GEORGE BARBASTATHIS: So I'd like to continue with what we were discussing last time. We were at the point where we're discussing the three dimensional wave equation. Before we start, let me take care of a couple of administrative things.

So for one, I'm very close to finishing grading the quizzes. So as soon as I do, I'll put them on a DHL and ship them back to Boston. Of course, those of you who are in Singapore will get them from-- locally. And so that's done.

The quiz solution, of course, has been posted. You must have seen it by now. So I would encourage you to go over this solution. It was a pretty good problems, actually, both of them were pretty good.

And the other thing I want to say is that I have posted the new homework. So has anybody looked at it yet?

OK, if you look at it, it might appear to be a little bit intimidating, especially the graduate version. It is about six or seven pages. But you may know from experience that if your homework is very wordy, it means that most of the work has been done for you.

So basically, what I've done is I've-- the reason I-- this is the first time I give this assignment. And basically I develop the analogy between the optical Hamiltonian and the mechanical Hamiltonian. And I guide you through a few derivations, a few proofs that basically, I think, in my opinion, elucidate the analogy and the difference. Of course, they're not exactly the same.

OK, and the reason I did it is because last year was the first time that they taught the Hamiltonian. It used to be-- I used to just skip that part and do other things. But last year I did it. And I got some pretty good feedback from the students who thought that, well, you know, this is kind of a cool analogy.

So this year I expanded it a bit. So that's why this homework is a little bit rich. But it's not to bad, really, once you get into it. And if you follow the instructions carefully, each equation should taken no more than two lines. So it is not like a long derivation or [INAUDIBLE] or anything like that.

It's just a step by step, and if you-- but read it carefully. Because there's some derivatives involved. So you will want to make sure you take the correct derivative each time. That's the only challenge. If you can manage to do it, which you will, if you follow the instructions carefully, then it should be painless.

And there's a little bit of MATLAB that you have to do. But that's not valuable. And the code, actually the MATLAB that you have to do, I also give you the code. It's posted to the website. It's a bunch of functions. And one of the things that I do in the assignment is I walk you through how to use the MATLAB code. So what you will have to do, basically, is take one of the functions and modify it.

Yeah, someone, turn on the mic. Have a question? Or-- OK. So going back to the-- I think there's a question. Somebody has turned on your mic, which is OK, actually, as long as you don't say anything embarrassing to [INAUDIBLE].

So let me go back. Let's go back now to the main topic. So, of course, we have moved on from Hamiltonians and all that stuff. We're doing wave optics now. And the last time we discussed the wave equation.

And without really a proof-- we did the proof of the one dimensional case. But then without the proof, I just added. Basically, the one dimensional case was this one, just the derivative of the Z term without any additional terms.

And then you can think of it simply by symmetry. If you accept that 3D space is symmetric, then the way you go from one dimension into the three dimensions is by just adding two derivatives similar to the one that you already had. And, of course, the waves that we'll get in 3D are much richer than the waves that we expected to get in 1D.

And here I plotted two cases. One is a plane wave, which I will play the movie. So you will see the fringes, in this case, of the plane wave are moving in some arbitrary direction now. It can go anywhere in 3D. Of course, I can only show a slice on the plane of your projector.

And there's a spherical wave where the wavefronts are not circles as you see here. But they're actually spheres. So they're spheres that are propagating clearly outwards in this case. So this is what we used to call it a diverging spherical wave in geometrical optics.

Well, you can also have the opposite. You can have a converging spherical wave if the wavefronts are moving inwards. So one of these are valid solutions to the wave equation. And the plane wave is very simple. I will do it next. The spherical wave is not really difficult, but it involves a transformation of the coordinates from cartesian that we have here to polar. And this is done in the book. I'm not going to do it in the class. It is a little bit of work. And you don't get any particular intuition.

If you take a class in electromagnetics or electrostatics, they will probably do it. But because in this class, we will use the spherical wave in its full form very radically-- we'll use an approximation to it. That's why I don't bother to do the full derivation.

OK, I'm going to skip now a couple of slides. And I'm going to go directly to this one, slide number 16, which is the plane wave. So what I would like to do-- so, for the rest of the class now, pretty much for the next, what is it, seven or eight weeks that we have left? We will be talking almost exclusively about plane waves and their superpositions.

Because what we will discover is that we can describe every optical wave and-- well, at least within the approximations that are valid in this class-- you can describe every optical wave as a superposition of plane waves. It's a very powerful property. Because plane waves, as we will see, they're very simple. And they obey some relatively straightforward properties. So if we learn how to describe those properly, then we can describe a great variety of other phenomena, including diffraction, interference, refraction, and a number of other things.

So I would like to make sure that we understand plane waves. It is a crucial part of the class. OK, so in the last lecture, we wrote a simple wave that we'll call the harmonic wave. And it used to look like this. From now on, I will start using a phasor notation almost without warning. So when I start throwing complex exponentials, you know that I'm using the phasor.

OK, so having said that, now let's look at the phasor of what I wrote last time. So last time I wrote something that looked like e to the ikz e to the minus i omega t. And I call this a harmonic wave. So this was in 1D. So the plane wave is basically the generalization of this one.

Now, I'm going to have to write terms of the form y and x. Do you remember what we call the quantity k in the case what I still had the one dimensional wave? What was the name of k?

AUDIENCE: It was the--

GEORGE BARBASTATHIS: Wave number, sorry, yeah, wave number, yeah. I'm sorry, someone answered in Singapore, so I-- OK, but he was diligent. So make sure to push your button.

OK, so k is the wave number. And it is fair to use it if you have one dimension. But if you have three dimensions, then we actually need, at least apparently, we need three such quantities. So I need to put the z subscript here. And then I need two more, one for the y and one for x.

So now the triad of these three-- I don't want to call them wave numbers. They're not really wave numbers. But these three quantities that have the-- I mean, these are essentially spatial periods, aren't they, in the three dimensions? So these quantities, if you take the triad of these quantities, it is called the wave vector.

And it is usually-- we use the same symbol as the wave number, but with the vector now. And it consists of the three components, ky, kz. It is a matter of taste how you write vectors. In this class, I actually-- I will alternate a little bit. Sometimes I will write vectors like this as a triad in a parenthesis. Sometimes I might write the vectors like this where I use the unit vector, the unit Cartesian vectors.

Of course-- OK, I hope that's not confusing. One can easily-- clearly they're both commonly used. So that quantity k, then, is the wave vector. And its significance, you can see it on the slide. It's probably better if I don't attempt to draw it there. But you can see it on the slide.

So the wavefront of the plane wave is a surface which is perpendicular to the wave vector. And what is a wavefront? Well, wavefront, you saw it in the movie that I played earlier. Wavefront is the surface upon which the phase of the wave is constant.

So what is this? This is basically a three dimensional sinusoid of which we only see a 2D slice. So this is sinusoid which alternates. The black regions are negative, minimum. I mean they're sort of a maximum negative value. The bright stripes are maximum positive value. And the wavefront is a surface on which the phase of the wave has the same value.

For example, if the phase is zero, then we're talking about the positive peak. If their phase is pi, then we're talking about the negative trough, and so on, and so forth, or anywhere in between, for that matter. And the wavefront has a sense of travel to it. Because if I play the movie again, then you can see that these surfaces of constant phase are moving. This is the sense that the wave transports energy, if you wish. Because the wavefronts are moving with the wave speed.

OK, so then we call it the plane wave, because this surface is of constant phase or planes. Of course, you only see them as lines here. Because, again, we're looking at the slice of the wave. But in 3D, they would be actually planes that would be propagating with the speed of the wave.

So that is-- so then the wave vector, we need it in order to tell us what is the orientation of these planes. And for reasons of convenience, we define it to be perpendicular to those planes. Now, that's one useful thing to know. But that's not all.

The wave vector, it appears to be composed of three elements. So the question is are these-- do I really have three parameters? Can I describe a plane wave with three parameters?

Well, even before I do any math, you could probably guess that I shouldn't. Because these are surfaces. These are planar surfaces in 3D space. So I cannot really fit 3 dimensions of planes in a 3D space, right? Something is wrong with that argument, even without doing any math.

So in actuality, these three numbers, kx, ky, and kz, they have to be related somehow. I cannot arbitrarily define all three of them. But I have to somehow relate them.

So the way I relate them is I actually go back to my wave equation. And recall this thing here, the wave, which I will call it something. What did I call it? I didn't call it anything, actually. I just put an amplitude coefficient in front. So the amplitude coefficient, it is good for completeness, I guess. And it must satisfy the wave equation.

So it must satisfy-- I'll call it something now, because-- f, I guess. So it is f of x, y, z, t. And it must satisfy this nasty thing, which looks like df dx squared plus df dy squared plus df dz squared minus 1 over the velocity now, d squared f over dt squared. It must equal to 0. This is what we call the wave equation.

And for future reference, sometimes, instead of writing all these deduced derivatives, there's a shorthand which collapses all of this with a symbol del. So you can just write del squared f minus 1 over c squared df dt squared is 0. OK, that's the wave equation.

Very good, so now what I need to do is I need to plug this expression into the wave equation. So what expression? This expression that I had for the wave, I need to put it into the equation. OK, almost by inspection, we can see what's going to happen.

So we have an exponential, right? And if I take one derivative, the first derivative, it will knock out-- it will multiply by ikx, right, if I take the x derivative. If I take the second x derivative, it will bring out another ikx. And since i squared equals minus 1, it means that the derivative will look like this, d-- let me write it separately here.

The second derivative, for example, of F with respect to x squared, it will equal minus kx squared times f itself, right? And then it can do the same for y, and z, and t. Except in the case of time, I will pull out omega squared.

So if I substitute all of that now and I put it in, I will get that minus kx squared minus ky squared minus kz squared-- these are the first batch of derivatives-- and then minus omega squared-- but I have to divide by c also because c, actually, c squared is in the wave equation. So all of this, according to the wave equation, must equal to 0. And, of course, it is possible to satisfy this by setting f equal to 0. But that's not very interesting.

So basically what this means, that this quantity here, it must equal to 0. So then, as advertised, the three parameters, kx, ky, kz, they're not independent. But I can specify two of them. The third one is completely determined by the two among the three plus the frequency of the wave. So if I give you omega, kx, and ky, then this equation will force kz to have a certain value.

And you can see from this equation, which is written bottom of the whiteboard and on the slide--

AUDIENCE: [INAUDIBLE]

GEORGE BARBASTATHIS: Oh, oh, oh, oh, thank you, thank you, yes, because I have a--

AUDIENCE: [INAUDIBLE]

GEORGE BARBASTATHIS: Yes, thank you. Because I have a minus sign here. So the extra minus will become a plus here. Yes, OK, now it is correct. And so if you look at this equation now, what does it describe? If you think of kx, ky, kz, if you think of them now as a new three dimensional space, what is the surface that the three of them are constrained to lie upon? Button.

AUDIENCE: A spherical shell?

GEORGE BARBASTATHIS: It's a sphere, that's right, a shell, the shell of a sphere, yeah. And this sphere is not new to us who have seen it. It actually popped up in a different context. We called it the Descartes sphere.

It is actually the same sphere that-- actually not even Descartes, an Arab scientist in the 11th century, guessed from Snell's law. So it is a little bit, I guess, magical, or soothing, or rewarding, whatever combination of those adjectives you like, that we found a nice analogy-- or not analogy, actually, we found an agreement with geometrical optics.

In geometrical optics, we said that the radius of this sphere equals 2 pi upon lambda. This is now what, in our new language, we call it the wave number, times the index of refraction, n. Now this is something that I haven't told you yet. But I will tell you later today.

In this equation, I haven't said anything about index of refraction. However, we know, again from geometrical optics, that c, the speed of light, depends on the index of refraction. So another way to derive this equation is to say-- to put the free space speed of light over here. And then in order for the equation to be correct, I would have to multiply by n squared, the index of refraction.

So you can see from this that, indeed, the radius of the sphere. Oh, and of course, to compute the radius of this sphere, I have to use, once again, these dispersion relations. So the radius of this sphere, if you look from this equation over here, the radius of this sphere is k, the magnitude of the wave vector. It must equal n omega over c free space.

OK, and now it is a matter of preference how you want to do it. For example, you can write omega as 2 pi times nu, where nu is of the frequency. Remember, omega is the angular frequency-- over cfs. Then you can use the dispersion relation, which says that c free space equals lambda nu. So nu will cancel. And you end up with n times 2 pi over lambda free space.

And, of course, nu-- I don't say anything about free space, because nu is the same everywhere, whether it is free space, or a slave-- I mean, not free space. I'm sorry. OK, so that's where this equation comes from. That's why the radius of the k sphere, it equals n omega over the free-- over the speed of light in a vacuum or 2 pi n over lambda, the wavelength in free space.

Now, those of you-- some of you may have taken solid state. This k sphere also comes up there. And it is called the Ewald sphere. In our language in optics, we seldom call it Ewald sphere. We call it k sphere or Descartes sphere. But all of these three terms, all of these three names, they actually denote the same thing.

And, of course, in the case of solid state, this sphere does not apply to light usually. But it applies to what? Button?

AUDIENCE: Electrons.

GEORGE BARBASTATHIS: To electrons, that's correct. So according to quantum mechanics, electrons are also waves. And they satisfy-- actually that don't satisfy this equation. They satisfy the Schroedinger equation, which is in one of the problems of the homework. So anyway, so Schroedinger equation similarly leads to a sphere like this one.

Is there one thing that-- OK, so this is a sort of preliminary introduction to plane waves. One thing that I would like to very strongly emphasize is this expression over here, the complex exponential-- I will rewrite it once again. e to the i kx x plus ky y plus kz z minus omega t, OK, this expression, when we see that, when we see an exponential with linear terms in the Cartesian coordinates, that's a plane wave. And now you know why, right? Because it corresponds to this moving wavefront.

And, of course, most of the time, we will not add it like this. We will omit this term because of the phasor business that we discussed last time. So we will actually see it like this. And also, very often, when-- several things that happen in here. For one thing, these three numbers, as I said before, they're not free. So they are given by kx squared plus ky squared plus kz squared equals omega over c squared or-- OK.

This is another way of solving the sphere. So we will see these kinds of expressions. For example, I might write something like e to the i kx x plus ky y plus. And there's a z here outside of the square root.

So this looks a little bit scary. But it doesn't need to be scary. It a plane wave. And we will see expressions like this one later in the class. But I want to sort of prepare you now so that they are associated with a physical meaning in your minds. It is not just an expression. It's actually denotes a plane wave. The two linear terms here-- actually, all three are linear. x, y, and z appear as first powers. And the reason this square root appears here is because of the Descartes sphere, or the Ewald sphere, or whichever.

OK, any questions about plane waves? The other wave that we'll be dealing with quite a bit is the spherical wave. And so clearly, as the name suggests, in this case, you have spherical wavefronts, which are either going away from a source, a diverging wave, or inwards, a converging wave.

Just as an aside, by the way, I should emphasize that the spherical wave is not a physical thing. So plane wave is not physical either, by the way. Why is the plane wave not physical? I mean, what's an obvious reason why a plane wave must be an approximation of something? Button.

AUDIENCE: Energy of the plane wave is infinite.

GEORGE BARBASTATHIS: That's right. It has infinite energy. It is infinitely large. So, clearly, I cannot create plane waves per se. I can only create sort of finite approximations to plane waves.

A spherical wave is-- but nevertheless, it is an interesting thing. And the reason we spend all this time defining it is because it provides a lot of mathematical convenience in dealing with more realistic waves, waves that can be realized. I can express them in terms of this fictitious entity, the plane wave.

The same is true for the point source, but for a slightly different reason. It is actually impossible to create an ideal spherical wave. The best thing you can do is you can create a wave that would sort of, if you-- well, the only-- the best thing that you can do is you can create a dipole source here. So if you aim the dipole the same, the vertical direction, then what you can do is you can create a radiation pattern that has approximately 60, or actually, 120 degrees opening, 126, to be-- 63, isn't it? I think it's 63 degrees, the dipole.

Anyway, so you can create a direction pattern like this, but never quite a spherical wave. In fact, someone as famous as Einstein said that spherical waves don't exist. Nevertheless, it is a very convenient mathematical quantity. And that's why we're describing it.

But again, by the way, this is true in any kind of physics that you might learn. For example, in mechanics, very often you deal with frictionless--

AUDIENCE: [INAUDIBLE]

GEORGE BARBASTATHIS: Is [INAUDIBLE]? Yeah.

AUDIENCE: Suppose you burst a small cracker in a water tank? Then it wouldn't-- wouldn't it be?

GEORGE BARBASTATHIS: Oh, I'm sorry, I meant only in optics.

AUDIENCE: Optics.

GEORGE BARBASTATHIS: Yeah, in water tank, possibly you can create a spherical wave. Even so, it's not a spherical wave. Because it is finite, right? It has a finite source.

Anyway, the reason in optics you can fundamentally not create this kind of thing is because of charge conservation. You cannot-- in order to create a spherical wave like this, you would have to create an oscillating charge here. And no such thing exists.

So the only thing you can do is you can create a dipole. That is two charges which are alternating, up and down, up and down, positive and negative. That would result in a wave that it looks approximately like a spherical wave, but only in the direction perpendicular to the dipole. In the direction along the dipole axis, it actually, the wave vanishes. So it is very different than a spherical wave.

To justify it a little bit why, even though it is non-physical, why we spend so much time discussing it, it has to do, actually, with systems theory. Yes, question?

AUDIENCE: Isn't a dipole actually canceling out because of the distance between the top and bottom source? You have like a different distance between them.

GEORGE BARBASTATHIS: The charge, the charge is 0. The charge is 0. But you still have a dipole moment. Then moreover, if the dipole is oscillating, that is, if it is going from positive to negative, positive to negative very, very fast, then you have charges accelerating. So therefore you generate electromagnetic radiation, as we will see in a little bit.

AUDIENCE: No, no, I mean the intensity of the field from the dipole isn't necessarily vanishing along the axis if the spacing's set the right way.

GEORGE BARBASTATHIS: Not everywhere along the axis, but you will definitely get nulls along the axis. If I remember correctly, yeah, the dipole has a radiation pattern that looks like this. It looks like a butterfly.

AUDIENCE: Yeah, but it's dependent on the distance between the plus and the minus charges.

GEORGE BARBASTATHIS: No, I'm talking about an infinitesimally small dipole now.

AUDIENCE: OK, cool.

GEORGE BARBASTATHIS: Yeah, if you have like a lambda over four and 10 or something like that, then yes, the radiation pattern looks different. But in the simplest possible case that you can approximate the best spherical wave is an infinitesimally small dipole whose radiation pattern looks like a butterfly.

So we actually went a little bit ahead of ourselves now. Because this entire discussion, it involved electromagnetics, which we haven't done yet. But by the end of this lecture, it will be a little bit more clear what I mean by dipoles and all of these things.

The thing I meant to say is that the spherical wave, mathematically, it corresponds to a point source. And that is something that most of you, probably, at some point or another, took a class on linear systems where typically you learn a term called an impulse response.

So the impulse pulse in the time domain is a very narrow excitation. In fact, it is infinitesimally narrow. And it is so narrow that, because it has to carry finite energy, it also has infinite amplitude. That's what it's called an impulse.

So the spherical wave is the equivalent of that, but in space now. The spherical wave originates as an infinitesimally small disturbance, if you wish, which then mathematically creates this uniformly or isotropically expanding spherical wavefront. And from that point of view, it is very useful. Because even the dipole that I said before, it can be described, actually, as a superposition of a spherical wave.

So I can be physically correct if I do my physics correct. I can get the proper answer. So the spherical wave provides this mathematical convenience.

Also, to be honest, the math that we will do here, it is a very good approximation to what you observe in the laboratory. That's another good justification for using it. Because even though we know that there is no such thing as a point source, many of the things that we will derive in the next two lectures based in this sort of crude approximation, they tend now to be very good approximations for what you see in the laboratory.

For example, if you take a pinhole, a pinhole is a finite thing, right? I mean, the smallest pinhole that people use in the laboratory, it may have a diameter of one micron. Sometimes we use bigger, three, five micron, something like that. Well, if you pass light through a pinhole, the light that comes out is, to a very good approximation, described by a spherical wave, but of course not everywhere, right?

The light coming out of a pinhole, it doesn't come out backwards. It only goes forward, right? So therefore, you have-- you know, clearly, the expression is approximately correct, but only for the part of the wave, the portion that is physical. So we have to be a little bit mindful of what is the meaning of everything that we write down, especially when things like that are involved.

The spherical wave has another problem. It has the problem that it blows up at the center. And the reason it blows up is because-- well, like the impulse in the time domain that you learned in linear dynamical systems, it has to have a finite energy. So in this context here, the finite energy, as you proved in the homework some time ago, it means that the amplitude of this spherical wave must decrease like one over the distance.

So this 1 over r that we get in the denominator, of course, at r equals 0, it explodes, right? So, again, that is the delta function, the impulse analogy. So the spherical wave has a few mathematical inconveniences of this sort that we have to be mindful with.

Now, there's one more mysterious thing about it, which I will just point out. And I will then move on without saying why. If you look at the expression of the spherical wave-- let me write it down. It looks like cosine kr minus omega t over r. And let's take stock here.

So the 1 over r we discussed. The 1 over r is energy conservation. We always like that. The cosine that you see in the numerator, if I go back and play the movie of the spherical wave, we have to wait for the plane wave to finish here.

So the kr minus omega t, it is this sort of explosion of the wavefronts that you see in the movie. So kr, if i is the radius, kr describes this sphere again.

So the cosine kr basically tells you that you have these alternating positive and negative valued spheres. And the minus omega t is the traveling aspect. It tells you that these spheres explode. They propagate outwards as a function of time. So this is the meaning of the kr minus omega t.

Then it bothers to put another term over there, minus pi over 2. And this might sound a little bit weird. I mean, in general, you can put any phase that you like there. It means that you have a shifted time axis. So I could put 5 there. Why did they bother to put pi over 2? What is the pi over 2 relative to?

OK, it turns out if you properly solve Maxwell's equations, which we haven't seen yet, but I'm sure you are aware that there is such a thing called Maxwell's equation that describes electromagnetic waves. OK, so if you solve it properly with this kind of point source, because of something having to do with magnetic fields, you pick up a phase delay with respect to what? You pick up a phase delay with respect to the source.

So this is now really weird. It tells you if you have a source that is oscillating, let's say, like a cosine, so it is maximum at t equals 0, the wave that comes out will actually be a sine. So it will be phase shifted by pi over 2. There's no easy way to explain this either than solving Maxwell's equation.

There is actually some papers in the literature-- this is very, very interesting for an-- I don't know. it's strange, right? It can be explained from the equation, but it's still a little bit strange. So people are actually working on this phase shift. There's some very strange aspects to it.

Anyway, for us, the phase shift, we'll just accept it. It is well established. So we'll just accept it and move on. When we write it as a phasor, I mean in the complex presentation, of course, e to the minus i pi over 2 is minus i.

So I skipped a few steps of the slide. So let me do them on the whiteboard. So to go from this in the complex presentation, you would write e to the i, kr minus omega t minus pi over 2. And, of course, you still have this thing outside. Nothing can-- this is not affected.

OK, then remember that e to the minus i pi over 2 equals minus i. And it equals 1 over i. So this is why the i appeared in the denominator over here. So because pi over 2 is kind of a pain to write down, when we write the spherical way, we'll just write it with the i in the denominator.

And this i is actually not-- I mean, it's not a big deal, you know. If you-- in fact, myself, sometimes I will just skip it. But it's kind of nice to know that it's there. And, of course, in the-- the other thing that we'll do commonly, but by now a thing where we're comfortable with that is that we neglect the temporal variation. And then we'll get the phasor.

The last thing that we'll do before moving on to a slightly different topic is-- remember in geometrical optics, we derived a lot of useful things using this paraxial approximation. So the paraxial approximation basically says that-- in this case, the wave originated relatively far away. And you have these spherical wavefronts which are nice spheres near the source. But if you go at a relatively long distance, basically, you cannot tell the difference between these spherical wavefronts and parabolas.

And it turns out that, because the sphere involves this nasty square root, it is much more convenient to represent this as parabolas. So the way you go from the sphere to the parabola, I did it on the slide. And I will do it again. Because this is a derivation I want you to be very familiar with. We've done it a few times. And we'll do it again and again.

You start with the expression for the wave. So it is e to the ikr, right? And there is bells and whistles. There is all the r's. There's the i's. Let me just write this.

OK, and then remember, so we have r. This is the polar coordinate. So it is x plus y plus z squared. And the paraxial approximation-- let me put this back-- in our convention, usually, is that this is the z-axis. So z is the axis of propagation, the optical axis. And then you have the other two axes perpendicular to it.

Typically, we'll denote x as being vertical. OK, and is this now right handed, the way I did it? Yeah, right hand, OK good. All right, so the paraxial approximation then means that z is much bigger than the values of the other two coordinates. Because, basically, I'm limiting myself to operate in this region. that's the meaning of the paraxial approximation.

So what you do then is you do a Taylor expansion on this square root. So you pull z outside. Then you use the Taylor property that says 1 plus small is approximately equal to 1 plus small over 2. And then you can see that r equals z 1 plus x squared plus y squared over z squared times 2, which is also known as z plus-- OK.

So that's the paraxial approximation for the polar variable. And now I can, of course, put it in my expression. So this means that e to the ikr is approximately equal to e to the ikz, e to the ik, x squared plus y squared over 2z. And sometimes we will leave it like this.

Or sometimes, we will remember that k equals 2 pi upon lambda. And then this expression now becomes e to the i 2 pi z upon lambda. e to the i, notice what will happen. The 2s will cancel. A pi will remain. And you'll get-- OK. So all of these are interchangeable expressions that we can use for spherical waves.

Another interesting question-- I omitted something. I omitted 1 over ir. And actually, there's also the amplitude. Let's put the amplitude for completeness. What happens to the ir? Should they also do some paraxial approximation over it?

It turns out it is sufficient to simply approximate it as z. For the part that appears outside of the exponent, we can simply approximate it as z. And the reason is because r, it is actually slowly varying. You know, if you compare-- what is r? Let's see if we can use the-- yeah, so r is this, right? So if you compare-- OK, so basically, you have to pick z equals constant. So I have to pick a plane that is tangential to the sphere at this place.

So I compare r1 to r2, right? So r1 goes all the way to the plane. r2 simply stops here. So think about it this way. Let's say that the error that they make-- when I say z is approximately equal to r, let's say that they make a small error. Let's say that they make an error of 1%. That means that they misrepresent the amplitude of the wave by 1%.

Now, let's look at the term that is inside the exponential, e to the i pi xy, x squared plus y squared over lambda z. If I miss by 1%, that is, if I had e to the 1,000 pi and I miss by 1%, then it means I went to e to the 990 pi, right? So this 1% error actually caused me, how many, 10 full oscillations. That's why you have to keep a better approximation inside the exponential than I can afford to keep outside.

So with the phase, I have to be much more accurate in order to make sure that I don't make a big error in the phase delay. And that's because the cosine varies very rapidly. As you increase r, the cosine is oscillating. But 1 over r is a slowly, what we call a slowly varying function of r. And therefore, we can afford to be sloppier with it, if you wish.

So the paraxial approximation, then, for the spherical wave is just like that with z in the denominator but this additional quadratic term in the exponent. And, of course, this additional quadratic term is the one where all of the action is happening. And this-- we will see this later in the class.

And that's the second important lesson from today, which is that if we see an expression which is quadratic in the Cartesian coordinates, we'll call it a spherical wave. So we went through all these adventures, I guess, to arrive at two very basic waves. One is the kind of wave in whose phasor you have linear terms in the cartesian coordinates. That we'll call the plane wave.

OK, so if you see something like this, it doesn't really matter what these things are, kx, ky, and so on. But because these terms are linear, it is plane. When you arrive at an expression that looks like this, you might have all kinds of junk. And somewhere you have something of the form e to the i pi x squared plus y squared over lambda z, this quadratic means that it is spherical, or, more accurately, the paraxial approximation to a spherical wave. Yes, button.

AUDIENCE: I'm curious what would be the propagation of an evanescent wave?

GEORGE BARBASTATHIS: Let's postpone that for a little bit later. That's a good question. But we don't have the tools to deal with that yet. [INAUDIBLE] is asking-- I wrote this square root here. Here it is.

[INAUDIBLE] is asking, what if I pick kx and ky so that the argument of the square root becomes negative? Therefore, this quantity would become imaginary, right? That is called an evanescent wave. But let's not go into it now. Yeah, we'll come back to it though. It's a valid question. And other questions? Or I should say, any questions that I'm willing to answer? Because that is a good question, but--.

OK, did you come up with any questions about spherical waves or plane waves during the break?

AUDIENCE: OK, here I have one question. So just now we mentioned that when the kx, ky is very small while the kz is very big, we call it plane wave, correct?

GEORGE BARBASTATHIS: Not quite. It is true, actually, what you say. kx and ky will be very small.

AUDIENCE: Yeah, at--

AUDIENCE: [INAUDIBLE]

GEORGE BARBASTATHIS: Yeah.

AUDIENCE: OK, but this one I'm talking about in the k dimension. How about in the space, in the real 3D space.

GEORGE BARBASTATHIS: OK, the paraxial approximation, I did actually in space. So this diagram over here, that was space, x, y, z. But what you say is true in the spherical wave. The reason I'm reluctant to admit what you say is because the spherical wave is not a plane wave. So therefore kx, ky, kz, they vary across the wavefront, right? So it's a bit more complicated what happens.

But yeah, if you took a small portion of the wavefront here, and you plotted kx-- well, this is the direction. If you plotted, kz would be big. And then kx would be very small. So it is true. kx and ky are also very small.

But you have to be careful with a spherical wave. Because, for example, here, we have a different kx and kz, right? So what it means here, kx and ky-- well, what does it mean? Let's not go into Wigner distributions, right?

OK, so let me say what I'm trying to say in plain language. The kx, ky, kz will define them for a plane wave. So a plane wave is this kind of thing, where you have a well-defined wave vector. And the entire wavefront is planar.

Now, what you are saying is that if I have a spherical wavefront like this, locally, I can define normals, right? And I can call those wave vectors as well. And that is actually true. But they're not plane waves.

You can think of them as rays. That's a good approximation. They would be the equivalent of geometrical rays, but not plane waves. But nevertheless, you are correct. They would-- these things are also paraxial. So they would also satisfy kx, ky much less than kz. That is true. Other questions?

I'd like to talk about a fun topic. Today, I was-- I want to do electromagnetics. So I don't want too spend too much time on this. But I'll talk about it, because it's fun and very useful in many contexts.

So in all of this business, we wrote the-- this is, again, the wave equation. But we didn't say anything about the speed of light. So actually, we did a little bit. I mentioned-- I guess this is better than a whiteboard. I have a history of what I've written throughout the lecture.

Yeah, so at some point, we did mention this. We did mention that the index of refraction actually enters in this equation over here. But if you remember diligently from your geometrical optics, the index of refraction also happens to be a function of wavelength. It's not a constant, right? It is a function of wavelength.

So what is means now is that in this equation over here, if I have a single frequency-- now, a single temporal frequency, a single omega-- then everything is fine. Because, sure, for this particular frequency, I can find the index of refraction. And I'm OK.

However, the index of refraction is a function of wavelength. And the wavelength, yes, it is a function of frequency. We do know that this equation is true. But we also said, if you recall, or if you go back to the video of the first lecture when I first presented this equation, I gave you a warning. And I said that this is the dispersion relation. But it is not the only possible dispersion relation. It is possible to have waves where the dispersion relation is different.

OK, first of all, just to make sure that the [INAUDIBLE] stuff about dispersion that we discussed about the glass, it is true. In other words, in glass, yes, the index of refraction is a function of wavelength. So this is a plot that we saw before.

But now, in addition to that, I'm talking about a slightly different type of dispersion. So this dispersion happens, for example, in waveguides. In everything that we'll do so far, we sort of assume that light propagates freely. We may put some lenses or whatever, prisms in the path of the light, that is refractors, and mirrors, and so on. But all of these are relatively large. We never try to confine the light to a very small space.

But you can do that. You can create waveguides. Actually, for visible light, you will never actually use a metal here. You would use a dielectric waveguide. But in microwaves, this is done very commonly.

In microwaves, people trap the wave in a metal tube as if it were water, I suppose, or some kind of a liquid. You just have a hollow tube that is a metal. And then you launch the microwave inside the tube.

And to give you an idea, the wavelength in microwaves is in the order of-- well, as the name suggests, it is the order of-- the frequency's gigahertz. So the wavelength should be between millimeters and centimeters. That is typically-- actually, less, no the wavelength is a little bit less, maybe in the order of millimeters, typical millimeter.

So the size of these tubes might also be on the order of a few millimeters. So now you have a guide that is approximately of size of a few wavelengths. So in that case, it turns this equation, c equals lambda nu, does not hold anymore. It is true for a free space. But it is not true in a waveguide.

The reason it is not true in a waveguide is because the wave must satisfy boundary conditions. If you recall when we gave the wave equation, we said we would need to satisfy an initial condition and boundary conditions. In free space, there is no boundary condition. Actually, there is a boundary condition that the wave must vanish with infinity. OK, fine, but that doesn't have any implication in our calculations.

But, for example, in the case of a metal, you have to force the wave to be 0 on the metal. Because, well, we'll see later in electromagnetics why this is the case. But take my word for it. On the metal, the field must equals 0.

So now if you have a sinusoid in this metal, the sinusoid is in kind of a straitjacket. Because it is sinusoidally varying. But also, it must be over such a period that it vanishes at the edge of the waveguide. So, for example, it can be-- if this is the waveguide, the sinusoid can be like this. That's fine, because it vanishes. It can be like this. That's also fine. It can be like this, and so on. But, for example, can it be like this? No, that wave cannot.

OK, so I will not do the full derivation. And again, it's a little bit-- well, we would have to solve the partial differential equation in a serious way. But it turns out that this straitjacket that we put the light in actually results in a different dispersion relation that looks like this. It is not the c equals lambda nu anymore, but it's this one.

AUDIENCE: I have a question?

GEORGE BARBASTATHIS: Yes?

AUDIENCE: What happens if a is smaller than lambda?

GEORGE BARBASTATHIS: Then the wave will not even get into the waveguide. It will become evanescent. So it will not go in.

AUDIENCE: Thank you.

AUDIENCE: What is shown in the diagram, isn't it a standing wave? Because it's not-- oscillation is-- the way you have drawn--

GEORGE BARBASTATHIS: Yeah, it is a standing wave in the vertical direction.

AUDIENCE: Yeah, but how does it propa--

GEORGE BARBASTATHIS: Well, if the [INAUDIBLE] of it is bouncing back and forth-- another way to think of the waveguide is that the light is-- you can think, actually, in terms of rays. You can think of rays bouncing back and forth. But you can also have the conjugate ray. These are both possible. And the result is a standing wave in this sentence. But you can still have propagation on the horizontal axis. Because there is a component that goes towards the right.

OK, so when you get an equation like this one which relates omega to k-- and, of course, as you can imagine, now there's a great variety of such equations that you could have. This is the equation that happens to be true for the metal waveguide. If you replace it with a different waveguide, for example, if you have dielectrics of different indices here, then this equation changes.

The bottom line is that if you have this equation, then you can plot one of the two. You have one term here that is determined by geometry. a is the size of the guide. And then you have the wave vector and the frequency. And, of course, you have the free velocity of light.

OK, omega is fundamental. As soon as you specify the frequency of the electromagnetic wave, nothing can change that anymore since we're in the linear region. But the question is, what is the wave vector? So the wave vector is really what you are after in this equation.

Omega is given. It is, say, 10 to the 15 Hertz. What is the k? Or in other words, what is the wavelength?

Now, for reasons which will become apparent in a moment, the people who invented these diagrams, they actually plot them kind of the other way around. Instead of omega on the horizontal axis, since omega is given, they put omega on the vertical axis. And the horizontal axis is k.

So the way you read this diagram is-- of course, you solve this equation. And the way you read it is that if I give you a certain omega, then you have to draw a horizontal line. And wherever this horizontal line meets the plot, this-- you read the abscissa and-- I never get this right. Is this the abscissa or the other one? That's the abscissa?

AUDIENCE: [INAUDIBLE]

GEORGE BARBASTATHIS: That's the ordinate? OK. I must come up with a mnemonic to remember. Anyway, so the value of the horizontal axis gives you the wave vector, and hence, the wavelength. Now, interestingly, if you were in free space, this would look like a straight line. Because, well, of course in free space-- I have written this equation many times. I don't feel like writing it again. In free space-- well, let me do it once more.

So in free space, I have that c equals k over 2 pi. That's also known as lambda times-- what am I doing? The other way around-- 2 pi over k, that is also known as lambda, times omega over 2 pi. So in free space, then, you have omega equals ck. This is an equivalent way of writing this equation here. These are basically the same equation.

So in free space, this is a straight line. Now, if you have an equation like this one, you can see that at relatively high frequencies, they approach. So, indeed, all of the-- oh, by the way, what are these curves? So this equation involves an integral parameter. You can plug in-- well, not quite 0, please erase that from your notes. I should not have put 0 here. 0 doesn't make sense. But you can have plus/minus 1, plus/minus 2, and so on. And these are known as modes.

So the index defines the mode that is launched into the waveguide. So each one of these modes, at very high frequencies, it actually kind of approaches the free space mode. But as you go to lower frequencies, you see that it bends away and then eventually hits the ordinate axis. And there's a gap here now. Because there's nothing in this place over here.

So what will happen if you come up, if you use an omega that falls in this range?

AUDIENCE: Evanescent wave.

GEORGE BARBASTATHIS: Correct-- actually, someone asked it before in Boston. Someone asked me before, what happens if you pick a to be smaller than a wavelength? Well, this is what happens. In fact, I wouldn't espouse it exactly like this. A more accurate way to say it is what happens if you pick an omega which is very small?

So therefore, given the value of a and given the value of omega, you cannot find the k anymore. The square root becomes imaginary. Well, if that happens, the wave does not enter the waveguide. It becomes evanescent, as we say.

So basically, it may penetrate a little bit for a couple of wavelengths. And then it will exponentially decay. Now, if you-- as you go progressively-- so, if you launch at very low frequency, the wave does not enter.

As you start increasing the frequency now, you will see that the intersection only meets one mode. So basically, at the range of frequencies-- I don't know what it is here. This looks like 3 or something. So between-- so the way you read this axis is you read the ordinate. And you multiply by three times 10 to the 14. So, for example, 5 actually means 15 times 10 to the 14 Hertz.

OK, so then between 3, the value f3, which is 9 times 10 to the 14, and the value, whatever it is, 7, you only launch one mode. As you go up in frequency, if your intersection here meets more than one line, then actually you can have multiple modes in this guide.

Anyway, so these modes, these little wiggles that I drew over here, the first mode is the field shape that looks like this, only half of the period of the sinusoid. The next mode, you allow one full period. The next mode, you allow 1 1/2 period, then so on and so forth.

So as you go higher in frequency, it means that the wave-- basically, what it means is that the waveguide now-- or, I should say the wavelength of the wave becomes progressively smaller than the size of the waveguide. So therefore, you can pack more modes into the guide.

So I only plotted three here. Of course, there's infinite. But anyway, as you go up in frequency, you have the possibility to excite more than one mode. Now, how do you excite the modes? Now, that depends, of course, on your initial conditions, right?

If you create a distribution in the entrance of the waveguide that looks like this, then you will excite the first mode. If you create a distribution that looks like this, well, if the second mode is allowable, that is if you are above the green line, then you will excite it. If you are below the green line, you will probably excite nothing. Because the first mode cannot be excited this way.

The reason I'm telling you this is not because I want to do waveguide theory, but because I but because I want to do something that's called a beat. And a beat is an extremely useful concept, both in the space domain and in the time domain.

So in the time domain, the beat is actually-- it is a very simple thing. It is defined as the superposition of two sinusoids of slightly different frequency. So we said this many times, that if the wave is linear, therefore a superposition is allowed. If I have two solutions to the wave equation. I take the sum. It is still a solution.

So now what you can see here very easily-- that's a diagram that I stole from the textbook. Because the two waves have slightly different frequency, they will misalign. So you can, for example, start at the point where the waves are exactly aligned. So they kind of oscillate together. When one is positive, the other is positive.

But because the period is different, or the frequency is different, a little bit later, they will kind of separate. And now, over here, you can see that this is the opposite. When one of them is positive, the other is negative.

And if you also happen to judiciously pick them to have the same amplitude, it means that here when, as we say, they oscillate out of phase, it means that the wave amplitude overall becomes very small. And so-- and of course, in the other case, where they are in phase, the amplitude becomes twice the amplitude of one.

So this is sort of a static picture from the book. I also made a movie out of it. So here you see the beat propagating. OK, so now, what is really interesting about this-- I'm going to play it again.

If you look at it carefully-- it takes a-- let me play it once more. Actually, I think I can just-- if you look at it carefully, you will see that there is two different velocities here. There's two things happening. One is you have this envelope that is moving to the right. But also, inside the envelope, you have small fringes, wiggles that are also moving to the right.

Now, I told you, of course, but if you look at it once again more carefully, you will see that the two are moving with different speeds. Can you see it? The wiggles are moving faster than the envelope. And you can see that the wiggles are moving faster than the envelop if you look here near the vicinity of the null. You will see that there's a little bit of a-- once in a while, there's a pulse, like a little pulse. That means that the fringe are passing by.

So the reason this is happening-- and now I will not derive it on the paper. Because it will end a little bit late. But the reason this is happening is because if you take the superposition of the two sinusoids, now we have to be a little bit careful. Because they have different frequencies.

So if we use phasor-- we could use phasors. But we might get a little bit confused. So for that reason, I just wrote it in the real space simply with cosines. So you do the same thing with it before. You take the sum of the two cosines. You apply the trigonometric identity to write it as a product. And then you find that the product contains two terms.

One of them-- actually, by the way, the-- oh, shoot. There's one more error in the notes. There should be a cosine between the two parentheses. There shouldn't be a cosine. I better write down the correct expression.

OK, where kc equals k1 plus k2 over 2, and the same for omega c, and km equals k1 minus k2 over 2, so one of them is the average. The other is the difference. So now each one of those, each one of these cosines-- OK, there's a missing cosine here. But imagine that there was a cosine. Each one of those is a traveling wave.

So basically, what you see now is the product of the two traveling waves. And you can see very clearly it's a product here. This is-- I have one sinusoid of relatively low spatial period and another sinusoid of relatively fast spatial period. And I multiply them. But they're both propagating. Because they're both of the form kz minus omega t. But they're propagating with different speeds. Because the speed of the-- oh, and-- so, what are the names?

Well, the fast sinusoid here is called the carrier. Those of you who are electrical engineers are very familiar with those. The carrier wave is-- in radiowaves, for example, it is the frequency upon which we'll modulate the signal. And the slow sinusoid, that's called the modulation.

And-- I lost my train of thought. All right, but each one of those is a propagating wave. So therefore, their velocities would be equal to the-- it will depend on which wave I'm talking about. So, for example, the velocity of the carrier will equal omega of the carrier divided by the k of the carrier. And the velocity of the modulation will equal omega of the modulation over k of the modulation.

OK, and so this is not magic. We'll just apply the theory that we learned. But now it is the two waves that are propagating. And, of course, in general, the two velocities are different. Because the two rations over here, they have no reason to be the same. In fact, it is quite the opposite. Most of the time they are different.

So this velocity, I actually used a slightly different notation. The velocity of the carrier is called the phase velocity. And the velocity of the modulation is called the group velocity.

So the group velocity, it equals, as you can see, the ratio of the difference of omega to the ratio of the difference of k. So if you imagine a beat-- this is called a beat, by the way. I think I said it before. So if you imagine a beat where the two frequencies of the two numbers are very close, then it will become the derivative.

So the group velocity is basically the derivative of the-- whatever omega is as a function of k, if you take the derivative of that expression, you get the group velocity. The phase velocity is the same velocity we've been talking about so far while we're doing waves. So there's nothing new about the phase velocity. It is still the ratio of omega over k.

But the group velocity is the derivative. And the reason now-- ah, now the reason is revealed.

AUDIENCE: George?

GEORGE BARBASTATHIS: There's a question?

AUDIENCE: Yeah, why did the derivative come about?

GEORGE BARBASTATHIS: Yeah, you're right. I kind of blasted through. So if you look at the group velocity, it equals omega of the modulation over k of the modulation. And this equals omega 1 minus omega 2 over 2 over k1 minus k2 over 2. That is omega 1 minus omega 2 over k1 minus k2. And if you take omega one very close but not quite the same and k1 very close but not quite the same, then v sub g will become d omega dk.

So that is the reason people put omega on the vertical axis of it here. Because the group velocity, then, is simply obtained as the slope of the dispersion diagram. And that's fascinating, if you think about it. Because it suggests some weird things.

For example, it suggests that if I could somehow create this pendulum diagram that looks like this, what is the strange thing about this? That's right, it has a negative group philosophy, which means that in the wave, the carrier and the modulation, they go in opposite directions. The phase velocity is going to the left. The group velocity is going to the right.

I meant to make a movie of this to show you. It actually looks a little bit weird. But I didn't have time to do it. And I'll show you one next time. But now, of course, it demands-- it's interesting. How can you do something like this? It turns out to be an extremely hot topic in optics research.

Now we know how to design, theoretically, structures. By the way, natural materials, they don't have this property. So you have to try really hard to make this property happen, for example, by patterning a dielectric. If you create patterns in a certain way, then you can cause-- you can make group, I mean dispersion diagrams that look like this.

And, of course, to fabricate them-- you know, it is one thing to design them and simulate them and one thing to fabricate them. So these are all very hot topics in optics research. So that's why I-- and also, beat is sort of a very important topic. So that's why I spend some time talking about.

And one of the problems-- again, in the new homework that I posted, one of the problems I'll get you to play a little bit with this dispersion relation so you can get a feel for it. Unfortunately, in this class, we do mostly optics in the space domain. This, what I'm discussing here is kind of time domain, right? Because we're talking about modulations, and carriers, and so on.

This class is mostly about space domain. So we don't spend too much time on this topic. But anyway, it is sort of important to know. That's why I covered it.

With that, unless there is any questions, I will move on to electromagnetics. Questions?

AUDIENCE: George, one question. So just now, we show that carrier and the modulation wave actually have different velocity. But in everyday life, common sense for [INAUDIBLE], for example, the red light and the yellow light travel at the same speed. I mean that how different frequency have different wavelengths. But the product of these two is constant.

GEORGE BARBASTATHIS: Not at all, for example, we know that is not true in glass. That is the reason why a prism analyzes white light. It is because red and yellow, they travel with different velocities in glass. So any medium that is dispersive, they have different velocities.

AUDIENCE: So only in a vacuum they have the same velocity.

GEORGE BARBASTATHIS: In vacuum, yes. So the linear dispersion relation that I showed before, that is for vacuum. Actually, it is also linear in a medium, but with a different slope. I should have said that.

COLIN SHEPPARD: I was just going to point out that, actually, you show in the phase velocity there for the free space, not for the--

GEORGE BARBASTATHIS: For free space, yeah.

COLIN SHEPPARD: Yeah, but the point where you've measured the group velocity, that you've labeled omega c kc, if you drew a line from the origin to that point, the slope of that would give the phase velocity.

GEORGE BARBASTATHIS: You are correct, actually. Yes, yes, yes, thank you, yeah. Yes, that's very true. The phase velocity is omega over k. So it should be that one. Yes, so I need to correct the slides. Thank you.

Yeah, this is the phase velocity of free space.

COLIN SHEPPARD: And it's greater than the speed of light.

GEORGE BARBASTATHIS: That's right. How could that happen, by the way? Because this is a phase velocity, right? It can do anything it wants. It can go faster than the speed of light. Because it carries no information. It is simply an oscillation.

The group velocity, as you will see actually-- in the homework, I get you to derive the group velocity for this case. You will see from the answer that the group velocity is actually smaller than the speed of light. You will see that the group velocity for this diagram, it is given by something like c square root 1 minus something. So the group velocity is always less than c.

OK, so I will post the corrected version. Any other questions?

AUDIENCE: So would a patterned metamaterial basically be the same thing as a special case of a gradient index lens, a GRIN?

GEORGE BARBASTATHIS: I don't know what you mean by a special case. I mean, they have quite different properties.

AUDIENCE: Well, I mean you can maybe imagine the idea of the gradient in the dielectric material where the gradient and the index of refraction sort of creating a negative group velocity or some case where you're actually diverting light away from what you're trying to image.

GEORGE BARBASTATHIS: I don't think with a simple gradient index you can-- OK, it depends on what you mean by gradient. If it is a slowly varying gradient like the GRIN optics usually, I don't think you can create sufficient dispersion to turn the light around.

If you have subwavelength patterns, then yes, because the resulting evanescent waves that can adapt in ways that-- yeah, I mean, this is the whole area-- yeah, so metamaterials. But those, usually the gradients. Are huge, right? Because within a space of less than the free space wavelength, you might have variation of index from 3 to 1, and then back to 3, and so on. So those are huge gradients.

AUDIENCE: You say that group velocity is the velocity with which the information is traveling?

GEORGE BARBASTATHIS: Well, that's one interpretation, yeah. This has been challenged too. Because people saw that group velocity in some cases can also exceed the speed of light. So therefore, even that is questionable.

But anyway, in most cases, group velocity is the velocity at which the modulation travels. So the modulation usually denotes information. If you exceed the group-- if the group velocity, if you manage to make it exceed the speed of light, which has been proposed, then it means that the modulation actually does not carry information. It is something that you artificially put in there.

AUDIENCE: So why we means-- what is the advantage of negative group velocity means opposite direction information travel?

GEORGE BARBASTATHIS: Well, it can make you a pretty good career and get you papers published in Nature. No, no, I mean, there's-- I'm joking. But [INAUDIBLE] it is interesting. It is interesting physics and very counter-intuitive. That is cool, right?

But also, the people who work on it, they have shown mathematically that you can get focusing of light that is sometimes tighter than traditional refractive-- you can get interesting kinds of dispersion for pulse shaping, very narrow-- you know, you can do some clever things with it. You can slow down light, which is-- slow down light means that-- if you go back to this diagram, you see that as you go towards lower frequencies, the group velocity becomes progressively smaller.

In fact, over here, well, if there's a point where it is actually 0-- so what does it mean? Well, it means that if you really had the light pulse, not the fringe of the light, not the carrier wave, but the envelope, the modulation, if you were to run into this frequency or very close to this frequency, that could stop, right? It would actually-- or it would move very, very slowly.

That's the principle of slow light, which actually also has potential applications. For example, you can store data. If you can slow light down, you can create an optical buffer. You can wait for something else to happen and then launch the wave again. So there is things that you can do with this in addition to interesting physics. It's a fascinating topic, this one. So I'm glad you're asking questions.

AUDIENCE: Yeah, and if you go for a higher frequency, we can see that we can excite many modes in the waveguide. So all the waveguides will-- it means all the modes will have different velocity?

GEORGE BARBASTATHIS: That's right.

AUDIENCE: So--

GEORGE BARBASTATHIS: Which is bad, right? Because it means that if your information sample split between those two modes, then they will separate. So they will arrive at different times. That is called modal dispersion. And it is a very serious problem in telecommunications. It means that you get distortion in the signal. Was that your question, or, I'm sorry, I kind of answered your question before you even asked it.

OK, here they are. I'm sure all of you, at some point or another, at some moment, probably a moment of a nightmare, you saw them, right? These are Maxwell's equations. So you can read them in the notes. I have the two slides preceding these where I sort drew them in the traditional way. And, of course, the book also has an explanation.

But I would like to ask you if you remember from your physics 802, those of you who are at MIT, or whatever it was, your basic electromagnetics class, do you remember each one of those, what is it telling us, and why does it look the way it does?

Let's start with the first one. Does anybody remember? What does the first one do? So actually, these are-- you know, these are the same equations. One on the left hand side is written in integral form. The right hand side is written in differential form.

Does anybody want to volunteer and say, for example, the first row? It is the same equation. So what does it tell us?

AUDIENCE: The-- you want to say?

GEORGE BARBASTATHIS: Toss a coin.

AUDIENCE: The first one says, the integral form says that the total electric flux passing through a closed surface is equal to the total charge enclosed inside that surface. And the equivalent differential form says that the total amount of electric field emerging out of a point is equal to the amount of charge at that point.

GEORGE BARBASTATHIS: That's correct. So basically, it is telling you that if you have a charge, then you have field lines radiating out of that charge. And if you enclose this charge with a surface and you compute the integral of the-- I don't know why this is a cross. By the way, there's another mistake. This should have been a dot, not a cross. But anyway, if you compute this integral, it is determined by the total amount of charge. OK, we can go back to the notes now. [INAUDIBLE].

So this is what your colleague just said. You have a charge. Then you have field lines radiating out. And then if you take all of these lines, dot product them with the elemental surface, and then integrate, then you get the amount of charge. And why? Why should it be so?

AUDIENCE: Principle of charge conservation.

GEORGE BARBASTATHIS: OK, but why do the field lines emanate out of the charges?

AUDIENCE: It goes from positive to-- I mean, it originates from positive. And we can say that field line represents the direction of the force on a-- you need positive charge.

GEORGE BARBASTATHIS: OK, that's what I was looking for. So the field lines are basically-- they tell you, exactly like you said, what is the direction that the small positive charge-- let me repeat. What is the direction of the force that the small positive charge would feel if you were to place it at any given position in that space? This is what the electric field says.

And then there is a relationship called Coulomb's law, which for two given charges, it tells you the force. So if you have a single charge here. And then you apply Coulomb's law, then you get these radiating lines. So then Gauss' law is basically nothing other than Coulomb's law.

And yes, it also has to do with charge conservation. Because if you somehow cancel all the charges inside this volume, then the integral will vanish. Because it means that you don't have any net electric-- actually, you can have electric charge radiating. But the integral will vanish. In other words, if you have the charges surrounding, somehow, this space, the space cannot pull them all inside.

OK, what about-- you have a question? What about the second one?

AUDIENCE: That means, [INAUDIBLE] line integral is 0 means b is a closed line.

GEORGE BARBASTATHIS: B is a closed line. And b is a closed line why?

AUDIENCE: It means that there is no magnetic charge.

GEORGE BARBASTATHIS: That's right. Magnets only come as dipoles. You cannot have an isolated north pole, for example, of a magnet. So actually, this is the same equation. In fact, they're also known by the same name. They're known as Gauss' law. This type of equation, except there's no such thing as magnetic charge.

What about the next one? This one looks awful, del cross c equals db dt. So let's look at the integral. Usually the differential form is mathematically better but not very insightful physically. So one gets more information by looking at the integral forms, even though the integral forms look really frightening.

But anyway, the third equation is known as Faraday's law. And it says that if you have a viable magnetic field inside a wire, then a potential will develop across this wire. And do you know of any places where this is used? Button. Even I cannot hear you. Imagine the people in Boston. Not working? Oh, maybe use someone else's. OK, [INAUDIBLE], you're on.

AUDIENCE: It's induction motors and--

GEORGE BARBASTATHIS: Yeah, voice coils and generators, they use the same principle. So, for example, if you-- people do this in hydroelectric plants. You have a giant magnet. And you somehow move a coil inside that magnet. And, of course, you have to arrange it properly. Then the coil will develop a voltage. So that's the principle of-- of course, you have to move the coil, for example, by pouring water over it, or by steam, or whatever. But anyway, that's the principle of a generator.

OK, so and the last equation that--

AUDIENCE: George, just as an aside, how does a nuclear power plant generate energy, electricity? Does it rotate something?

GEORGE BARBASTATHIS: Actually, I don't know.

COLIN SHEPPARD: I think it just generates steam, and then you--

AUDIENCE: Oh, OK.

GEORGE BARBASTATHIS: Then you move a coil, I guess, yeah.

AUDIENCE: Right, right, OK.

GEORGE BARBASTATHIS: And then the last one is actually the most interesting one. Because this law actually was involved with a breakthrough in electromagnetics. It can be known for a long time that if you have a current, a magnetic field develops around that current. That is actually Lorentz-- it is known as Lorentz force.

If you have a moving charge, and then you put a little magnet nearby, then the-- what am I saying. Anyway, if you have a moving charge, and if you have a current, it creates a magnetic field around it. That has been known for a long time.

But what was not known and was noticed by Maxwell was that if you have a capacitor-- well, in a capacitor, you can not have a current per se. Because the capacitor, well, inside there's a dielectric. So there can be no current.

But you can have a variable charge density in the capacitor plate. So what I'm trying to say is that if you apply a-- think of charge in a capacitor. When you charge a capacitor, the capacitor, at the beginning, the voltage across the capacitor is 0. And by the time you have totally charged it, the voltage has reached some value.

Well, while the voltage is changing, there is actually charge flow into the capacitor. There is a positive charge going to the positive electrode, then negative leaving from the opposite electrode. And at the same time, what is happening, the electric field inside the capacitor grows. Because as you get more charge accumulating in the capacitor plates, then the electric field of the capacitor is changing. So in a sense, you can think of it as a current as well.

And what Maxwell-- he guessed it, actually. He didn't observe it. But he guessed it. He guessed that this kind of variable electric field should also generate a magnetic field. And the only reason he guessed it is by symmetry from the previous law.

What does that say? It says that if you have a variable magnetic flux, you generate an electrical potential. Well, he looked at this equation. And he said, wait a minute. It's fine that if I have a moving charge, I have a magnetic field. OK, fine. But if also I have a moving-- if I have a variable potential, then why should I not generate a magnetic field? I should.

And it turned out to be true. I mean-- Maxwell, by the way, he was-- do you know what was his day job? James Clerk Maxwell, he was a fluid mechanic. He was [INAUDIBLE]. And that's why the terminology-- not the terminology, but the notation in electromagnetic magnetics is actually very similar to the fluid mechanics notation, rho, and epsilon, and all that stuff.

I mean, they're not the same things, but the-- and also, these theorems, the mathematical theorems, they come from fluid mechanics. They apply to incomprehensible flows. For example, the Gauss theorem, it applies to incomprehensible flow in the pressure potential.

Anyway, so Maxwell did that first. So he guessed that you should have this form in here. And then he actually wrote down all of these equations in a coordinated form. And the reason this was a very valuable thing, and I guess we're all employed because of this [INAUDIBLE], is because if you manipulate Maxwell's equations-- and I will not do it now, because I ran out of time. But if you manipulate them, you can actually show that you arrive at the wave equation for the electric or the magnetic field. I did it here for the electric field. But you can also derive the same equation for the magnetic field.

So therefore, what this means is that if you have electric and magnetic fields which change in time, Maxwell's equations tell you that they change in a coordinated way. And the way they change is as an electromagnetic wave.

I think it's time to quit. In fact, we're already five minutes late, so.