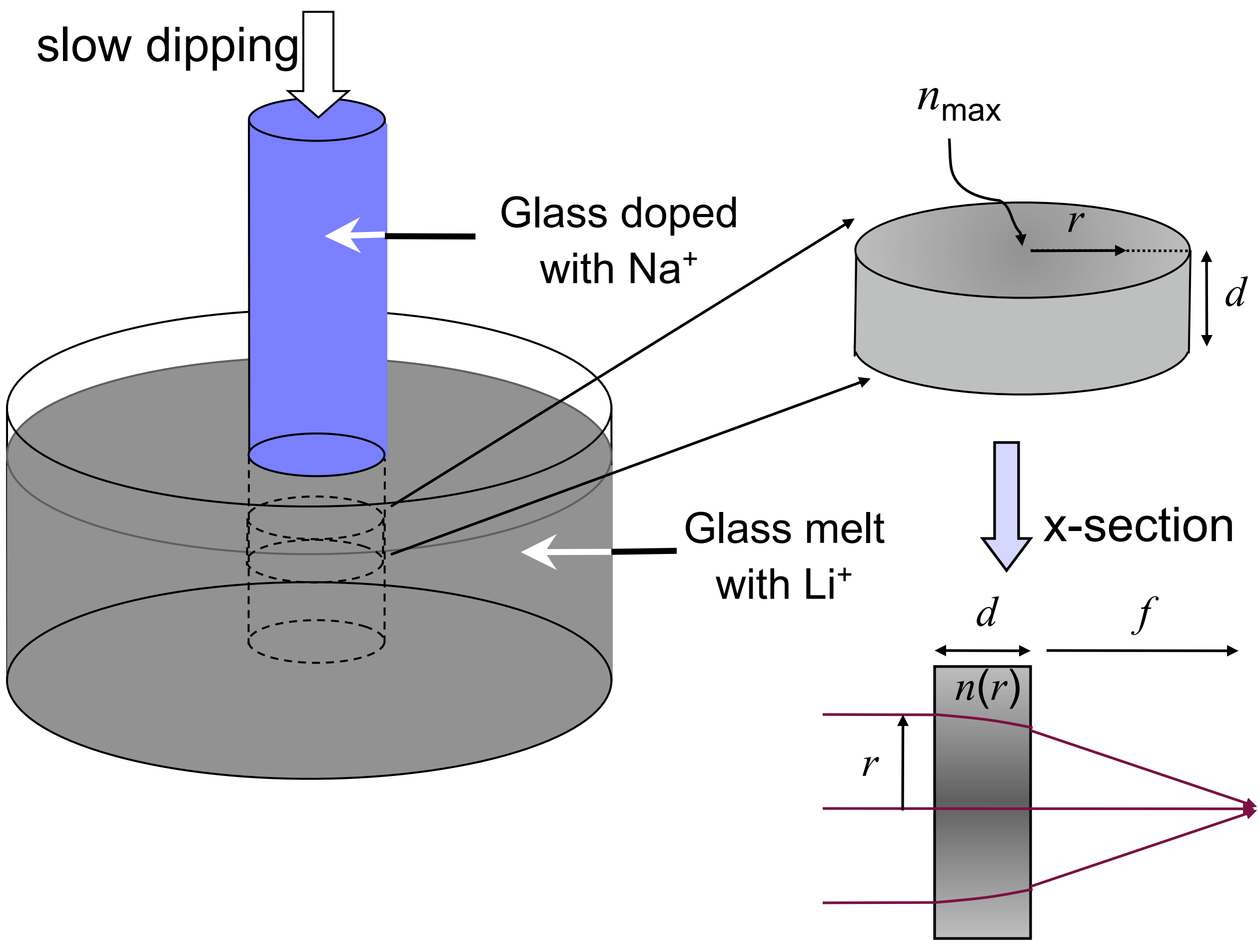
Gradient Index (GRIN) optics: radial quadratic

<u>Radial index profile:</u>

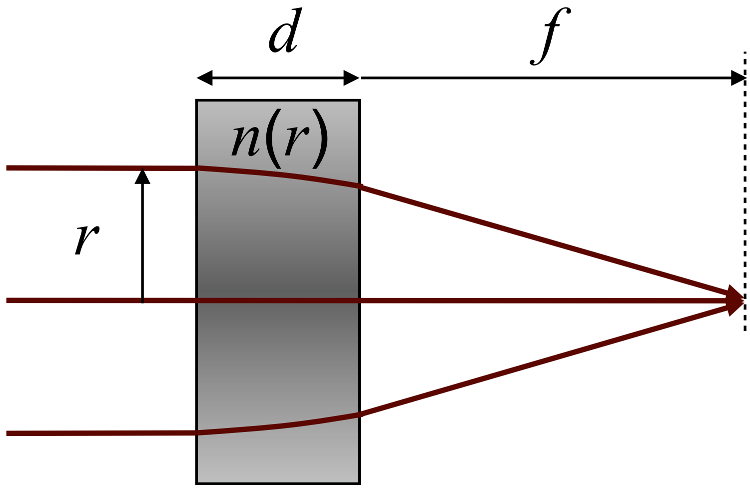
fabrication by ion exchange



- Diffusion driven → parabolic index profile $n(r) = n_{\text{max}}\left(1 - \frac{\alpha r^2}{2}\right)$
- Index contrast n_{max}-n_{min}≡Δn ~ 0.1 (commercial)

- Focal length $f = \frac{1}{n_{\text{max}}\alpha d}$

Paraxial focusing by a thin quadratic GRIN lens



Consider a ray from infinity entering the GRIN at elevation r. If focusing is to be achieved, this ray must meet the on–axis ray at a distance f from the exit face; therefore, their optical paths must be equal according to Fermat's principle.

For the on–axis ray,

$$\mathrm{OPL}\,(r=0) = n_{\max} d + f.$$

For the ray at elevation r,

$$\mathrm{OPL}\,(r) \approx n_{\max}\left(1 - \frac{\alpha r^2}{2}\right) d + \sqrt{r^2 + f^2},$$

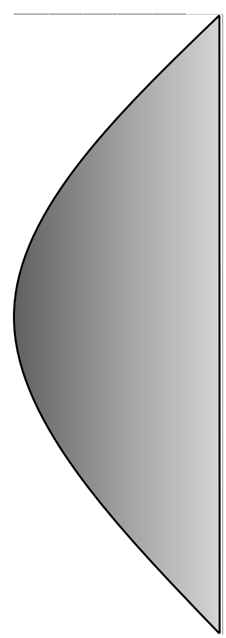
where we have neglected the small elevation decline due to the bending of the ray inside the GRIN.

Applying Fermat's principle in the paraxial approximation,

$$\begin{aligned}
n_{\max} d + f &\approx n_{\max}\left(1 - \frac{\alpha r^2}{2}\right) d + \sqrt{r^2 + f^2} \Rightarrow \\
f + \frac{n_{\max}\alpha d}{2} r^2 &\approx \sqrt{r^2 + f^2} \approx f\left(1 + \frac{r^2}{2f^2}\right) \Rightarrow \\
\frac{n_{\max}\alpha d}{2} r^2 &\approx \frac{r^2}{2f} \Rightarrow \\
f &\approx \frac{1}{n_{\max}\alpha d}.
\end{aligned}$$

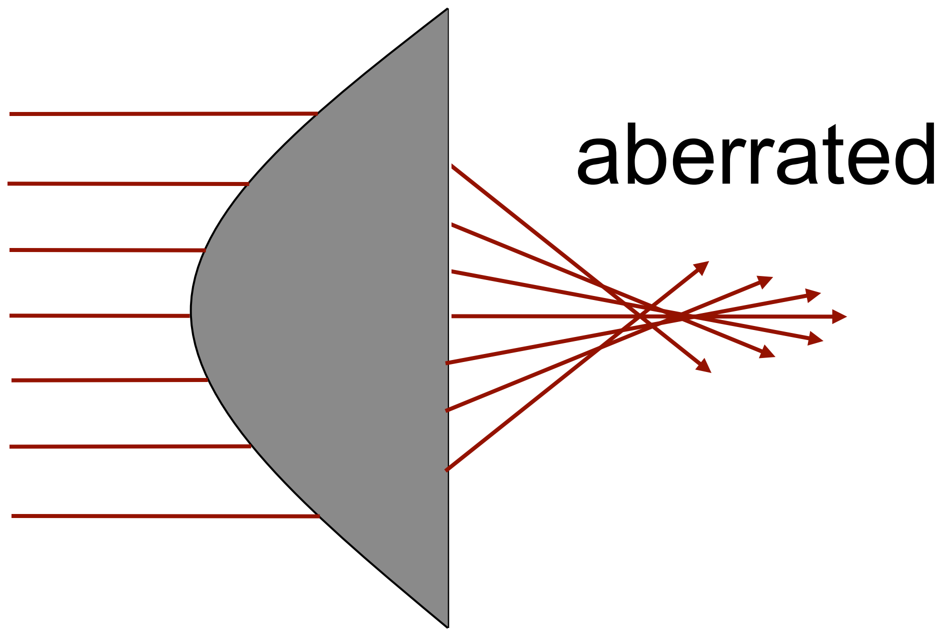
Gradient Index (GRIN) optics: axial

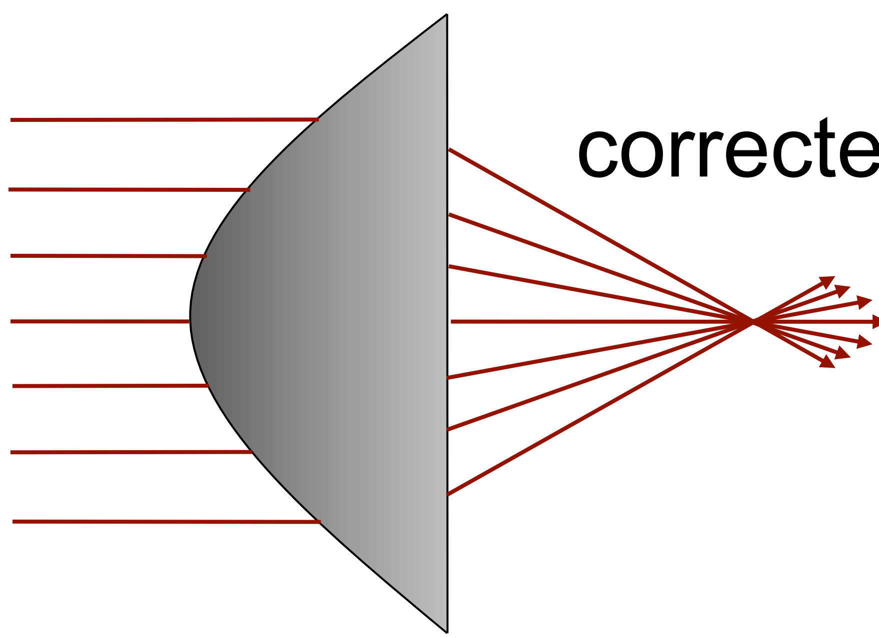
Axial index profile:
fabrication by melding & grinding



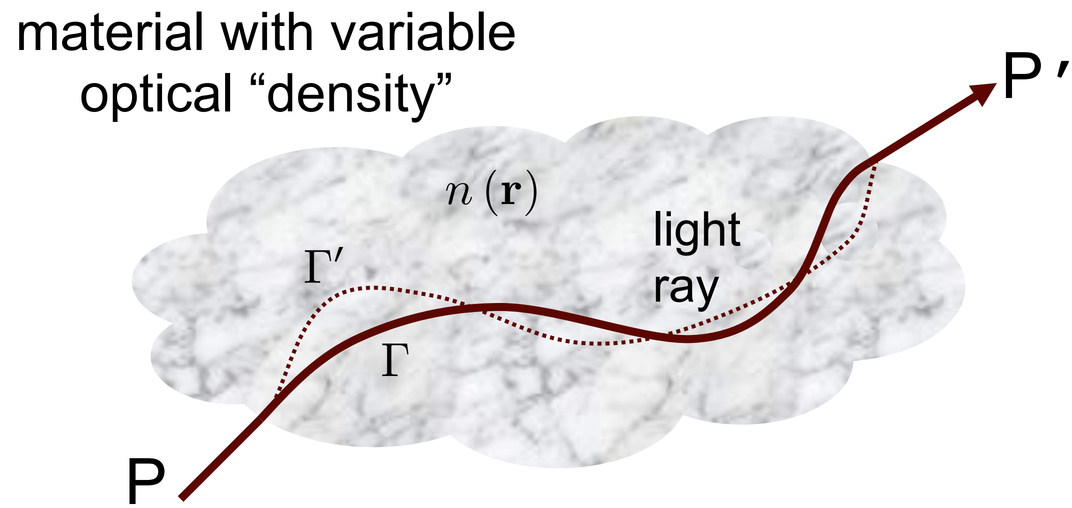
- Stack
- Meld
- Grind & polish to a sphere
- Result:
  Spherical refractive surface with axial index profile $n(z)$

Correction of spherical aberration by axial GRIN lenses





Generalized GRIN: what is the ray path through arbitrary *n*(*r*)?



"optical path length"

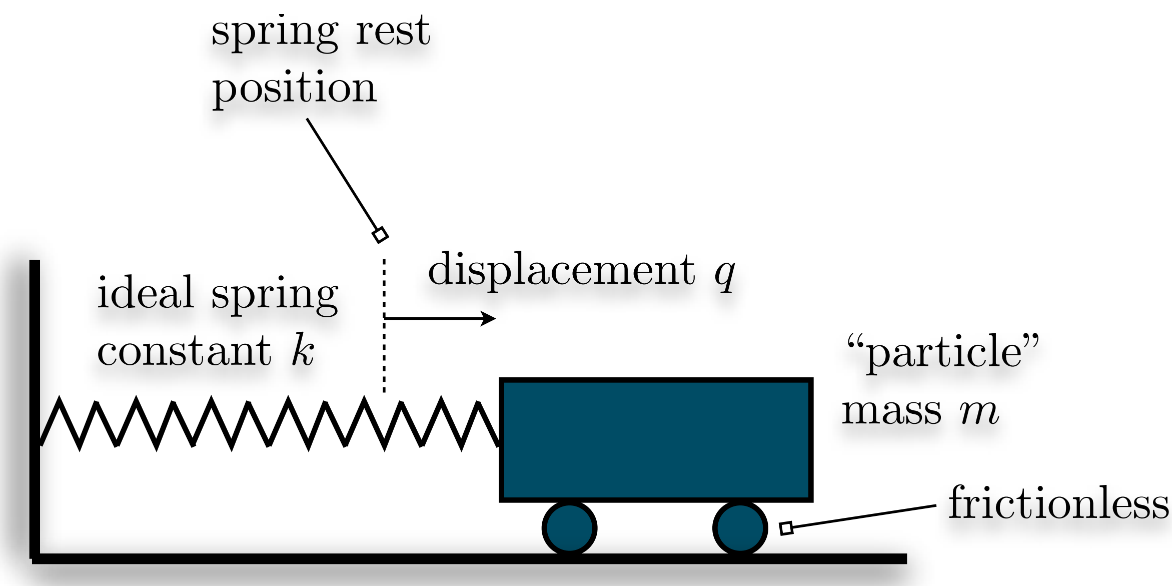
$$\int_\Gamma n(\mathbf{r})\,\mathrm{d}l$$

Fermat's principle:

The path Γ that the ray follows is such that the value of the path integral of refractive index $n(\mathbf{r})$ along Γ is smaller than all other possible paths Γ'.

Let's take a break from optics ...

Mechanical oscillator



Potential energy: $V = \dfrac{1}{2}kq^2.$

Kinetic energy: $T = \dfrac{1}{2}m\dot{q}^2 = \dfrac{1}{2}\dfrac{p^2}{m},$

where $p \equiv m\dot{q}$ is the momentum.

Since there is no dissipation, the total energy

$$H = T + V$$

must be conserved.

Introduction to the Hamiltonian formulation of dynamics

The Hamiltonian formulation is a set of differential equations describing the trajectories of particles that are subject to a potential (force.) The trajectory is described in terms of the particle position $\mathbf{q}(t)$ and momentum $\mathbf{p}(t)$. The Hamiltonian is the total energy, *i.e.* the sum of kinetic and potential energies, and it is conserved if there is no dissipation in the system. For example, for a harmonic oscillator the Hamiltonian is expressed as

$$H(\mathbf{q},\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + \frac{k\mathbf{q}^2}{2}. \tag{1}$$

The first term is the kinetic energy for a particle of mass m, and the second term is the potential energy for linear spring constant k.

The Hamiltonian equations in general are

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial H}{\partial \mathbf{p}}, \tag{2}$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{q}}. \tag{3}$$

The expressions on the right–hand side are the gradients of the Hamiltonian with respect to the vectors $\mathbf{p}$ and $\mathbf{q}$, respectively.

Let us consider the simplest case of a one–dimensional harmonic oscillator. In this case the position and momentum are scalars q, p. The Hamiltonian equations become

$$\left.\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}t} &= \frac{p}{m} \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -kq. \end{aligned}\right\} \Rightarrow \frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = \frac{1}{m}\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{k}{m}q \Rightarrow \frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \frac{k}{m}q = 0. \tag{4}$$

We have arrived at the familiar 2^{nd}–order harmonic differential equation. For example, assuming a particle that is initially at position $q(t=0) = q_0$ and at rest, $p(t=0) = 0$, the solution to the Hamiltonian equations is

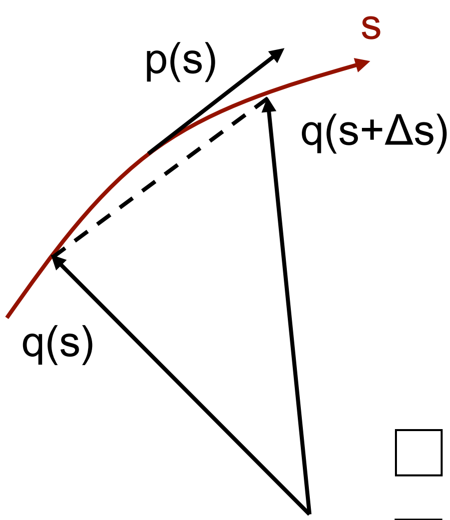
$$q(t) = q_0 \cos\left(\sqrt{\frac{k}{m}}t\right), \tag{5}$$

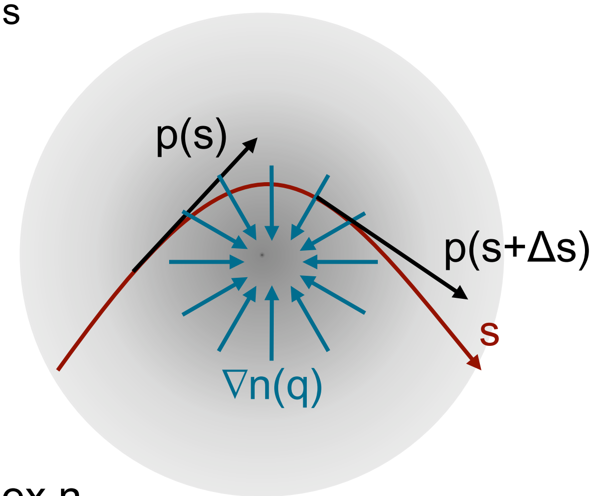
$$p(t) = -q_0\sqrt{km}\sin\left(\sqrt{\frac{k}{m}}t\right). \tag{6}$$

The solution set $\left\{q(t), p(t)\right\}$ is the trajectory of the particle. The motion represented by the trajectory that we found is clearly a harmonic oscillation.

Hamiltonian Optics postulates

s: parameterization of the ray trajectory
q(s): position vector for the ray trajectory at s;
p(s): tangent vector to the ray trajectory at s





□ = low index n_{min}

■ = high index n_{max}

<u>Geometrical postulate:</u>

Rays are continuous and piecewise differentiable

$$\Delta\mathbf{q}(s) \approx \frac{\mathbf{p}(s')}{|\mathbf{p}(s')|}\Delta s$$

$$\Rightarrow \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}s} = \frac{\mathbf{p}(s)}{|\mathbf{p}(s)|}$$

<u>Dynamical postulate:</u>

Momentum changes along trajectory arc length in proportion to the local refractive index gradient

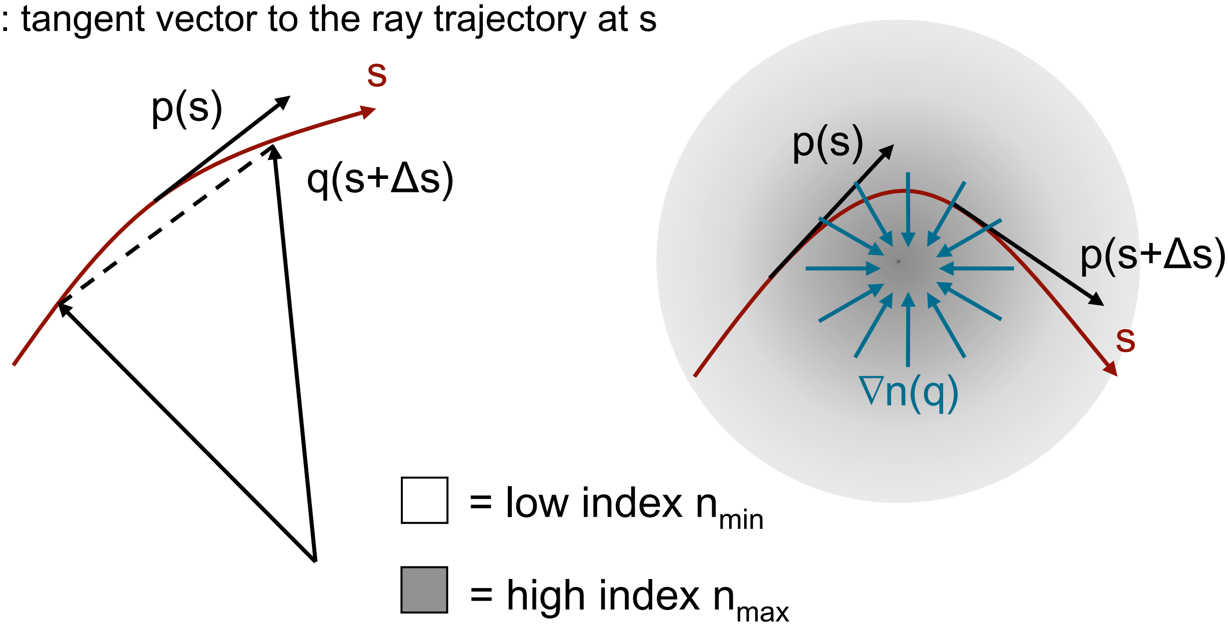
$$\Delta\mathbf{p}(s) \approx \nabla n\Big(\mathbf{q}(s')\Big)\Delta s$$

$$\Rightarrow \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = \nabla n\Big(\mathbf{q}(s)\Big)$$

These are the “equations of motion,”
i.e. they yield the ray trajectories.

The ray Hamiltonian

s: parameterization of the ray trajectory
q(s): position vector for the ray trajectory at s;
p(s): tangent vector to the ray trajectory at s



The choice $H = \left|\mathbf{p}\right| - n\left(\mathbf{q}\right)$ yields

$$\nabla_{\mathbf{q}} H \equiv \frac{\partial H}{\partial \mathbf{q}} = -\nabla_{\mathbf{q}} n\left(\mathbf{q}\right) \qquad \nabla_{\mathbf{p}} H \equiv \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}\left(s\right)}{\left|\mathbf{p}\left(s\right)\right|}$$

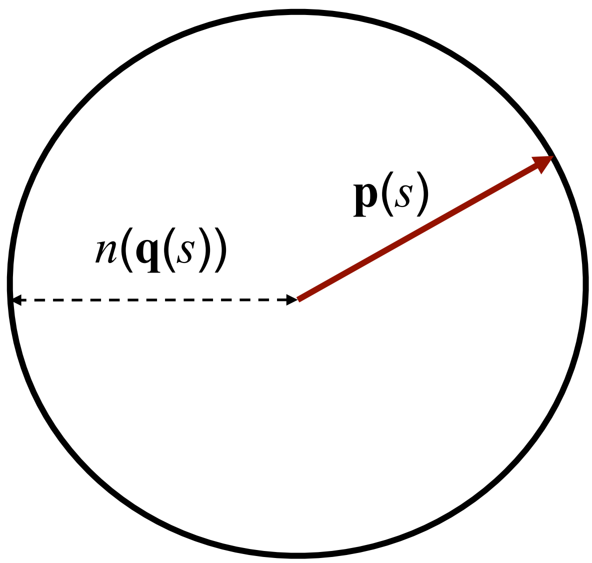
Therefore, the equations of motion become

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}s} = \frac{\partial H}{\partial \mathbf{p}} \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = -\frac{\partial H}{\partial \mathbf{q}}$$

Since the ray trajectory satisfies a set of Hamiltonian equations on the quantity *H*, it follows that *H* is conserved.

The actual value of *H*=const. is arbitrary.

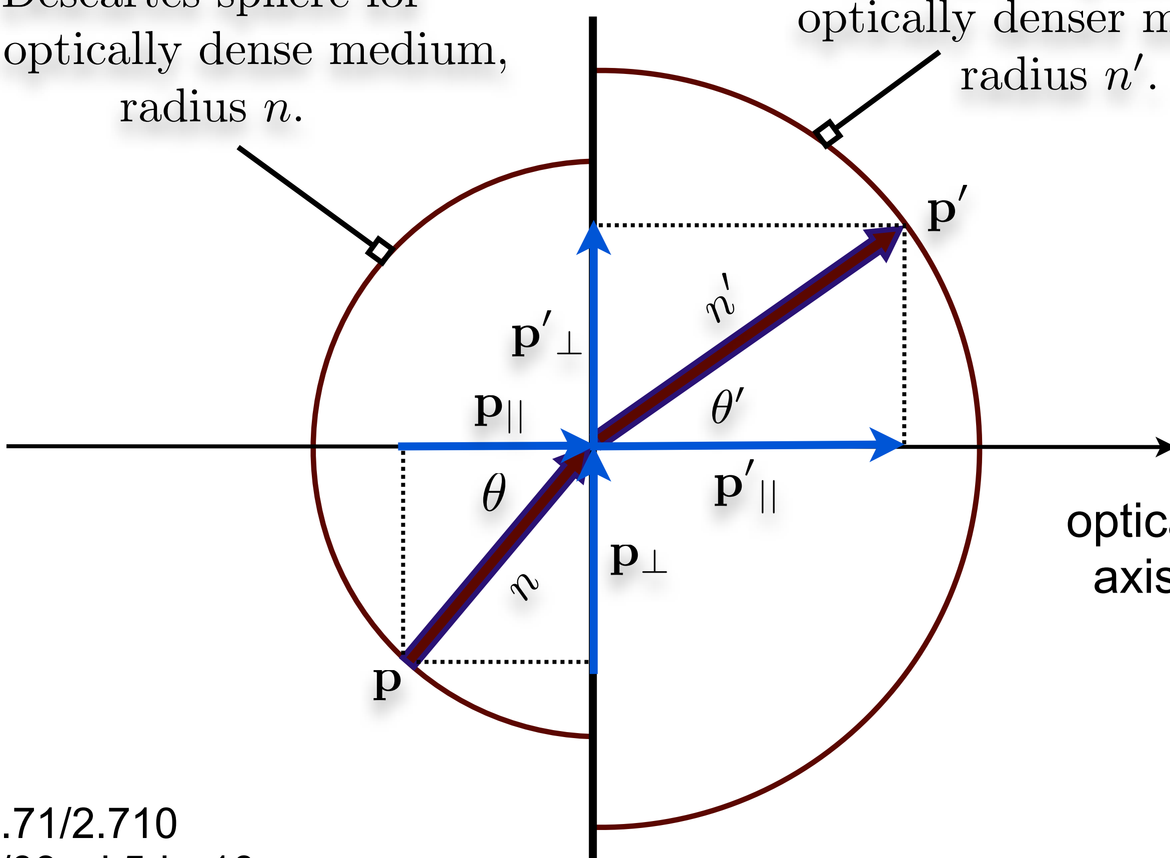
The ray Hamiltonian and the Descartes sphere



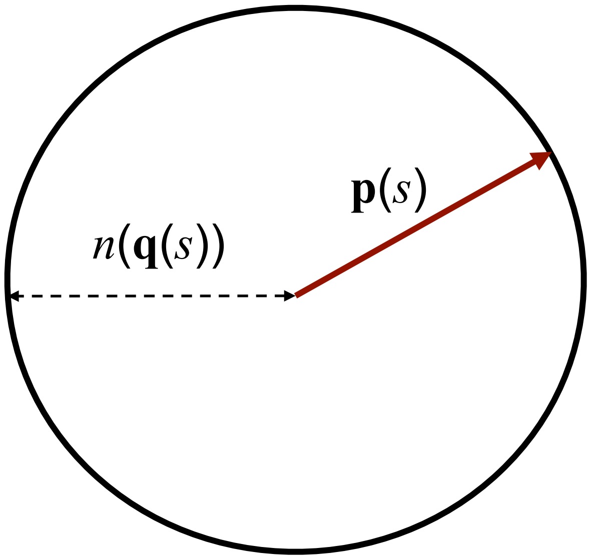
$$H = |\mathbf{p}| - n(\mathbf{q}) = 0$$

The ray momentum $\mathbf{p}$ is constrained to lie on a sphere of radius n at any ray location $\mathbf{q}$ along the trajectory s

Application:
Snell's law of refraction



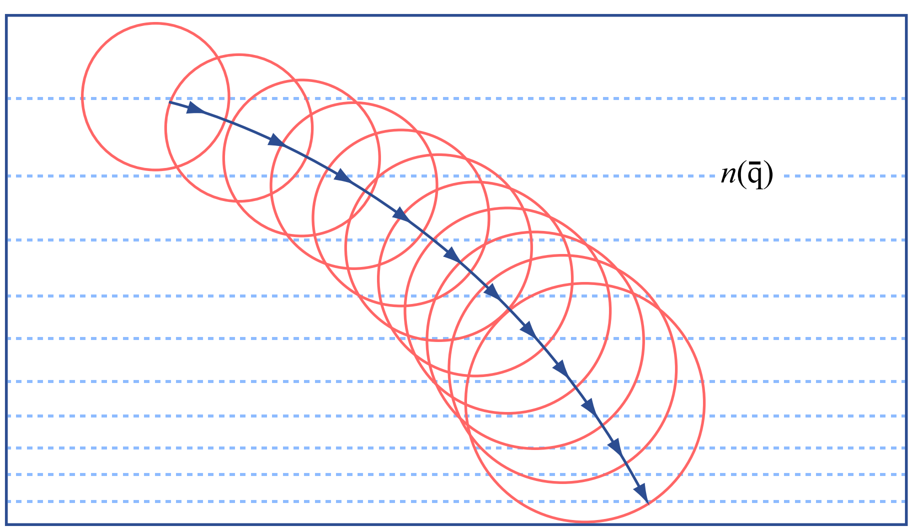
The ray Hamiltonian and the Descartes sphere



$$H = \left|\mathbf{p}\right| - n\left(\mathbf{q}\right) = 0$$

The ray momentum $\mathbf{p}$ is constrained to lie on a sphere of radius n at any ray location $\mathbf{q}$ along the trajectory s

<u>Application</u>:
propagation in a GRIN medium



The Descartes sphere radius is proportional to n(**q**); as the rays propagate, the lateral momentum is preserved by gradually changing the ray orientation to match the Descartes spheres.

Figure by MIT OpenCourseWare. Adapted from Fig. 1.5 in Wolf, Kurt B. Geometric Optics in Phase Space. New York, NY: Springer, 2004.

Hamiltonian analogies: optics *vs* mechanics

Hamiltonian of mechanical system $$H_{\mathrm{m}} = \frac{|\mathbf{p}|^2}{2m} + V(\mathbf{q})$$

Energy $$E = H_{\mathrm{m}} \quad \text{(conserved)}$$

Momentum $$\mathbf{p} = m\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{|\mathbf{p}|}{m}$$ Velocity

Mechanical Hamiltonian equations

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}s} = \frac{\mathbf{p}}{|\mathbf{p}|} \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = -\frac{m}{|\mathbf{p}|}\frac{\partial V(\mathbf{q})}{\partial \mathbf{q}}$$

Optical Hamiltonian equations

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}s} = \frac{\mathbf{p}}{|\mathbf{p}|} \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = \frac{\partial n(\mathbf{q})}{\partial \mathbf{q}}$$

Analogous if: $$\frac{\partial V(\mathbf{q})}{\partial \mathbf{q}} \leftrightarrow -\frac{|\mathbf{p}|}{m}\frac{\partial n(\mathbf{q})}{\partial \mathbf{q}} = -\frac{n(\mathbf{q})}{m}\frac{\partial n(\mathbf{q})}{\partial \mathbf{q}} =$$
$$= -\frac{1}{2m}\frac{\partial n^2(\mathbf{q})}{\partial \mathbf{q}}.$$

$$\Rightarrow V(\mathbf{q}) \leftrightarrow -\frac{n^2(\mathbf{q})}{2m} + \text{const.}$$

Choose const. $$= -\frac{E+1}{2m} \Rightarrow 2m\left[E - V(\mathbf{q})\right] \leftrightarrow n^2(\mathbf{q}) - 1$$

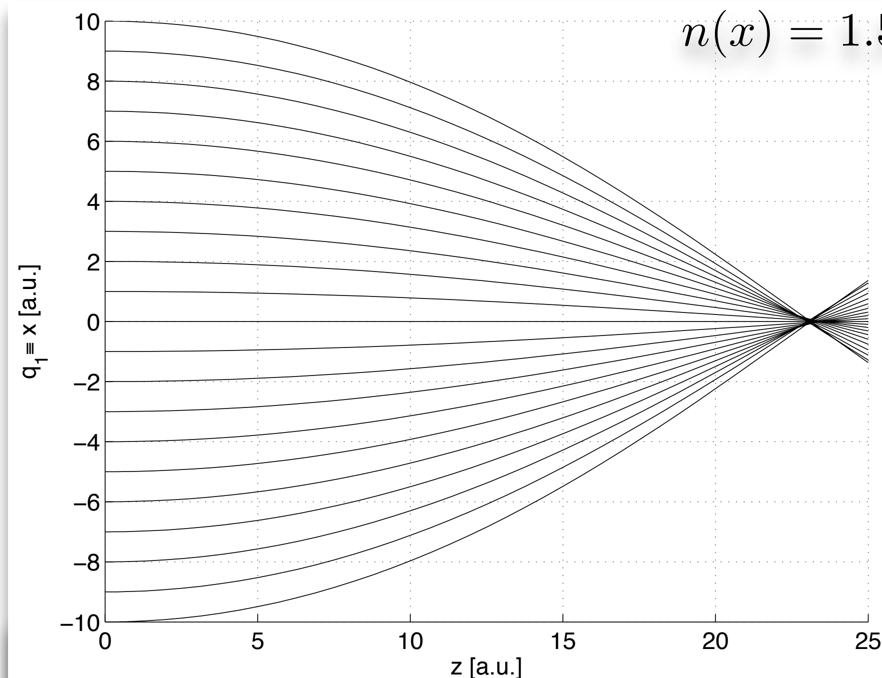
and $$E_{\text{kinetic}} \propto E - V(\mathbf{q}) > 0 \leftrightarrow n(\mathbf{q}) > 1.$$

physically allowable kinetic energy

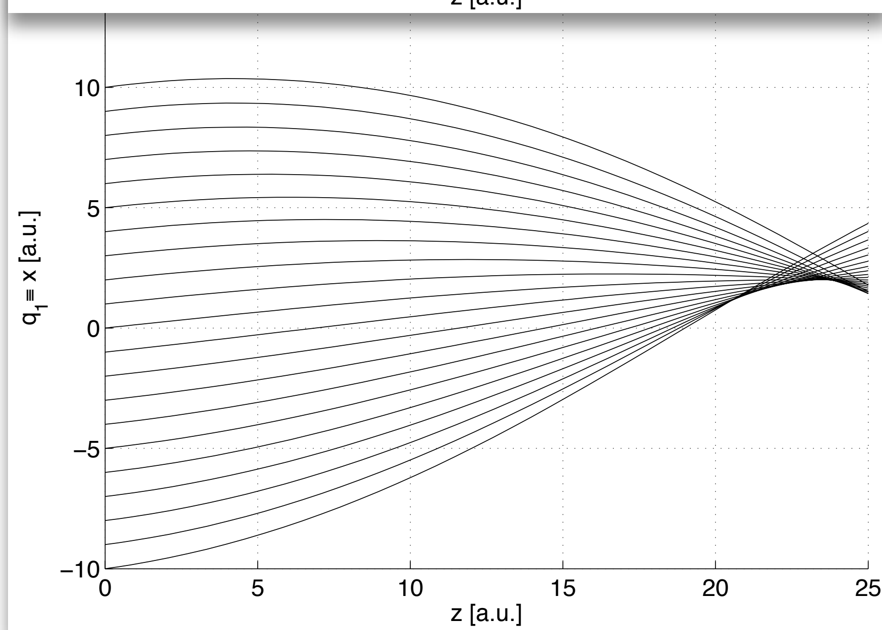
physically allowable refractive index

Example: Hamiltonian ray tracing of quadratic GRIN

$$n(x) = 1.5 - 0.0033\, x^2$$



on-axis incidence



off-axis incidence, $p_x(0)$=0.2

<u>Further reading:</u>

- M. Born and E. Wolf, *Principles of Optics*, Cambrige University Press, 7th edition, sections 4.1-4.2
- K. B. Wolf, *Geometrical Optics on Phase Space,* Springer, chapters 1, 2
- K. Tian, *Three-dimensional (3D) optical information processing,* PhD dissertation, MIT 2006.