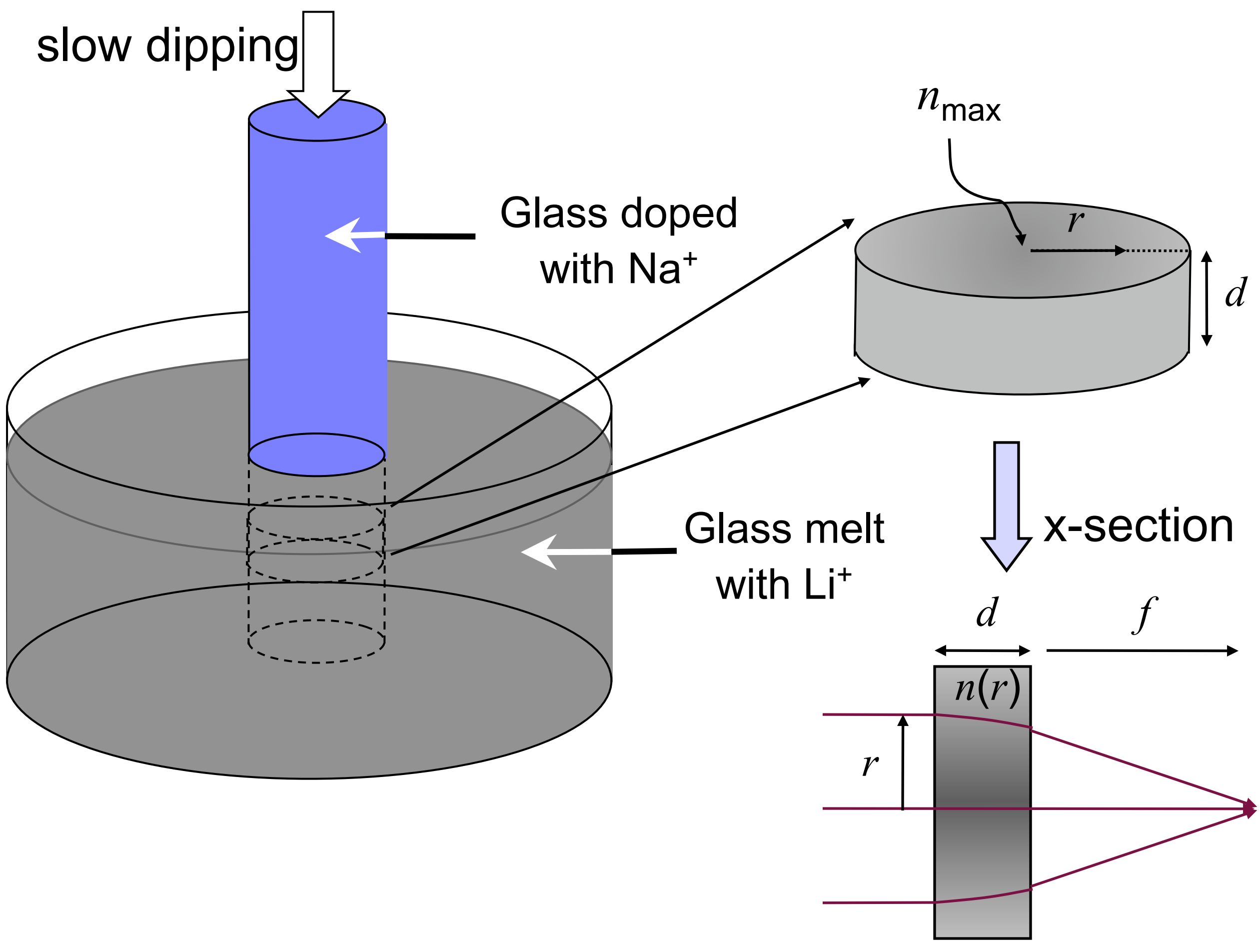
Gradient Index (GRIN) optics: radial quadratic

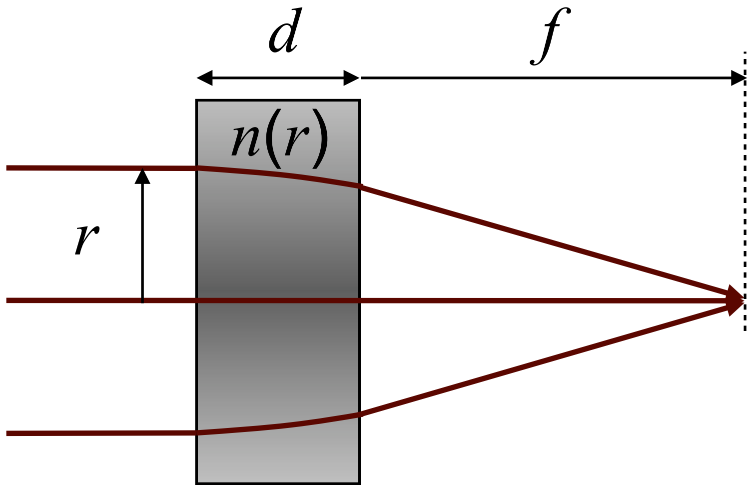
<u>Radial index profile:</u>

fabrication by ion exchange



- Diffusion driven → parabolic index profile $n(r) = n_{\max}\left(1 - \frac{\alpha r^2}{2}\right)$
- Index contrast $n_{max}-n_{min} \equiv \Delta n \sim 0.1$ (commercial)
- Focal length $f = \dfrac{1}{n_{\max}\alpha d}$

Paraxial focusing by a thin quadratic GRIN lens



Consider a ray from infinity entering the GRIN at elevation r. If focusing is to be achieved, this ray must meet the on–axis ray at a distance f from the exit face; therefore, their optical paths must be equal according to Fermat's principle.

For the on–axis ray,

$$\mathrm{OPL}\,(r=0) = n_{\max} d + f.$$

For the ray at elevation r,

$$\mathrm{OPL}\,(r) \approx n_{\max}\left(1 - \frac{\alpha r^2}{2}\right) d + \sqrt{r^2 + f^2},$$

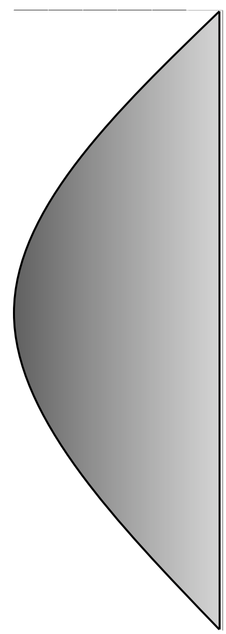
where we have neglected the small elevation decline due to the bending of the ray inside the GRIN.

Applying Fermat's principle in the paraxial approximation,

$$\begin{aligned}
n_{\max} d + f &\approx n_{\max}\left(1 - \frac{\alpha r^2}{2}\right) d + \sqrt{r^2 + f^2} \Rightarrow \\
f + \frac{n_{\max}\alpha d}{2} r^2 &\approx \sqrt{r^2 + f^2} \approx f\left(1 + \frac{r^2}{2f^2}\right) \Rightarrow \\
\frac{n_{\max}\alpha d}{2} r^2 &\approx \frac{r^2}{2f} \Rightarrow \\
f &\approx \frac{1}{n_{\max}\alpha d}.
\end{aligned}$$

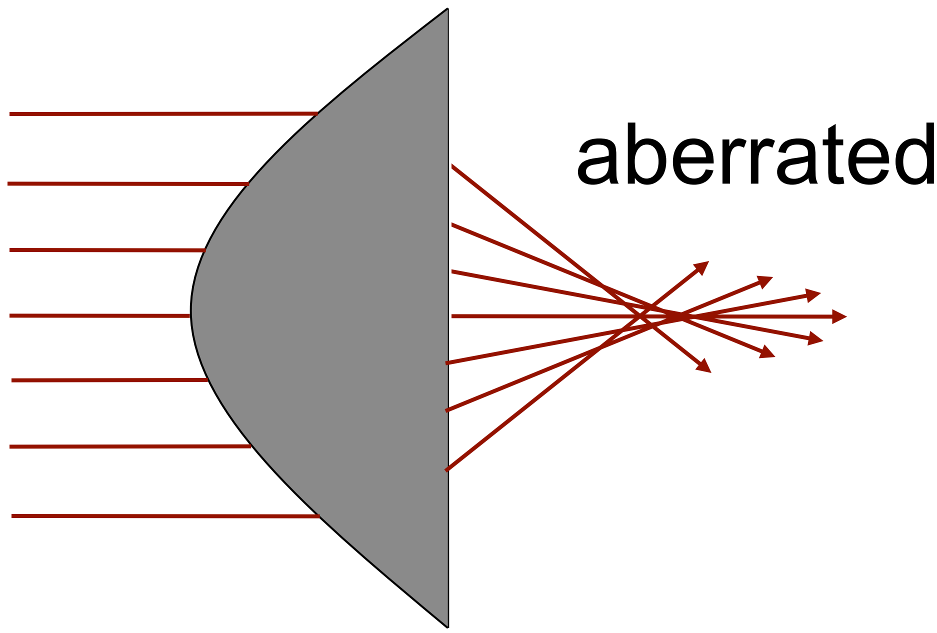
Gradient Index (GRIN) optics: axial

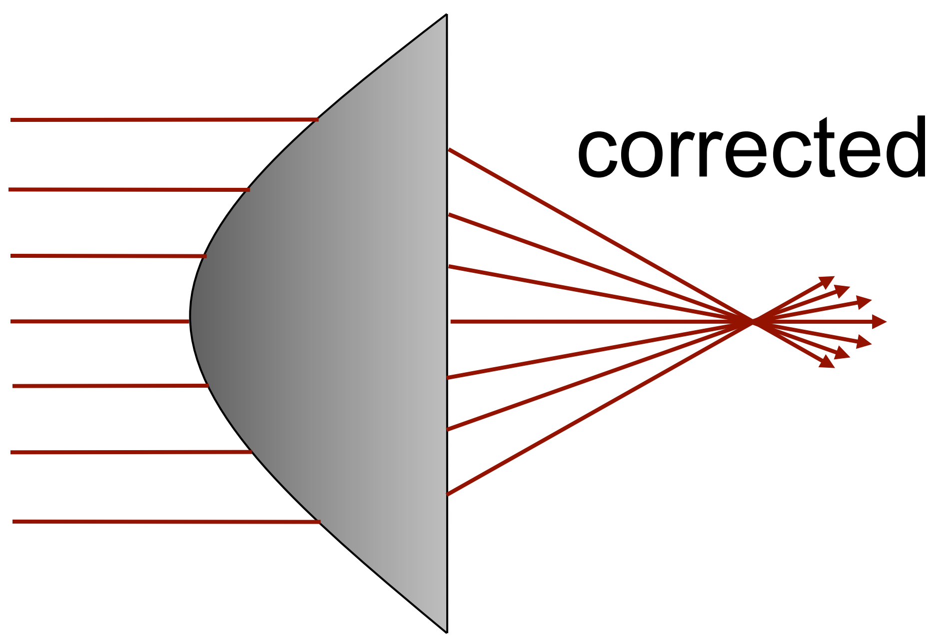
Axial index profile:
fabrication by melding & grinding



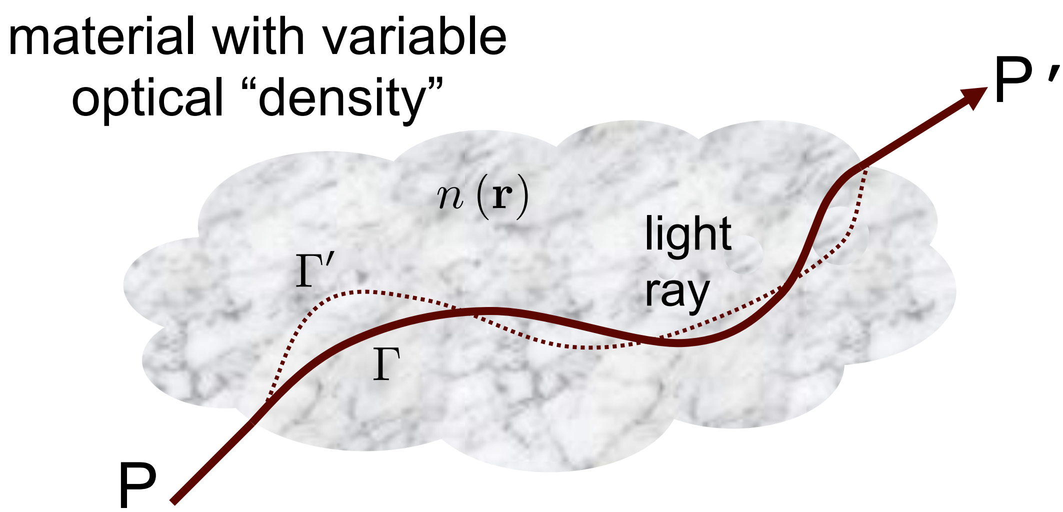
- Stack
- Meld
- Grind & polish to a sphere
- Result:
  Spherical refractive surface with axial index profile $n(z)$

Correction of spherical aberration by axial GRIN lenses





Generalized GRIN: what is the ray path through arbitrary *n*(*r*)?



"optical path length"

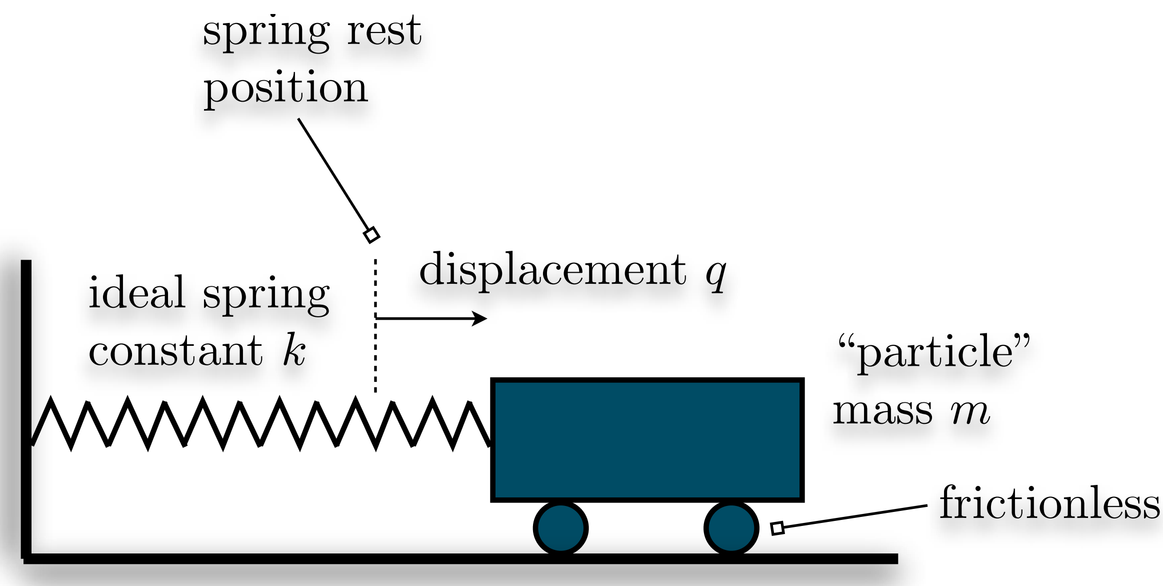
$$\int_{\Gamma} n(\mathbf{r})\,\mathrm{d}l$$

Fermat's principle:

The path Γ that the ray follows is such that the value of the path integral of refractive index $n(\mathbf{r})$ along Γ is smaller than all other possible paths Γ'.

Let's take a break from optics ...

Mechanical oscillator



Potential energy: $V = \frac{1}{2}kq^2$.

Kinetic energy: $T = \frac{1}{2}m\dot{q}^2 = \frac{1}{2}\frac{p^2}{m}$,

where $p \equiv m\dot{q}$ is the momentum.

Since there is no dissipation,
the total energy

$$H = T + V$$

must be conserved.

Introduction to the Hamiltonian formulation of dynamics

The Hamiltonian formulation is a set of differential equations describing the trajectories of particles that are subject to a potential (force.) The trajectory is described in terms of the particle position $\mathbf{q}(t)$ and momentum $\mathbf{p}(t)$. The Hamiltonian is the total energy, *i.e.* the sum of kinetic and potential energies, and it is conserved if there is no dissipation in the system. For example, for a harmonic oscillator the Hamiltonian is expressed as

$$H(\mathbf{q},\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + \frac{k\mathbf{q}^2}{2}. \tag{1}$$

The first term is the kinetic energy for a particle of mass m, and the second term is the potential energy for linear spring constant k.

The Hamiltonian equations in general are

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial H}{\partial \mathbf{p}}, \tag{2}$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{q}}. \tag{3}$$

The expressions on the right–hand side are the gradients of the Hamiltonian with respect to the vectors $\mathbf{p}$ and $\mathbf{q}$, respectively.

Let us consider the simplest case of a one–dimensional harmonic oscillator. In this case the position and momentum are scalars q, p. The Hamiltonian equations become

$$\left.\begin{aligned}\frac{\mathrm{d}q}{\mathrm{d}t} &= \frac{p}{m}\\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -kq.\end{aligned}\right\} \Rightarrow \frac{\mathrm{d}^2q}{\mathrm{d}t^2} = \frac{1}{m}\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{k}{m}q \Rightarrow \frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{k}{m}q = 0. \tag{4}$$

We have arrived at the familiar 2^{nd}–order harmonic differential equation. For example, assuming a particle that is initially at position $q(t=0) = q_0$ and at rest, $p(t=0)=0$, the solution to the Hamiltonian equations is

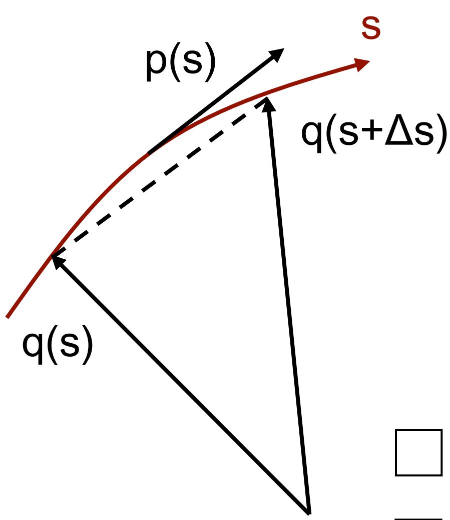
$$q(t) = q_0 \cos\left(\sqrt{\frac{k}{m}}t\right), \tag{5}$$

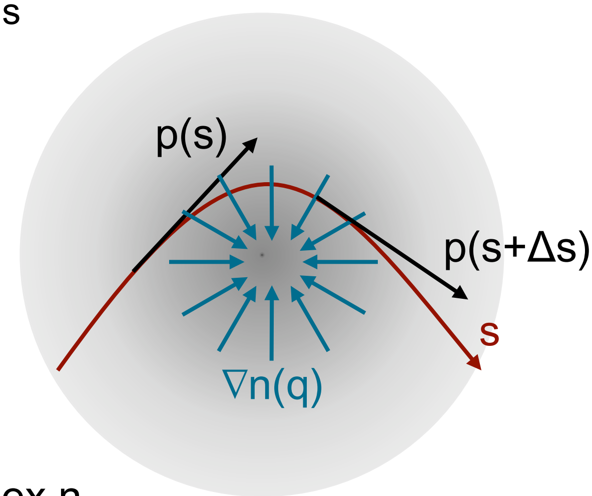
$$p(t) = -q_0\sqrt{km}\sin\left(\sqrt{\frac{k}{m}}t\right). \tag{6}$$

The solution set $\left\{q(t), p(t)\right\}$ is the trajectory of the particle. The motion represented by the trajectory that we found is clearly a harmonic oscillation.

Hamiltonian Optics postulates

s: parameterization of the ray trajectory
q(s): position vector for the ray trajectory at s;
p(s): tangent vector to the ray trajectory at s





□ = low index n_{min}

■ = high index n_{max}

<u>Geometrical postulate:</u>

Rays are continuous and piecewise differentiable

$$\Delta\mathbf{q}(s) \approx \frac{\mathbf{p}(s')}{|\mathbf{p}(s')|}\Delta s$$

$$\Rightarrow \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}s} = \frac{\mathbf{p}(s)}{|\mathbf{p}(s)|}$$

<u>Dynamical postulate:</u>

Momentum changes along trajectory arc length in proportion to the local refractive index gradient

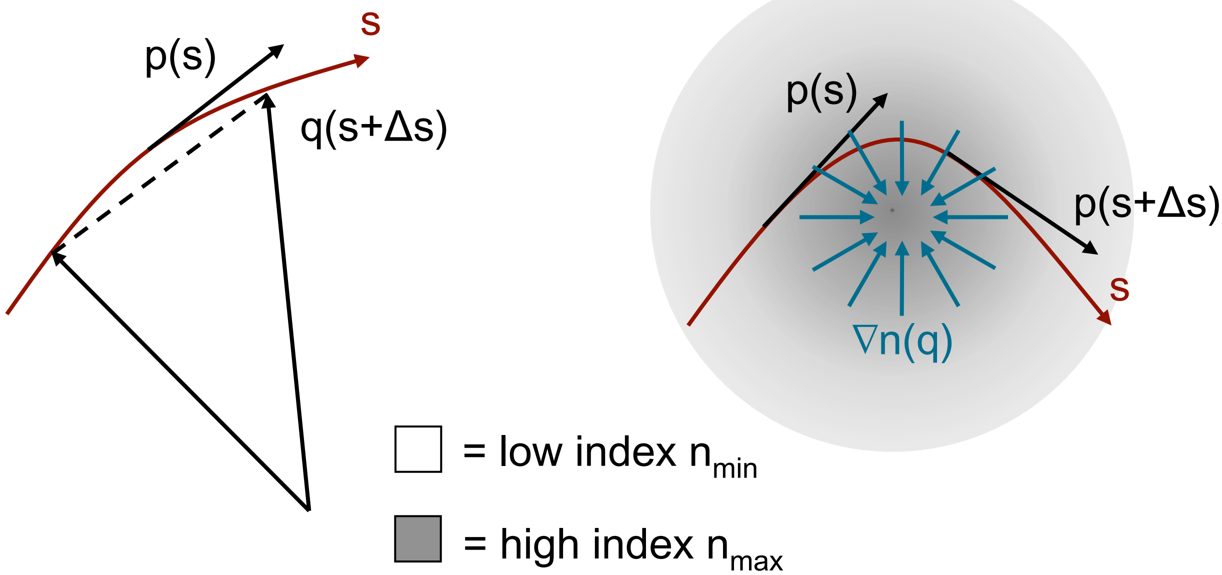
$$\Delta\mathbf{p}(s) \approx \nabla n\Big(\mathbf{q}(s')\Big)\Delta s$$

$$\Rightarrow \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = \nabla n\Big(\mathbf{q}(s)\Big)$$

These are the “equations of motion,”
i.e. they yield the ray trajectories.

The ray Hamiltonian

s: parameterization of the ray trajectory
q(s): position vector for the ray trajectory at s;
p(s): tangent vector to the ray trajectory at s



The choice $H = \left|\mathbf{p}\right| - n\left(\mathbf{q}\right)$ yields

$$\nabla_{\mathbf{q}} H \equiv \frac{\partial H}{\partial \mathbf{q}} = -\nabla_{\mathbf{q}} n\left(\mathbf{q}\right) \qquad \nabla_{\mathbf{p}} H \equiv \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}\left(s\right)}{\left|\mathbf{p}\left(s\right)\right|}$$

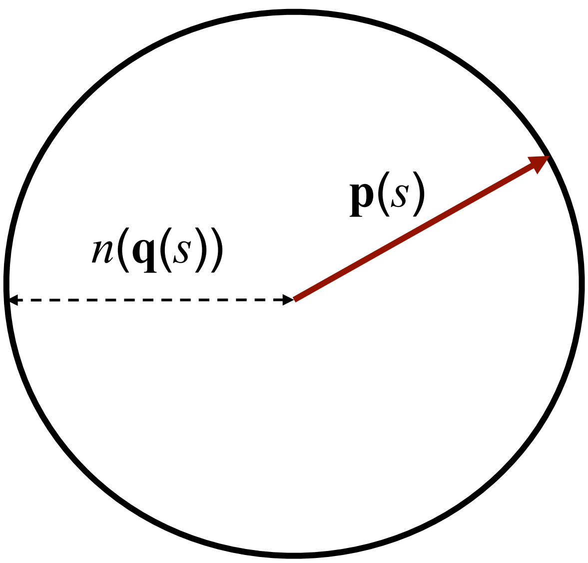
Therefore, the equations of motion become

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}s} = \frac{\partial H}{\partial \mathbf{p}} \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = -\frac{\partial H}{\partial \mathbf{q}}$$

Since the ray trajectory satisfies a set of Hamiltonian equations on the quantity *H*, it follows that *H* is conserved.

The actual value of *H*=const. is arbitrary.

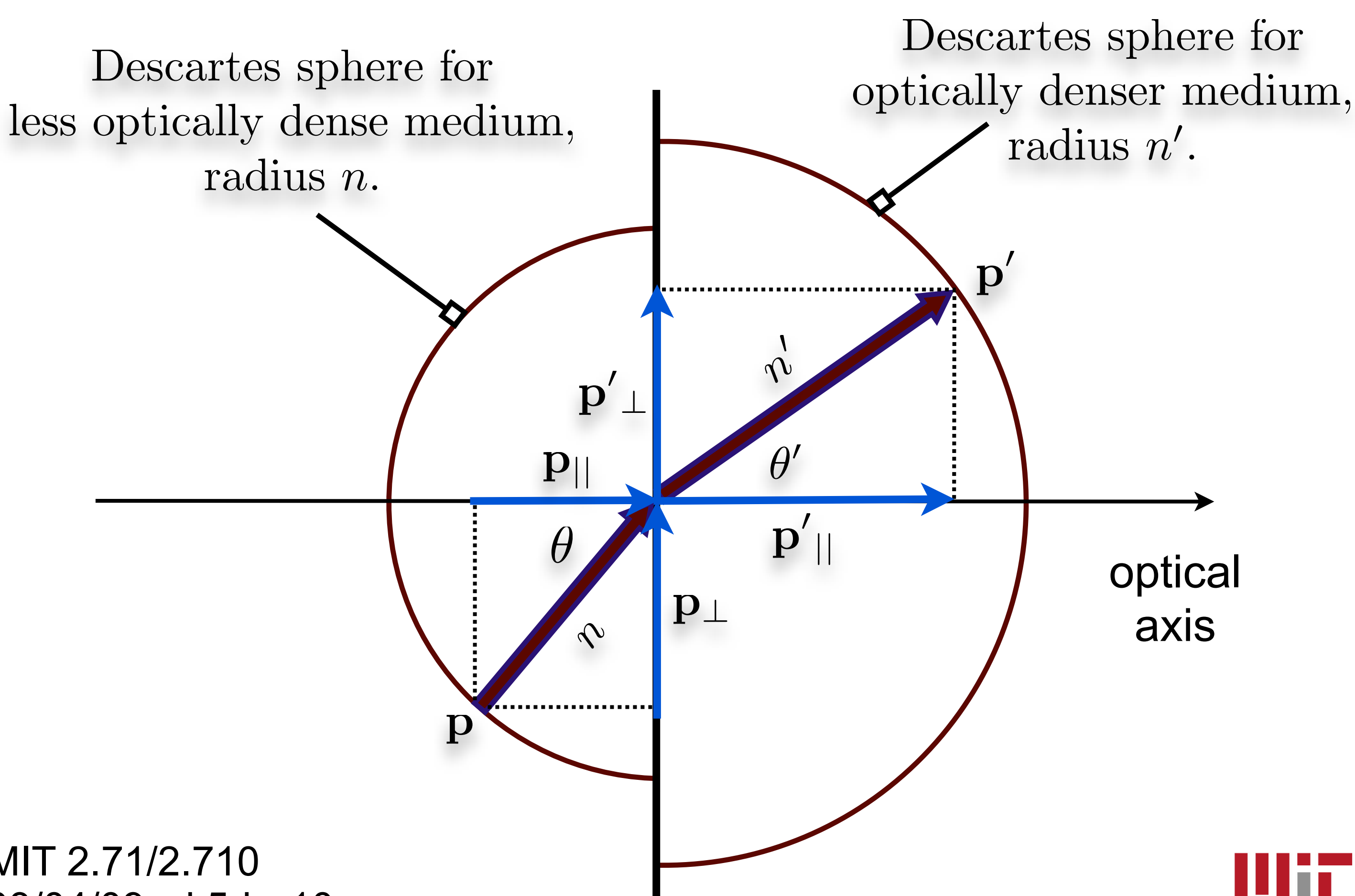
The ray Hamiltonian and the Descartes sphere



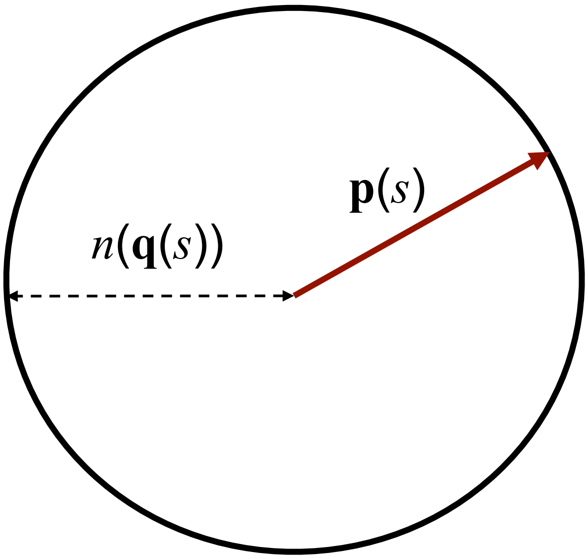
$$H = |\mathbf{p}| - n(\mathbf{q}) = 0$$

The ray momentum $\mathbf{p}$ is constrained to lie on a sphere of radius n at any ray location $\mathbf{q}$ along the trajectory s

<u>Application</u>:
Snell's law of refraction



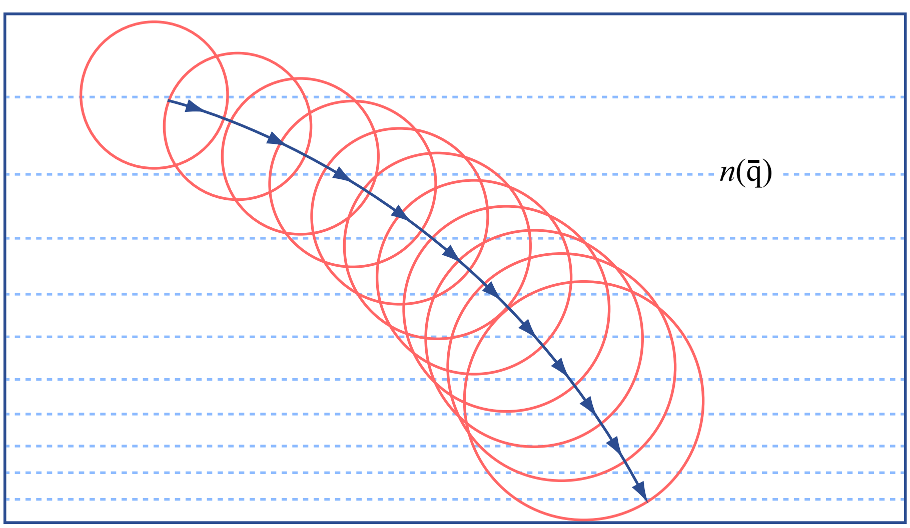
The ray Hamiltonian and the Descartes sphere



$$H = |\mathbf{p}| - n(\mathbf{q}) = 0$$

The ray momentum $\mathbf{p}$ is constrained to lie on a sphere of radius n at any ray location $\mathbf{q}$ along the trajectory s

<u>Application</u>:
propagation in a GRIN medium



The Descartes sphere radius is proportional to n($\mathbf{q}$); as the rays propagate, the lateral momentum is preserved by gradually changing the ray orientation to match the Descartes spheres.

Hamiltonian analogies: optics *vs* mechanics

Hamiltonian of mechanical system $H_{\mathrm{m}} = \dfrac{|\mathbf{p}|^2}{2m} + V(\mathbf{q})$ Energy $E = H_{\mathrm{m}}$ (conserved)

$$\text{Momentum} \quad \mathbf{p} = m\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{|\mathbf{p}|}{m} \quad \text{Velocity}$$

Mechanical Hamiltonian equations

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}s} = \frac{\mathbf{p}}{|\mathbf{p}|} \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = -\frac{m}{|\mathbf{p}|}\frac{\partial V(\mathbf{q})}{\partial \mathbf{q}}$$

Optical Hamiltonian equations

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}s} = \frac{\mathbf{p}}{|\mathbf{p}|} \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = \frac{\partial n(\mathbf{q})}{\partial \mathbf{q}}$$

$$\text{Analogous if: } \frac{\partial V(\mathbf{q})}{\partial \mathbf{q}} \quad \leftrightarrow \quad -\frac{|\mathbf{p}|}{m}\frac{\partial n(\mathbf{q})}{\partial \mathbf{q}} = -\frac{n(\mathbf{q})}{m}\frac{\partial n(\mathbf{q})}{\partial \mathbf{q}} =$$

$$= -\frac{1}{2m}\frac{\partial n^2(\mathbf{q})}{\partial \mathbf{q}}.$$

$$\Rightarrow V(\mathbf{q}) \quad \leftrightarrow \quad -\frac{n^2(\mathbf{q})}{2m} + \text{const.}$$

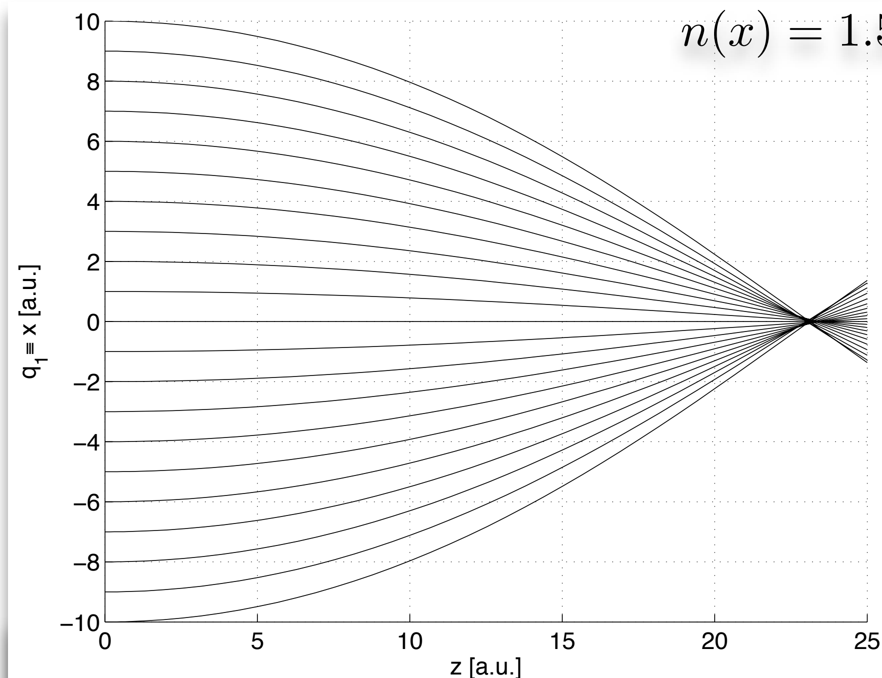
$$\text{Choose const.} = -\frac{E+1}{2m} \Rightarrow 2m\left[E - V(\mathbf{q})\right] \quad \leftrightarrow \quad n^2(\mathbf{q}) - 1$$

$$\text{and} \qquad E_{\text{kinetic}} \propto E - V(\mathbf{q}) > 0 \quad \leftrightarrow \quad n(\mathbf{q}) > 1.$$

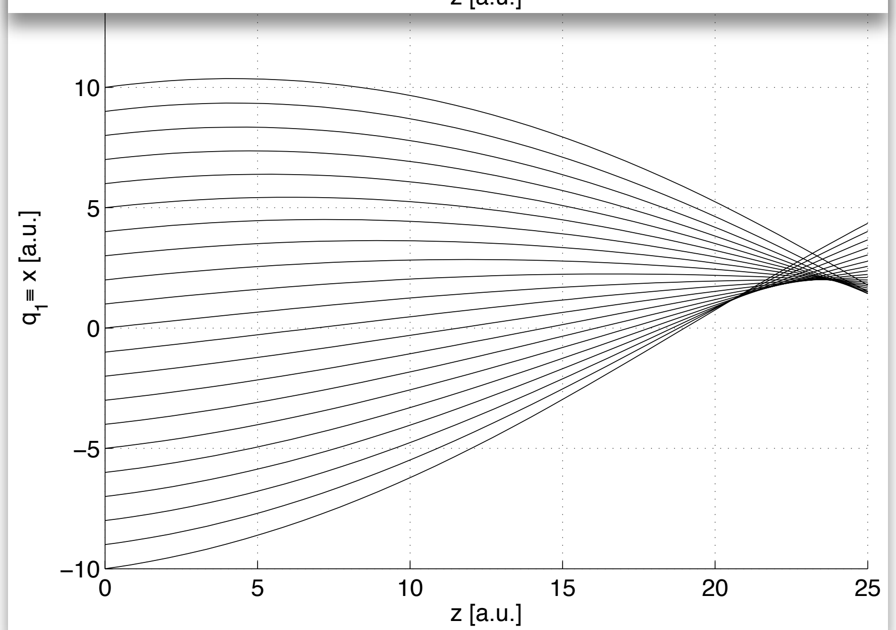
physically allowable kinetic energy — physically allowable refractive index

Example: Hamiltonian ray tracing of quadratic GRIN

$$n(x) = 1.5 - 0.0033\,x^2$$



on-axis incidence



off-axis incidence, $p_x(0)=0.2$

Further reading:

- M. Born and E. Wolf, *Principles of Optics*, Cambrige University Press, 7th edition, sections 4.1-4.2
- K. B. Wolf, *Geometrical Optics on Phase Space,* Springer, chapters 1, 2
- K. Tian, *Three-dimensional (3D) optical information processing,* PhD dissertation, MIT 2006.