GEORGE BARBASTATHIS: So a little bit of housekeeping before we continue. First of all, you may have noticed that in the reading assignments, I have started Boston from Goodman's book. So there is some pros and cons about this. Goodman is very good if you are an engineer, especially electrical or mechanical engineer. Because then you are very used to thinking about systems, block diagrams, transforms, and so on. So it is very nice living this way. But it's a little bit mathematical.

Hecht is more on the physics side. So I feel Hecht is written for junior or sophomore physics students. And, of course, they're very nicely complimentary. The only real downside is that they use different notation. So if you tried Stanley from both Hecht and Goodman, you have to be a little bit careful to keep their notation-- I mean, the notation is not consistent, but you have to keep yourself from getting confused by the inconsistent notation.

Nevertheless, the diagrams are, of course, consistent because they both calculate the same. Fresnel, the fraction pattern is the same from [INAUDIBLE] pattern and so on. But you have to be a little bit mindful of the coordinates. For example, they may be using different symbols. Anyway, I highly recommend that you're starting from both books. Hecht also has a much more intuitive explanations, and many more figures, and so on. But Goodman is more rigorous, and also better suited to an engineer's way of thinking. So that's why I use both textbooks.

By now, it's actually closer to Goodman from this point on. So that's an additional benefit. Anyway, the reading assignments are from both books. If you decide to follow just one book. For example, either Hecht by itself or Goodman by itself. You don't miss significantly, you can follow either book. But I think until you get the complete picture, if you follow both books and in the way they serve to reinforce each other.

Anyway, so that's the study about the textbooks. A little bit more housekeeping. I have posted the slightly revised version of Monday's notes. One minor correction, if you look at this. This is the very last slide from Monday. There was an error, at least one that they found. In the expression for the Fourier coefficient c sub q, it was the function sinc of 2 over 2. This is correct. On Monday, there was an extra pi inside the argument of the same function. That pi should not have been there, so I've removed it. We will see today later the derivation of this expression, actually, a similar expression. So hopefully, that will clarify matters.

The thing I did compare to Monday. Piper reminded me on Monday, when he discussed the grading, about the dispersion. So if you look at the-- go back a little bit to the expressions, or the diffraction angle from a grating. OK, that's a good one. OK, so last week, we focused on a discussion about the effect of the period. So we said that if you make the period smaller, the diffraction order is spread out more, so the diffraction angle is inversely proportional to the period. Actually, the sine of the diffraction angle.

We didn't say anything about wavelength. So, of course, the wavelength appears in the numerator there, which means that if you have light of multiple colors, then longer wavelengths will focus at the longer angle. So this is what you see on the last slide that they posted today in the revised notes. So I have a grating here which is illuminated by white light.

So, of course, white light is composed of a broad spectrum of colors ranging from somewhere in the infinite to somewhere in the ultraviolet. Anyway, let's take the discussion. Let's keep it to the visible wavelength. So, of course, the red wavelength has a longer color. I'm sorry, the red color has a longer wavelength. So therefore, the red color is diffracted at the larger angle than the blue.

So this phenomenon is called dispersion, very similar to the grating dispersion. But it is referred to as anomalous, because it is the opposite of the grating. I'm sorry, it is the opposite of a prism. What am I saying? Let me start over. So in the case of a prism, as well as in the case of a grating, the phenomenon of analysis of white light, where white light after propagating through the element gets decomposed into its color component. And each one color component propagates at the different angle. This phenomenon is referred to as dispersion.

If the two are different in the sense that in the case of a grating, the blue light is refracted at the smaller angle than the red light. That is called the normal dispersion. In the case of a prism, it is not diffraction anymore. It is refraction. And we saw why it happens. This has to do with the dependence of the index of refraction of glass on wavelength. And the dependence is such that the blue light is refracted at the larger angle, in the case of a prism. That is called normal dispersion, as opposed to the anomalous.

Now, there is nothing anomalous about the grating, I suppose. Historically, this had to do with the fact that people observed this phenomenon with prisms first, so they call it normal dispersion. Then they noticed that the grating does the opposite thing, so they said, well, this is abnormal. They called it anomalous. Anomalous in Greek means abnormal. Anyway, so this is what I wanted to add to Monday's lecture. Are there any questions about gratings?

Actually, between today and next Wednesday, we will cover the basics of diffraction theory. And the rest of the class will be basically applications on what we cover in these three lectures. So needless to say, these three lectures are very, very important. So let us start with a little observation on our Fresnel propagation formula.

So to remind you very briefly, today, I will be using the whiteboard a lot. But the equations that I write are all of them, either in the notes or in the textbooks. So feel free to copy them if you like, but you don't have to. It may be better. It's up to you whether you want to copy the derivations or not. But there will be a lot of them. Probably by the end of the lecture, your wrist may be a little bit tired.

OK. So to remind you, we did this about a week ago. We said that if you have a complex field at the plane xy, and then you propagate by a distance, z. Then at the output plane x prime, y prime, the field is given by this convolution integral, which I will rewrite here. So the field at the output after propagating by distance z equals some constants-- and we'll spend some time talking about these constants-- times what we'll call the convolution integral.

So the convolution integral is really written something like this, e to the i pi. OK, so this is the integral within the Fresnel and scalar approximations. Then what we can do is we can-- let's take this exponent, and expand it. So I will only do it for the x case. That's not rocket science. They've just expanded the binomial.

Now, let me remind you. So the x prime coordinate is at the output plane. And also, you can see that it is not participating in the integration. The integration is with respect to x. So the x prime part will actually be thrown out of the integral. And then I have this remaining part. So the next thing I'm going to do is I'm going to assume that the input field is finite.

So finite means that the field, if you look at the input field here. The field is non-zero in a relatively small region near the optical axis. But then away from the optical axis, the field becomes zero. That is a reasonable assumption, because most objects that we're going to create in real life are finite. We've said this before, I think. We deal with things like plane waves and spherical waves in this class, which are infinite. But these are idealizations that we use in order to make the math simpler.

In this case, let's use actually a real life assumption, which is that the object is finite. And since the object is finite, x will be confined to relatively small values. So what I will do now is I will look at this expression, x prime over lambda z. This is this part of the exponent. And what I will do now is I will allow z to become very large.

So what does this mean? It means that I start propagating further and further and further away from the transparency. Of course, x is limited by the size of the input field. So x square is also limited, but z grows as I propagate further away. So there will come a point where x square maximum will become less than lambda z. That is, the maximum value of this fraction over here will become less than one.

And in fact, if I keep propagating further and further, this term will actually grow less and less and less. OK, if that is the case, then the coefficient-- then this term over here will become negligible, which means that I can drop it. OK, so this is what is written in your transparency over here. Basically, I actually did this for both x and y. I pulled out the x prime, y prime. And I'm left with this expression over here.

So let me also do it on my whiteboard here. So what I'm saying is that the g out. In the whiteboard, I will not write the y-coordinates, because it is too much. Well, maybe I write them to avoid confusion. OK, so this will now take this form. This got pulled out of the integral because they don't participate in the integration. And then what I have inside looks like this.

And what I was saying is that if I let z become long enough, then it can basically neglect those two terms. Now, why do I neglect only these two terms, and not the products of y over y times y prime and x times x prime. Why should they keep those two terms? And, of course, I forgot something in both cases. In here, I should have g sub in of x comma y, and the same in here. OK, you see what I did? Over here, I need to put the input field, and the same here. So why am I keeping this term, and I am not dropping it? If I drop it, it's very easy, right? If I drop it altogether, then what I will get is the integral of g sub in. It's like the average of the input. Why am I not doing that?

AUDIENCE: If you remove the cross term, then we actually neglect the complete spatial variation of the input field, and it becomes like a pointed [INAUDIBLE].

GEORGE BARBASTATHIS: OK. So I would actually disagree with that. If I neglect it, I will get that g sub out of x comma y prime equals g sub in of x comma y dx, dy. So this is a constant, right? It is like the average of g sub in. So I haven't really neglected it. I have said that it gets averaged out at the output. And by the way, that's wrong. That's not true, right? This is a wrong result.

My question is why is that wrong? Why do I have to keep the entire formula, z out of x prime, y prime, z. Let me write it properly now. So the question is why can I neglect the quadratics, but I must keep the cross terms? Actually a very simple answer. All you have to do is look at it.

AUDIENCE: Does it mean you are restricting the output field to a limited area?

GEORGE BARBASTATHIS: That's right. If you look at these cross terms, they actually depend on the output coordinate, x prime. I haven't made any assumptions about x prime. x prime can be as large as I want. So this term can actually dominate the quadratic term. That's the point. This term, I can control by making the input transparency relatively small, and making z arbitrarily large.

However, this term has the x prime in it. And x prime can be actually very large itself. So I can never neglect this term. That's the point of this argument. Everybody clear on that? So if I indeed neglect this term, then what I will get is that g sub out-- I think this marker is dead, so I'll move on to the next one. It has some terms which I will neglect, but it is proportional to a quantity that looks like this.

Which I can also rewrite as. I didn't do anything. All I did was I-- these parameters here, if you look at this coefficient. It does not depend on the variable of integration. Therefore, I can call it something. I call it u. And this integral now. It is probably familiar to you. If it were in one dimension, it would be immediately familiar. It is a Fourier transform. But it is in two dimensions, so it appears a little bit more complicated with two variables. But actually, well, it is a convention to write-- what happened here-- to write Fourier transforms as uppercase.

So this is a g sub in of u comma v. The Fourier transform computed at frequencies. Something magical happened if I let the field propagate at a relatively long distance. They field that I get at that output plane actually equals approximately the Fourier transform of the input field, which is interesting. And we'll see a lot of ramifications of that in the next few hours.

OK, let's see this now in a calculation. Actually, Piper solved it in practice, in real life. Last time at the demo, you actually saw this kind of thing happening. What I will show now is some movies which are also posted on the website, so you can go back and produce them. So in the movie, you will actually see the Fourier transform slowly developing. In this case, it is a rectangular aperture. So the Fourier transform is very easy to compute.

Of course, in the movie, there's no Fourier transform. In the movie, all I did is I convolved with a Fresnel propagation coordinate for progressively longer distances. And then I put all those frames together, and I made the movie. But you will see that as z grows larger, the aperture develops a small oscillation. But then eventually, it develops this pattern that well, it is called the sinc function. And we'll see in a moment that this is actually the special Fourier transform of that function. Let me play that once again.

If you notice carefully, you will see that you start with a nice, clear, sharp aperture. As we will propagate, first, we'll see some diffraction ringing. Professor Sheppard described in detail last time. But then this ringing slowly gives away to this cross like looking pattern. So you can see it here. It has a very distinct pattern. It has a central lobe, and then side lobes expanding along the x and y dimension.

OK, so what is this now, and how did it come about? This function, the aperture that I started with. I can describe it mathematically as g sub in of x comma y equals 1 if-- basically, let me go back. So this function equals 1 if x and y are within this rectangle. Mathematically, we can describe this as x less than the size of the aperture, and y also less than the size of the aperture. Generally, it may be a rectangle, not a square. So use the different variables. And it is zero otherwise.

OK, so it is convenient to define a function. And actually, I goofed. If the size of the aperture is x over naught, then the value of the function is 1 for x less than x0. So I will explain this in a second. OK, so if I defined the function rect this way. So that's the function. Sometimes, it's also known as a boxcar. And so this would be rect of x.

And if I went to plot rect of x over x0, then if I substitute x over x0 0, 0 here. Then it is 1 if x over x0 is less than 1/2. So basically, this extends from minus x0 over 2 to x0 over 2. And it equals the value 1 over there. And therefore, the total size of the boxcar is x0 as advertised.

OK, and, of course, if I define the rect function this way, then my original function, g sub in. I can simply write it as was done here, as a product. So now, what is the Fourier transform of that product? Let me write it down to the Fourier transform definition. OK, that's the Fourier transform definition.

The first thing that I notice in this case, which is very convenient, is that the integral is separable. I can write it, really, as a product of two integrals. One of them is in the x dimension. And the other looks very similar in the y dimension. This is not always the case. Many, or actually, most 2D functions are not like that. But in this case, we're lucky. Means you don't have to do the whole thing. We can just do the integral for one coordinate, and then we immediately have the answer for the other coordinate as well.

OK, so let's write it then. So I will just write the one dimensional integral. OK. So now, let's do one more simplification. I will assume that x0 is unit, is unity. We'll come back and rectify this one. But for now, I will just assume it this way to make my life a little bit easier. So if that is the case, if x0 is one. Then [INAUDIBLE] what is the function rect of x over one. It is this one over here. So it is 0 outside, and it is only 1 between x equals minus 1/2 and 1/2. So in that case, then, the integral. That's the integral.

Now, there's a simple one to calculate. I don't know how I picked up a naught here. This would not be no naught. Drop this one. Just e to the i to pi u, right? Nothing here. So I have u times 1/2 minus e to the i 2 pi, u minus 1/2. And these two minuses cancel. And if I flip this around, it is actually the definition of a sine. So this equals 1 over minus i 2 pi u0, times minus 2i sine of what was here. The twos also cancel. And I get pi u.

So finally, after dropping the remaining constants. I still got a u0. There's no u0. For some reason, my brain puts a naught there. There shouldn't be. So after finally canceling whatever is left, I get sine of pi u over pi u. OK, so that function, by definition, is called a sinc. And I'll jump ahead in the naught a little bit. You can also look it up in the textbook.

The sinc function looks like this. It is in page-- I forget which page, between page 12 and page 14 of the Goodman textbook. This is the sinc function. One argument equals 0. 2 and it has a peak and then it kind of oscillates, but the amplitude of its oscillation drops inversely proportional to the argument. So the oscillation comes from the sine. The inversely proportional comes from the u in the denominator.

So this may be a little bit boring for you. For those of you who have taken signal processing, you're probably ready to go to sleep now. For reasons of completeness, we have to do it, to go through it. Then I will not compute too many Fourier integrals. But in any case, if you are to compute one Fourier transform, that's the one to compute.

So that's it, then. This is also the definition of the original rect function that we had. And its Fourier transform is the sinc function. Now, we're not done yet, because I made one more simplifying assumption. I said that x0 equals 1. So what do we do about this x0? Well, does anybody know what I can do about this x0?

OK what I'll do is a change of variable. And I will do it in the general case. Let's say that they have a G of u equals the Fourier transform of some general function, g of x. So this is then the Fourier transform of g of x. What is the Fourier transform of a scale version of g of x? Well, that would be something like this. It would be from minus infinity to infinity, g of ax, e to the minus i two pi ux dx.

And to get rid of this ugly thing here, I can make a change of coordinates. For example, let's say that c equals ax. Then this means that dx e equals a dx. And I can write the integral. So they actually become dx e over a. I pick up 1 over a out here. Nothing happens to infinity. It remains infinity. This would be g of c, e to the minus i two pi. Now, x is also c upon a, big C.

I haven't cheated. This is the transformed-- the integral. So the 1 over a basically keeps me honest here. Makes sure that the area of the differential is preserved. It's also known as a Jacobian. But anyway, that's what it is. And I can do one more little manipulation here. I can rewrite it like this. And we can recognize now that this integral-- let me see if I can fit them both on the screen. OK, so recognize that this integral is the same as this integral, except with a different variable, with a different argument. So therefore, what I derived here is one over a G of, u over a.

OK, so this is a property of Fourier transforms known as the scaling theorem. Or sometimes, people call it the similarity theorem. And let's see how we can apply it to the question at hand. We derived that the rectangle function. If you Fourier transform it, you get the sinc function.

OK, what I really wanted to get is the Fourier transform of a rectangle, which has a size, x0. Now let me write down the scaling theorem. It says that g of ax Fourier transforms to 1 over a G of u over a. So in this case, a is identical to 1 over x0. So therefore, the Fourier transform will be x0, sinc of x0 times u.

And this is intuitively satisfying because the units here inside the sink. The units are naught. x0 has dimensions of space, meters. u is a frequency, so it has in dimensions of inverse meters. So therefore, what I have inside the argument has no dimensions at all, which is, of course, of the way it should be. OK, so this is them.

So [INAUDIBLE] for one last time. This is how we obtain this function with the central lobe. But the side lobe is actually not quite the sinc function itself, because I'm blocking the intensity here. It is actually sinc squared. But anyway, this is where this came from. So, of course, if you multiply the two dimensions, you get a sinc in the x dimension, and sinc in the y dimension, and then, of course, you get the product.

And the Fourier transform theorem says that the final field will actually be the Fourier transform, but with the coordinate, the special frequency coordinate replaced by x prime over lambda z. This is where this came from. I substituted u with x prime over lambda z. So the bottom line is that this is perhaps easier if you look at it heads up. So here is a rectangular function. I only saw the x dimension here with a size of x0.

Then you can see that the Fourier transform. Actually this square, the intensity of the Fourier transform. It has this characteristic sinc pattern with a central lobe and then side lobes. And the size of the central lobe is inversely proportional to the size of the rectangle. So if I make this rectangle smaller, this size will become bigger. So this is then our first Fraunhofer diffraction pattern. The Fraunhofer diffraction pattern of a rectangular function.

Of course, there is many different apertures that are of interest in this business. Oh, and this, by the way, is called the sinc pattern, as I already mentioned. So there's many different patterns of interest. For example, very often in optics, we use circular apertures. Lenses, irises, in cameras, most optical systems have a circular aperture. In this case, we talked about the blinking or the Poisson spot here. But that's not what I'm interested now. I'm interested in the far field diffraction pattern.

And in this case, you also get a pattern with a-- kind of looks like a sinc, but a sinc with circular symmetry. It is not exactly a sinc. It is given by a rather nasty formula here. It is the ratio. First of all, it is all done in polar coordinates. So you see that you get the square root of the sum of the Cartesian coordinates squared. But the function itself is given by the ratio of a Bessel function of the first kind and order 1 divided by its argument.

I will not go into to the detail of the derivation here. Goodman describes it in great detail. So if you're interested, you can go and check it out over there. I do want to emphasize a couple of things. First of all, that is this sometimes by analogy to the sinc. This pattern is referred to as a jinc. So the J comes, of course, from the Bessel J. So we call it a jinc function. And more commonly it is referred to as the Airy pattern. Airy not because it sucks air or something like that. Actually, it is named after someone, some Englishman, whose name was Airy.

So Airy pattern. And if you compare it with the previous one. The previous one. I'm sorry, you have endure this animation again. OK, so the previous one. The null actually occurred that lambda l divided by the size of the aperture. In the case of the jinc, there's a factor of 1.22 that [INAUDIBLE] the calculation. So the null basically occurs at the very similar looking variable. If you make the diameter shorter, the size of the jinc will grow. But the null, the zero of the jinc, occurs at this function, at this value, 1.22, which, of course, comes from the zero of the Bessel function. So there's no intuition here. It's just where this function happens to reach value zero.

OK. Let me skip this slide, and perhaps you can go over it and talk about it later. It basically elaborates a little bit on the issue of-- I said before that in order to observe the Fraunhofer diffraction pattern, have to let z become long enough. Have to propagate the field far enough. So this slide answers the question, well, how far is far? Let me skip it four now. And if we have time later, I will come back to it.

But what I would like to get started now is a few comments on Fourier transform the cells, and how they apply to different apertures. So calling the Fourier transform is a topic in applied math, really. I don't want to convert this to 18085, or whatever it is at MIT that you'll learn those things. But I will remind you of some of the basic properties.

So one is the definition of the Fourier transform. I already wrote it down. Many of you are more familiar with the time domain definition, where the Fourier variable is actually a frequency measured in hertz. Of course, because here, we're talking about signals in the space domain. The frequency variable is the spatial frequency, so the units are actually inverse meters. Hertz is inverse second. The units here are inverse meters.

And, of course, because we're dealing with two dimensional special variables, it is a two dimensional Fourier transform because it is a double integral. But other than that, it's very similar. The other thing I wanted to remind you is that there is an inverse Fourier transform which looks very similar, except for a minus sign, so into the exponent here.

And, of course, the inverse Fourier transform takes you back to the original function. So it's like a dance. You start with a initial function. You compute the Fourier transform, then you plug it into the into the inverse Fourier transform, and you get back what you started. That is sometimes referred to as the Fourier integral instead of an inverse Fourier transform.

So what is this really, this Fourier transform? If you look at its surreal part, and if you have a real function here. Basically, what the Fourier transform does is it multiplies this function. It is denoted as red here, g of x. It multiplies with a sinusoid. The real part of this complex exponential is a cosine. So you multiply the function with this cosine, and then you integrate. OK, so why do you do something like that?

Actually, does anybody know why Fourier came up with this kind of transform? What was the context that Fourier-- what was Fourier? Fourier was a French mathematician, or a French applied physicist, I guess. And he was trying to solve a particular problem. Does anybody know what's the problem he was trying to solve when he came up with this business?

OK, it was a problem of heat transfer. Fourier was trying to solve the problem of what is that temperature distribution between two hot plates, one of them at temperature t1, that at temperature of t2. And actually, the answer is not given by a Fourier integral. It is given by a Fourier series. And if you make the plates go. If you increase the distance between the plates, the Fourier series becomes an integral. So this entire mathematical arsenal here, it actually came from the field of heat transfer, interestingly enough.

Anyway, that is of no concern to us here. The Fourier transform, as many of you know-- especially those who do acoustics or signal processing-- it has tremendous applications in signal processing nowadays. And, of course, it is still used in heat transfer. But in our context here, it is more signal processing that we will use it.

OK, so why do we multiply by a sinusoid? Well, the reason is the following. Suppose that G, our transformed function, is itself a sinusoid. OK, so here is G with a particular frequency, u0. So G is the red sinusoid. The Fourier transform kernel is another sinusoid. And in general, they have different frequency, like shown here.

So what does the value of this integral? Do you know? If the two frequencies are different. If you multiply two sinusoids and integrate them over a very long distance, actually infinite. Actually, by convention in this class. I don't know if I mentioned it before. The convention, if I don't put bounds to an integral, that mean it goes from minus infinity to plus infinity. So this is an infinite integral of two sinusoids with different frequency multiplied. What is the answer?

AUDIENCE: Zero?

GEORGE BARBASTATHIS: Zero, yeah. Because the various oscillations that will cancel eventually. So you'll get nothing. OK, however, there's a singular case when the frequencies are the same. And what is the value of this integral in this case? Well, infinite, right? Because if you multiply them, this will be positive. This will also be positive, because you are multiplying two negative quantities. So you actually get infinity, which is not very good.

But in mathematics, we have a way of dealing with this kind of abrupt infinities. We call them delta functions. And, of course, I'm severely abusing the mathematics here. The way the delta function comes up. Does anybody know? It comes as a limit. The way you get a delta function is you actually bound this integral, so that you get a finite value. And then you let the bound go to infinity, and the limit is a delta function.

Anyway, without going into these mathematical intricacies, we can represent this situation here as-- OK, forget for forget the second delta function for a moment. But this situation where the value of the integral is zero for all frequencies except one. Because the integral assumes a huge value. Then we write it as a delta function.

And the why we get two delta functions. Well, we'll get two delta functions because the way this works is if you take the Fourier transform of an exponential, this is a single delta function. Now, if you taking the cosine. Of course, the cosine is a sum of two complex exponentials. And now we know how to deal with this. It's one of those that's given by an expression like this one. So you actually get two symmetric delta functions.

OK, so what is the one half here? Well, the one half is actually the energy contained in this delta function. So that's normal. The thing is that is a little bit weird about this is that this sort of situation implies that there's such a thing as negative frequency. Of course, there's no negative frequencies. The frequencies can only be positive. The reason that we need a negative frequency is actually for mathematical rigor, because we insisted on using phasors.

You remember a long time ago, when we started talking about waves. We said that waves are real, so they are actually cosine functions. But for mathematical convenience, in order to avoid complicated trigonometric calculations, it would represent this cosine function as a complex exponential. Well, if you really had the simple cosine transform. So you use the cosine into the kernel for the integral. That is known as a Fourier cosine transform, and then it contains only positive frequencies.

But it's nice to calculate. Gives you very ugly formulas. So that's why we'll use the complex exponential. It is simpler formulas, but the price we pay is this weird negative frequency. So there's nothing to worry about. It is not wrong physics in any way. It is simply a matter of mathematical convenience that leads to these negative frequencies.

And, of course, I will not go through all these derivations over here. But several functions, their Fourier transforms can be computed in [INAUDIBLE] form. In fact, all of these functions, you can go ahead if you like, and do the Fourier transform by yourselves. It is relatively simple mathematical exercise. So we will use some of these very often. Mostly, we'll use the rectangular function. I already talked about this one. We'll use the circular function. I talked a bit very briefly. Then there is the triangular function, which has a shot of a grayscale. It starts from zero, then progressively it goes to one, and then drops back down to zero.

In linear fashion, and the com. The composite sequence of delta functions, that is very useful in sampling. I don't use it very much in this class, actually. I sort of bypass the issue of sampling. But I'm sure all of you are familiar with Nyquist sampling rates, Nyquist frequencies, and so on and so forth. So these all can be explained by the com function. And Goodman has a section in the book. I forget where it is. It's a section two point something. Yeah, section 2.4, two dimensional sampling theory that goes over it. I will not go over it in the class. But it may be good idea for you to review it. OK.

So as I said, there's several functions here who's Fourier transforms can be computed. I will not go through all of these, but it is good for you to know where this kind of thing is in the book, so when necessary, you can refer to them, and you can get the answers for values [INAUDIBLE].

So for example, here is the rectangle function that we computed before. And, of course, it gives the sinc response. Another one worth remembering is Gaussian, a Gaussian function. Actually, also Fourier transforms to Gaussian, which is interesting. And another useful one that we will deal with later is this one. You should look at them all before last. It also looks like a Gaussian, but with a J here. So this we recognize. Physically, what is this function? It is a complex quadratic exponential. Physically, what did we call it? If I write it in a slightly different form, you will recognize it right away. What is this?

AUDIENCE: Spherical wave along z?

GEORGE BARBASTATHIS: It is a spherical wave, propagating a distance z. So what you see over there is actually a spherical wave with a slightly weird definition, a squared equals 1 over lambda z. So this expression here in the row before last is a spherical wave. So a Fourier transform is also a spherical wave. And we will use this Fourier transform quite a bit in the next two lectures.

It might be good if you start studying, by the way, if you don't know what this is, it means you haven't studied. And I don't know how you did the homework without studying, possibly by copying from the last year. But I strongly recommend that you don't do that, because you're presumably here in order to learn. And you don't learn unless you study. So it is about time, not because of the quiz, but anyway. The quiz is also coming up, so it is about time to start studying it. So this is like a friendly advice, I guess, from an older guy. Study.

OK. [INAUDIBLE] that the Fourier transform has. Once we have this basic Fourier transforms that are shown here, then we can compute even more Fourier transforms by using the various properties of the Fourier transform. So one of those we wanted to derive. This is the scaling theorem. I did this at the beginning of the class. And it tells you that if you scale the argument that goes inside the Fourier transform, then the Fourier transform itself scales the opposite way.

So for example, in the case of the Fraunhofer diffraction, it says that if you make an aperture smaller, its Fraunhofer diffraction pattern becomes larger. So this is the scaling theorem, physical and mathematical. Physical, it tells you that the Fraunhofer diffraction becomes bigger. Mathematically, it comes from this scaling property of the Fourier transforms.

Another important one is the scaling theorem, which will prove a little bit later. But it's also very important one. Actually, all of these properties are very important. Number four is actually energy conservation. It relates the modulus-- the integral of the modulus of a function. We recognize this as energy. If you look at number four, magnitude g squared is actually intensity. And if you integrate intensity over the entire plane, then you'll get, of course, energy flux, you get power. And power has to be conserved, so this is what this theorem says, very important, Parseval's theorem.

And the convolution theorem is also very important. We'll see an application a little bit later today, or maybe Monday if we run out of time today. But anyway, all of these are very important. OK, so I will show you some Fourier transforms to sort of give you some of the properties. Are we still on?

AUDIENCE: Yes.

GEORGE BARBASTATHIS: Thank you. So this is a sinusoid. Of course, this is not a physical transparency. Well, I can make a physical transparency like this, but this assumes negative values, which means that to make a physical transparency like this one, you would have to have a phase delay, as well as a grayscale variation.

What I'm trying to say is that if you have a cosine, two pi ux. Its magnitude goes like this. And its phase. What is the phase? What is the phase of the cosine? What is the phase of a positive real number? What is the phase button? Someone said zero here. And that's correct. What is the phase of a negative real number? Someone here suggesting zero. Negative real number.

AUDIENCE: 180 degrees?

GEORGE BARBASTATHIS: Fine, that's right. So therefore, the phase of the cosine is zero, where the cosine is positive, and jumps to pi, where the cosine is negative. OK, so that would be a very difficult transparency to make, right? Because you would have to have the grayscale variation like this to impose the variation the amplitude modulation. And then you would have to impose some variable phase delay also, in order to impose the phase delay.

So that is difficult to do. But anyway, mathematically, we can write anything we like. So this is the cosine, and its Fourier transform, of course, consists of two delta functions. So this is what these bright dots indicate, delta functions whose spacing equals the inverse of the period of the cosine. And, of course, if you squeeze the cosine, since the spacing equals the period, then the two delta functions will go further away.

Another way to describe the same is, of course, by the scaling theorem. If you squeeze, it's equivalent to scaling by a quantity larger than one. And therefore, the spacing will also scale by a quantity larger than one. What is a more physical transparency that we saw in the previous lecture? I said that this is difficult to do, because you have to impose both amplitude and phase variation on the transparency.

AUDIENCE: A binary transparency?

GEORGE BARBASTATHIS: A binary, that's right. That's right. If you add a transparency whose magnitude looks like this. Goes between zero and one. That is fine, right? I can do it very simply by taking a piece of glass. And I can deposit some metal, for example, aluminum, or chromium, or more something like that in these regions. Oops, so you can not see what they were. Yeah.

Sorry about that. I pushed a button here that I should not have pushed. So in these regions where I have the [INAUDIBLE] to the metal, the transmissivity goes to zero. Another transparency that we saw, and it is also physical, is this one. That was actually the first example that we did on Monday. OK, how do I express this transparency? Is it cosine?

AUDIENCE: Is it 1/2 plus 1/2 cosine?

GEORGE BARBASTATHIS: Yeah. Because it swings between zero and one, right? So this, this will do it. Each Fourier transform of this one. How would it be different than the Fourier transform of the cosine that I have on my slide here? What is the Fourier transform of this one?

AUDIENCE: So you have a DC component. You have a DC component, and then yeah. The magnitude of that frequency is half of it.

GEORGE BARBASTATHIS: So in this representation, I would still have the tool delta functions at spacing. But also in addition, I would have an extra spot in the center. And this part would be brighter. So the power that goes into this part correspondingly would be 1/2, 1/4, 1/4 squared.

So the spot that goes into the center is what you very correctly refer to as the DC thermal. And now, of course, we know why we call it DC. I think I mentioned it also last time. It's because it corresponds to zero frequency. So in electrical signals, the zero frequency is known as the direct current, or DC, DC component. OK, now without cheating, that is without looking at the next page of the notes. I would like to ask you, and see if someone can guess. If I rotate this grating by some angle what will happen to the Fourier transform? Yeah?

AUDIENCE: If you rotate it by 90 degrees, I'd expect the frequencies to rotate by 90 degrees, as well.

GEORGE BARBASTATHIS: That's right. So if you rotate by 90 degrees, you expect the two spots to appear along the V-axis rather than the U-axis. If you rotate somewhere in between, where would this [INAUDIBLE] go? They will also rotate in what fashion? OK, so the observation to make here is that the two spots if you draw a line that connects the two Fourier delta functions. These lines would be perpendicular to the fringes of the grating. And this will remain true as you rotate the grating then, because actually, the Fourier transform does not know what the coordinates are.

So the Fourier transform knows that you have a variation along this direction. And that gives rise to the two delta function in this direction. In the vertical direction, there's no variation. So therefore, the Fourier transform is confined to the zero frequency. So if you rotate the grating, then these spots will rotate so that the line connecting them remains perpendicular to the fringes.

This may not show quite right because the projector actually squeezes my slide. So it may not show quite right. But if you think about it, you should convince yourselves that the two spots should be located along a line perpendicular to the grooves of this gradient. And, of course, if you squeeze the grooves in this rotated grating, then the two spots will also move away, again, along the same line perpendicular to the grooves. Any questions about that?

The other property of the Fourier transform farm which is listed in the table of formulas that they showed earlier is linearity. And linearity says that if you have a function that is the linear superposition of two functions whose Fourier transform you know, then the Fourier transform of this function is the linear superposition of the two Fourier transforms.

So for example, here is a function consisting of two gratings of period lambda one and lambda two. Which one is the Fourier transform? That's the Fourier transform of the long period, right? Because the two spots are close together. If you take the Fourier transform of the short period, the Fourier transforms are further apart.

If you do the superposition now. What you get, well, it is a bit. If you superimpose two frequencies, you get the bit pattern. Here it is. Looks kind of messy. The Fourier transform is relatively cleaner. It is the two dots that you get from this one, plus the two dots that you got from the other one. So therefore, you get four dots total. That's what the superposition theorem says.

Of course, you can generalize. I don't know if you can see in both of them. On the top right here, there's a bunch of dots. These dots actually, each one of those. They're symmetrical along the axis, so therefore, they correspond to sinusoids. And the superposition of sinusoids looks very messy here. It is still periodic, but messy.

Of course, if you look at it in the Fourier domain, each one of those is it represented by its own individual pair of delta functions. But, of course, this is discrete now. What is even more interesting is that if you were to connect all these delta functions and get the continuous Fourier transform, then your original pattern over here, this page domain would become nonperiodic. So you can see very clearly. Here, I have discrete-- a discrete Fourier transform that corresponds to periodic pattern.

AUDIENCE: Could you draw the Fourier transform in the overhead projector? Because we can not see it. It looks dark, totally dark here.

GEORGE BARBASTATHIS: OK. I cannot quite draw it, but I can sort of cartoon it. So the cartoon would be dots like this. Something like this. So each pair of dots corresponds to a cosine. And then what you see on the top left is actually the superposition of all of these cosines. OK.

And, of course, you can have sort of more general transparencies. You guys are too young to remember this, but about in 2006, I believe. The Boston baseball team beat the Yankees After 85 or 86 years. They finally managed to beat them again. And the night of the game, this is the Prudential Tower. For those of you who live in Singapore, this is the-- I think it's the tallest building in Boston. And so that night, they lit up their lights in the offices in a way that if you looked at the pattern, you could see the sign, Go Sox, the Boston famous called Red Sox.

And, of course, the Fox 25 is the TV channel that sponsored the match. So I took a picture with my camera. I can see this tower from where I used to live in Boston. So this is a picture. And if you represent it as a transparency, so that is the bright spots correspond to transmission of light. The dark spots correspond to blocking the light. Then you can think it's Fourier transform.

And you can see sort of a more general pattern that looks like this. What is interesting is that if you look carefully at this pattern. And I don't know if you can see it in Boston. But the pattern here looks kind of diffuse. But there's some distinct spots, actually quite a few of these spots. Can anybody guess where these spots came from?

AUDIENCE: Some of the features in the image, I guess, have straight lines that kind of act like a box function, but not completely. Sorry, other way around. You're seeing basically periodic structure in the image on the left gets reflected as spots in the Fourier domain on the right.

GEORGE BARBASTATHIS: That's right. The building has irregular spacing between the windows. So you see a very clear periodic pattern here. It is modulated by the Go Sox illumination, but nevertheless, even the dark windows are visible in the picture, right? Dark windows. Some of the windows, they turned on the lights, some they didn't. But still, you can see the windows, even if the lights were off.

So this gives rise to a periodic pattern. And, of course, the Fourier transform of a periodic pattern as I said before is a sequence of dots corresponding to the Fourier series coefficients. So that's why you see this very nice distinct dots over here. It is the windows in the high rise.

There's also more periodicity. This is a roof that also is periodic. You can see a grating here. can you still hear me? I keep dropping their microphone. Thanks. The grating here should be visible as-- it must be one of these pairs of dots that do not correlate with the building. That is the pattern on the roof over here. This is a roof of another building.

Now, let's look at the various theorems. I've already said this before, so that's the similarity theorem. If you compare the Fourier transform of two rectangles, one small, one big. The Fourier transform will have the opposite behavior. The small rectangle will give rise to a large Fourier transform.

The other one that I wanted to describe is this one, which is the shift theorem. So the shift theorem, we briefly glanced over it in the earlier slide. So let me remind you what this earlier slide said. So the shift theorem goes like this. I will do [INAUDIBLE] in one dimension only. Then let's say that g of x has a Fourier transform, G of u. The question is now, if I shift g of x by some amount, x0, what is the Fourier transform?

OK, so we do the same thing. We know that since this is true. Since this is true, we know that G of u equals-- then that is the definition of the Fourier transform. Now, the question is what is this one? So this one, the Fourier transform of the shifted function will be given by something like this. This, of course, the same Fourier transform, but now I plugged in the shifted function.

And in order to bring it to order, again, I will do a coordinate transform. And this is very easy because, again, the bounds of the integral are minus infinity, infinity. They don't change upon the transformation. The integral doesn't change either. I mean, the differential doesn't change either. The only thing that changes is here. So you'll get the integral. So x equals c plus x0, right? Because this you recognize is the same as this.

OK, so this is the shift theorem. So now, why is it related to this one? Well, this one, if I do a cross section. It will look like this. What I did is I drew a cross section along the vertical axis. So let's call the vertical axis x. So this is one. What is this? Well, this is a rectangle. OK, we know this one. And we computed this Fourier transform.

If you look at this one, it is also rectangle. The size is the same. If this is x0, this is also x0. But it is displaced. Let's use a symbol for this displacement. Let's call it a. Actually, this would be minus a. And this one. OK. So let's see if we can apply the shift theorem.

Actually, we have to apply two theorems here, the shift theorem and the scaling theorem. Which one should I apply first? OK, let me start. Let's do one thing at a time. So let me write this function down. So g of x equals. Each one of those corresponds to the three rects.

OK. Now, I want to take the Fourier transform. So one, I've already done. It's this one. I guess I use the red pen. This one, we concluded earlier. It is x0, sinc of ux0. What about the other one? First of all, linearity says that I can just add them, right? So that's easy. Yes?

AUDIENCE: They're going to be the x0, sinc of u of x0. But shifted by e to the i 2 pi, a, and all that other stuff.

GEORGE BARBASTATHIS: That's right. So I will get this one for this term times the shift according to the shift theorem. And similarly, with a minus sign. Because here, the shift is in the negative direction. I'm sorry. The minus sign belongs here, and the plus belongs here. OK, so get a common term in all of this. OK, one is a plus. That is a minus. OK, so does this explain now what you see here?

AUDIENCE: It's a 1 plus cosine.

GEORGE BARBASTATHIS: That's right. This is. And indeed, in this calculated pattern. Actually, the way I did this. I used the fft2 function in Matlab. And you can see that fft2 correctly produced as a sinusoidal modulation here, which is imposed by the shift theorem, really. So that's very interesting. If you translate the original function, you get this sinusoidal modulation in the Fourier transform. And now because we have a superposition-- an interference, really-- of sinusoidal modulations, that's why we'll get the-- well, these fringes in the Fourier transform pattern.

And, of course, if you rotate this pattern. Then the fringes also rotate by the same token we said before, right? Because now in this case, the displacement is both x and y. So you will get the complex exponential in the rotated case. When you go to the Fourier space, due to the shift theorem, you will get the complex exponential of the form, e to the minus i two pi ax plus by. Where, for example, this is a. And this is b.

So when you do this, a preposition of these complex exponentials, you will get rotated fringes in the Fourier transform pattern. OK. Any questions? The last thing before we quit for tonight, or for this morning, is the evolution theorem. And that's a really, really important one that you probably remember. I don't know, maybe you remember it with horror. Or maybe you remember it with fondness. But anyway, whatever the case may be. You may remember from your systems classes.

So the convolution theorem says that if you have a system whose input is g sub in of x, and the output is g sub out of x prime. A linear system is actually-- the output is expressed as a convolution. And you may be more familiar with seeing these convolutions in the time domain, but it doesn't matter. In the case of optics, we're dealing with space domain signals. So we simply swap t with x. But it's actually the same idea.

So one of the [INAUDIBLE] example of this convolution in the case of Fresnel propagation. If you remember, Fresnel propagation was g out of x prime, y prime, proportional. It had some additional terms in front. But the integral of that we got goes something like this. G sub in of x comma y. So it is earlier convolution, where this function, h of xy. What is this again? What is this physically?

AUDIENCE: It's a spherical wave.

GEORGE BARBASTATHIS: Thank you. So the convolution here emphasize that if you take Fourier transforms of everybody. So you Fourier transform this one. You call it G sub in of u. You Fourier transform this one. You call it G sub out of u. You Fourier transform this one. OK.

Then the convolution theorem says that this equals G sub in of u times H of u. OK, that's the convolution theorem. So it says that in the space domain, you have a convolution relationship. Then in the Fourier domain, you simply get a multiplication. And actually, that also goes the other way around. If you have a multiplication in the space domain, you have a convolution in the frequency domain. We'll get to use that a little bit later. Does anybody want me to prove this? Do you believe it, or should I prove it?

Well, let me prove it. Since we're in the mood of a must today. So let me write the convolution integral. Actually, before I do that, let me write down the Fourier transforms. OK, similarly. OK, so these are really all the same. Now, let me write the output.

OK, and by the same token, these are the Fourier transforms. I can also write the Fourier integrals in the inverse fashion. So for example, I can neither g sub in of x equals integral G sub in of u, e to the plus i two pi ux du. If you recall, we call this the inverse Fourier transform, or the Fourier integral. And by the same token, I have h of x equals a similar looking integral for H of u. And g sub out, again, similar looking integral for G sub out of u.

OK, these are just definitions. So far, I haven't really done anything. Now, let me write out the convolution integral. What I will do now is a little bit horrible, but you will see the logic of it in a second. I will substitute the Fourier transform. Actually, I'm sorry. I will substitute the Fourier integral inside this relationship. So how many integrals do I get? I get three, right? I get one that I had, and then each one of those will be written as an integral.

So here are the three integrals. That's the original one. Then for g sub in of x, I substitute its own Fourier integral. And the same for h. Have to be a little bit careful. h is computed in this shifted coordinate, so it is x prime minus x du. OK.

Now, what I'll do is assuming that these functions are well-behaved and so on and so forth, I will actually interchange the order of integration. Let's see if I can do it in a way that it all fits here. OK, let me be a little bit more careful here. [INAUDIBLE] variable, u is in the same in the two integrals. So to avoid confusion, I will actually label them. I will call this u1, and this u2.

OK, so now, I have the du1, du2 integrals. What's inside g sub in of u1? H of u2. And all of this is multiplied by a x integral. So what do I have? So for x, I have u1 from this term, and minus u2 from this term. And what's left? This thing left over, right? So let me not forget it, e to the i two pi u2 x prime. x prime, of course, is not plain. So I'll just leave it there. It is not plain in the integration, that is. OK.

So now, what is this? I put one too many dx's. So what is this? It is the Fourier transform of an exponential. Remember, these integrals without bounds, they really go from minus infinity to infinity, right? So if I integrate an exponential from minus infinity to infinity, what do I get? We said it earlier this morning. Your tuition is ticking away one second at a time. Well, it's 9:25, according to my clock. So I guess we stop here. And I'll let you ponder this on your own. See you on Monday.