perhaps. But of course, physicists can do much better than everyday intuition.

So let's resume for another 20 minutes or so, and we'll continue on Monday. So what I would like to do today is start the description of wave optics. So basically, we followed geometrical optics and it was good enough to explain a number of interesting things for us, imaging systems, gradient indices, and so on.

But geometrical optics also has some serious failures. The most important of those is that when light interacts with very small features, geometrical optics fails. So for example, I used it when I answered [? Kalpesh's ?] question.

If you limit light to a single ray-- that is to an infinitesimal, small spot-- than geometrical optics pattern fails. So the picture that it gives us are completely wrong. So in order to deal with that, we have to deal with light as a wave phenomenon.

And there's an interesting history there. Newton was a big proponent of geometrical or particle description of light. Some other contemporaries of his, like Huygens, Fresnel, and sort of the Continental school, they were proponents of the wave theory.

And for many years, the question remained unsettled until Maxwell-- in the end, I think 1899, or 1896, or something like that, James Clerk Maxwell, actually a Professor of Fluid Mechanics, came up with a set of equations that proved very convincingly that the light is a wave. And then 10 years later, quantum mechanics proved that light is also a particle. So both were correct. Both Newton and the Continental school were correct.

So anyway, so you can think now that we're basically now moving from England to the continental Europe, because we're abandoning the geometrical optics theory, and we're moving to wave optics. Now, waves, of course, are a very pervasive phenomenon that we see around us. And they manifest themselves in many different forms.

So we're familiar with waves in seawater or rivers, and so on. We're familiar with sound waves. It's actually Singapore's founder, Lee Kuan Yew giving a speech.

So we can communicate by sound waves. That's another form of a material wave. So like water waves, sound waves are material waves. You set the air particles in motion when you speak.

We're also familiar with electromagnetic or radio waves, not because we can see them. These are actually not something that you can touch. But all of us, we use cell phones, old-fashioned radios, and so on. We use them.

This is actually an old-fashioned antenna. It's not that old-fashioned if you go to places like Europe, like Greece, for example. Many homes still have those antennas in their roof. So this is the way people used to receive television signals before cable TV.

And finally, there's other waves that they're sort of not quite obvious that they're waves. This is a very interesting geological phenomenon in the Philippines. It's called the "Chocolate Hills." There's a number, it's actually hundreds of very small hills, not very tall.

You might say, what kind of a wave is that? The hills are not moving. Actually, they're not moving in our time scale, but in the geological terms, they actually move. And sometimes, this motion-- as you know, the motion of the earth-- it can actually happen in timescales that are perceived by humans, and then it's called an "earthquake." It's a very nasty phenomenon.

So the point is here that waves manifest themselves in many different forms. Of course, I omitted another important manifestation, which is quantum mechanics. Quantum mechanics says that everything is actually wave, including yourselves and me. And every particle is also described as a wave. So electrons can be thought of as waves, and so on, and so forth.

Anyway, the waves are in fact so pervasive that there's no proper definition. And to paraphrase something that an artist said once about art, he said that art cannot be defined, but you know it when you see it. That's actually a very good way to put it, at least as far as art is concerned.

But it's also true about waves. Because there is a great variation in the way waves occur, there's no proper definition. What I attempted here is the simplest possible, all-inclusive definition as a traveling disturbance.

As far as I can tell, it is certainly inclusive of all waves. Of course, it includes other things that may not be waves, like a car moving down the road is also a traveling disturbance. It is clearly not a wave.

But anyway, another way to perhaps describe waves is if you-- there's some evidences of wave behavior. And a very prominent one is what you call "interference." So what you're seeing here is, we will see, these are called "spherical waves."

And we see two point sources that meet in these spherical waves. And as they propagate out, you see this very characteristic alternation between bright and dark spots. I have not even really said what this thing is. Is it a sound wave, a water wave, an electromagnetic wave?

In fact, it could be all of the above. In all of these cases, if I represent as bright the high value of the disturbance and as black the low value of the disturbance, then these sort of not quite periodic, but periodic-like looking behavior is called "interference." And when you observe this sort of interference, then it is a pretty good indication that you are actually observing a wave phenomenon.

In fact, the reason Fresnel in the Continental school insisted that light is a wave is because they actually observed light interference in experiments. They observed the light doing something like that in the experiment. So they said, well, wait, it must be a wave, then.

Another phenomenon which is-- actually, it is related to interference. When we cover it, it has quite a different name. It's called "diffraction." But when we cover it, we will see that clearly, it is not a new phenomenon. It is interference in this guise.

And diffraction basically says that the geometrical definition of shadows is incorrect. So what you see here is a wave that is impinging on a block. So it is blocked above and below. And a small portion of it is allowed to pass through.

Geometrical optics would have you believe that you would basically get a continuation, unperturbed continuation of the wave in the geometrical shadow of this slit. You can think of it as a slit there that the wave goes through.

But diffraction says that's not the case. As you can see, there's a rather more complicated pattern that emerges here. And that is another evidence of a wave phenomenon that we will describe quantitatively in great detail later.

Let's take a look a little bit in some more waves. So for example, the first one-- it meets my definition as a traveling disturbance, but not a very interesting one, because it is just a shape moving. The one that is on the right is perhaps more interesting, because now you see sort of-- it is more believable that it is a wave.

It has some oscillation in addition to the traveling disturbance. By the way, oscillatory behavior is also characteristic of waves. But it is not necessarily periodic. In fact, you can have a periodic wave, but that's not very interesting. It is the non-periodic waves that are the most interesting ones.

Now, if you let me go back and play this movie once again, that's not very interesting, but what you will see here is you will see that it's actually the same envelope that you saw moving on this diagram. You will see it is the same envelope moving over here, but inside, you have some wiggles that are also moving. And if you observe it very carefully, you'll see that the envelope is moving with a different velocity than the wiggles.

The wiggles have a name, by the way. They're called fringes. Can you see? So the wiggles are actually moving faster than the envelope.

So this brings us to the definition of what I used once Professor Sheppard's question about n earlier. I used the term "phase velocity." So what you saw in the previous movie, in the one that was in the right-hand side, was an envelope that was moving.

So here is the blue one. That's the envelope. And inside, there was a sinusoid. The sinusoid is multiplied by the envelope.

A wave that is composed of a sinusoid multiplied by an envelope is said to have a carrier. So the carrier is a sinusoid itself. The carrier can move at its own velocity, which, surprisingly, is usually different than the velocity that the envelope is moving.

Why I say surprisingly, well, some simple sort of [INAUDIBLE] might have said, they better move at the same velocity, shouldn't they, when in fact, the truth is the opposite. More often than not-- in fact, you have to try very hard to make them move with the same velocity, but more often than not, they move at different velocities. So the velocity at which the envelope is moving is called the "group velocity." The velocity at which the carrier is moving is called the "phase velocity."

And right now, we'll just leave those like this. Later, we will actually derive them properly. And we will see how they come about, and how they are defined. But I can say for now, without any proof, that what we wrote in geometrical optics, c over n, that is the phase velocity of the light.

If you were to put an envelope on the light somehow, which is done in special lasers called "mode-locked lasers," you can have a light wave that looks like this. It has an envelope as it propagates. In those, the group velocity is generally different than the phase velocity of the light.

Of course I said that earlier that the other waves are not very interesting. Nevertheless, for educational and analysis purposes, they're simple. So therefore, we start with periodic waves when we analyze waves, and periodic is also known as "harmonic waves."

And of course, in a harmonic wave, there's no envelope to speak of. And therefore, there's no group velocity to speak of because the envelope is flat. There's no profile going anywhere.

The only thing you have is a sinusoidal disturbance moving with the phase velocity. Here you only have the carrier, so the only velocity to speak of is the phase velocity. That's a characteristic of the harmonic wave.

And I will start putting some notation here. So the time, you can see that time is run in index here. And the horizontal axis, I will call it z. So we will stick to z as the actual wave propagation for a while.

So now let's do some math. Since this is a harmonic wave, I can expect to describe it as a sinusoid. And the sinusoid, in order to characterize it, I need to set a number of quantities, which have certain buzzwords. So we'd better start defining our buzzwords right now, so we don't get confused when we refer to them later in the class.

So the constant that goes in front of the sinusoid is called the "amplitude." The reason I use e as a symbol for the constant is because electromagnetic waves are usually electric fields. We will see that later, after we do Maxwell's equations.

For now, this could be any symbol, actually. In fact, very quickly I will refer to a different symbol. But in any case, this quantity that appears here is called the amplitude.

Then the other quantity that we need to describe a sinusoid is the frequency. But now the frequency, we have to be a little bit careful because there's two frequencies going on. You can see very clearly-- you can think of this as a snapshot of the wave.

If I take this picture and I freeze it, I will get something that looks like this. So in this snapshot, the axis is space coordinate. It is z. So the period that you observe here, the period that you observe in the spatial axis, it is called the "wavelength." We saw that before. We did the definition of the wavelength in, I believe, lecture number one.

Now suppose I do like this-- suppose I fix my position somewhere on this axis, and I observe the value of the wave as it is passing by me. So now I'm fixed at the given z and observe the Wave.

What do I observe if I stand over here? I will have another sinusoid, don't I? Because if I am sitting here, the wave that I see will be oscillating. It will go in the sinusoidal fashion. It will oscillate from a positive value to a negative value.

So that actually associates with a temporal frequency of the wave, for which we simply use the term "frequency." So we'll say frequency, we mean the temporal frequency. And some people use f. I prefer to use the symbol nu for that quantity.

Finally, I have not defined my time axis yet. So when I'm looking at the oscillation, if I'm fixed and I look at the oscillation of the wave, at time 0, the wave might be absolute positive. In that case, it means the wave is a cosine. Or it may be 0. In that case, it means it is a sine.

And how do I correct for this discrepancy? By adding a phase delay, by adding, for example in this case, pi over 2. So this additional phase that I need to add in order to fix the origin of time is called the "phase delay." And for a while, I will use the symbol phi for the phase delay.

So we defined basically all of the quantities that we need, except nu is not anywhere to be seen in this equation. What I've written instead is this expression z minus ct. And what is the meaning of this expression?

AUDIENCE: Traveling wave.

GEORGE BARBASTATHIS: It's a traveling wave, or another way to put it is a traveling coordinate system. If I write-- we have no doubt what this is. That's a sinusoid.

Now, that's a sinusoid in an axis of coordinate xi. If I now say xi equals z minus c t, where t is time, what I have done is I've taken this axis and I've started moving it with respect to its rest position that equals 0. So this is what I've done in the expression for the wave. I've put this traveling coordinate system into the sinusoid.

And I don't really have the frequency in here, but you can see that time, the time variable, is in the expression for the sinusoid. So I can get the frequency very easily. And the way I get it is by observing that since the wave is periodic in the time domain, it will repeat one period later.

So the period is the inverse of the frequency. So if I look at this expression over here, let me rewrite the wave. I can actually write it-- I haven't done anything, just expanded the parameters.

So when this quantity becomes 2 pi or any integral multiple thereof, the wave will repeat, won't it? So that will happen when the time equals the period of the wave. Or another way to put it is that 2 pi over lambda times c over nu, where nu is the frequency, the inverse of the period, here comes the pi.

Of course, the pi goes away, and we derive something that we've seen before, c equals lambda nu. And that is what we used to call the "dispersion relation." So this is how we connect the phase velocity of the wave with the wavelength and the frequency.

And the phase delay is actually a very important property of waves, but it's not a real big deal. This is what I was saying before.

AUDIENCE: George.

GEORGE BARBASTATHIS: Yes.

AUDIENCE: Why is it called "dispersion relation?"

GEORGE BARBASTATHIS: Yes, there's no dispersion to be seen here, isn't there? However, if I put, for example an index of refraction, then the index of refraction is a function of lambda, so then you'll get dispersion. And of course, another case is this is the dispersion relation in free space. If you put boundary conditions to the wave-- for example, if you confine it in a wave guide-- the dispersion relation changes. I will not do this in this class, but that's another way where the wave becomes dispersive.

So that is the explanation of the phase delay. I don't want to spend too much time. If it equals 0, you see this. This means the wave is a cosine, the phase delay is 0.

And then, depending on what you see at t equals 0, you may have different phase delays. Now, the reason the phase delay is an important quantity is because I want you to look at it very carefully and tell me what you see. So anybody want to volunteer what has happened here? I'm going to play this once again.

How are the two different, on the left and on the right? On the left, again, look at it very carefully. On the left, the two spherical waves, they're always bright together or dark together. So they oscillate, as we say, in phase.

On the right, the two waves are out of phase. So if you look at it carefully, you will see that when one is bright, the other is dark, and vice versa. I'll play it again so you can see it.

And because of this, again, I'm going to play it once again. And now don't look at the centers of the waves anymore, but look at the general patterns. You will see that the general patterns are actually different.

They have the same general shape, but whether the pattern is dark or bright at any given instant actually depends on how the two waves started. So when we have-- this is quantum superposition, because I have waves originating from two sources simultaneously. When we have a superposition, the structure of the overall wave that we observe depends very strongly on the relative phase between the two, the two sources in this case.

So with that, I think I will stop, and we will continue on Monday. And I will not post any new notes for Monday, because what I think we're far enough behind that we should catch up on Monday. What I would like to say is that it is about time to start the project for the graduate class, for 2.710.

So soon, but not for-- basically by Thursday morning Boston time and hopefully earlier, I will post in the website a list of possible projects and some instructions on what to do. And then basically, what I would like you guys to do is spend the next week selecting your project, and also contacting your mentors. Each project will be mentored by one of my graduate students, and some by me.

So possibly, I don't know if Colin, you might be interested in mentoring one, but anyway, we'll figure it out. Maybe Sam from Singapore, maybe not. But anyway, we will assign mentors for the projects, and then we'll get them started. The presentation is in May, May 6 or May 7. But anyway, the sooner we start, the better.