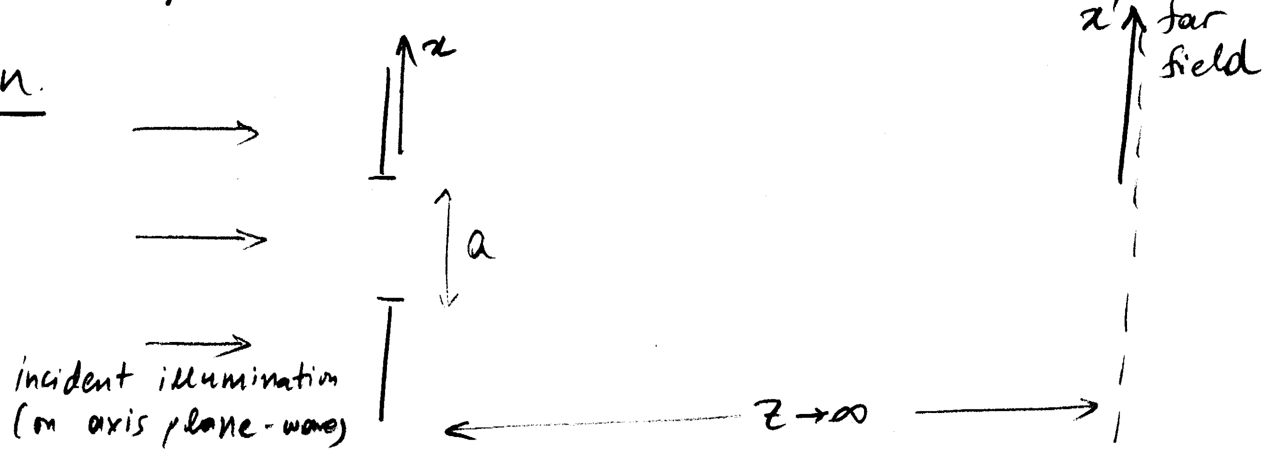
# Practice Problem Set #2.

(1) What is the Fraunhofer diffraction pattern of a 1-D slit of size a?

Soln.



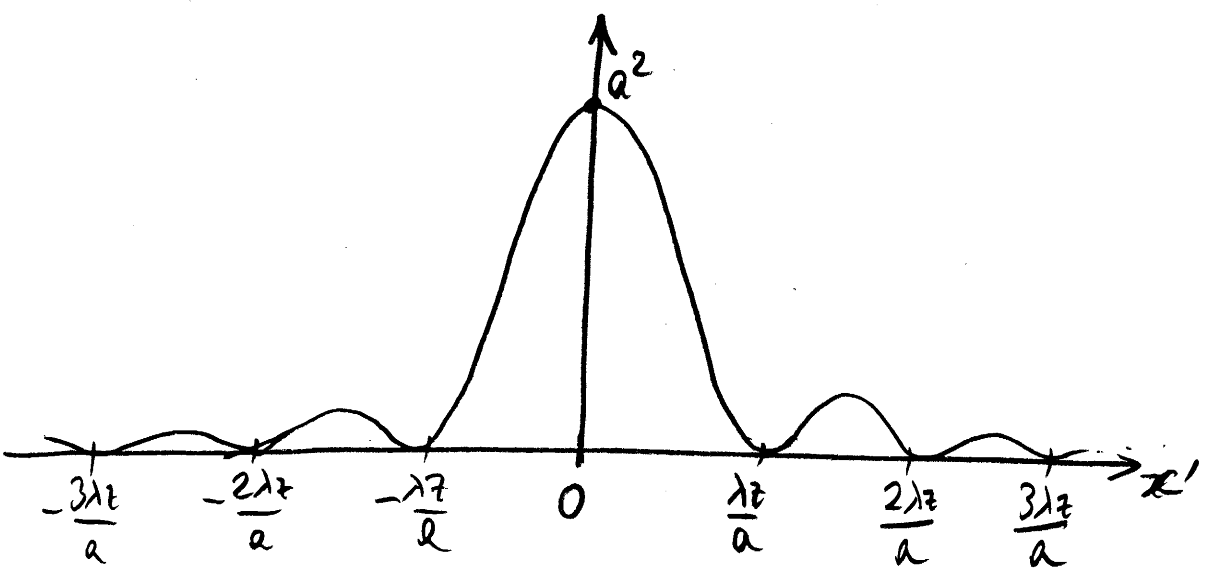

slit description (1D) $\quad f(x) = \mathrm{rect}\left(\frac{x}{a}\right)$

Fourier transform of slit: $\quad F(u) = a\,\mathrm{sinc}(au)$

Diffracted far field : $\quad g(x') = e^{(i\pi)\frac{x'^2+y'^2}{\lambda z}} F\left(\frac{x'}{\lambda z}\right)$

Fraunhofer diffraction pattern (Intensity)

$$|g(x')|^2 = a^2 \mathrm{sinc}^2\left(\frac{a x'}{\lambda z}\right)$$



② What is the Fraunhofer diffraction pattern of a sinusoidal amplitude grating

$$f(x) = \frac{1}{2}\left[1 + \cos\left(2\pi \frac{x}{\Lambda}\right)\right]$$

$\Lambda$ is the grating period.

<u>Soln</u>

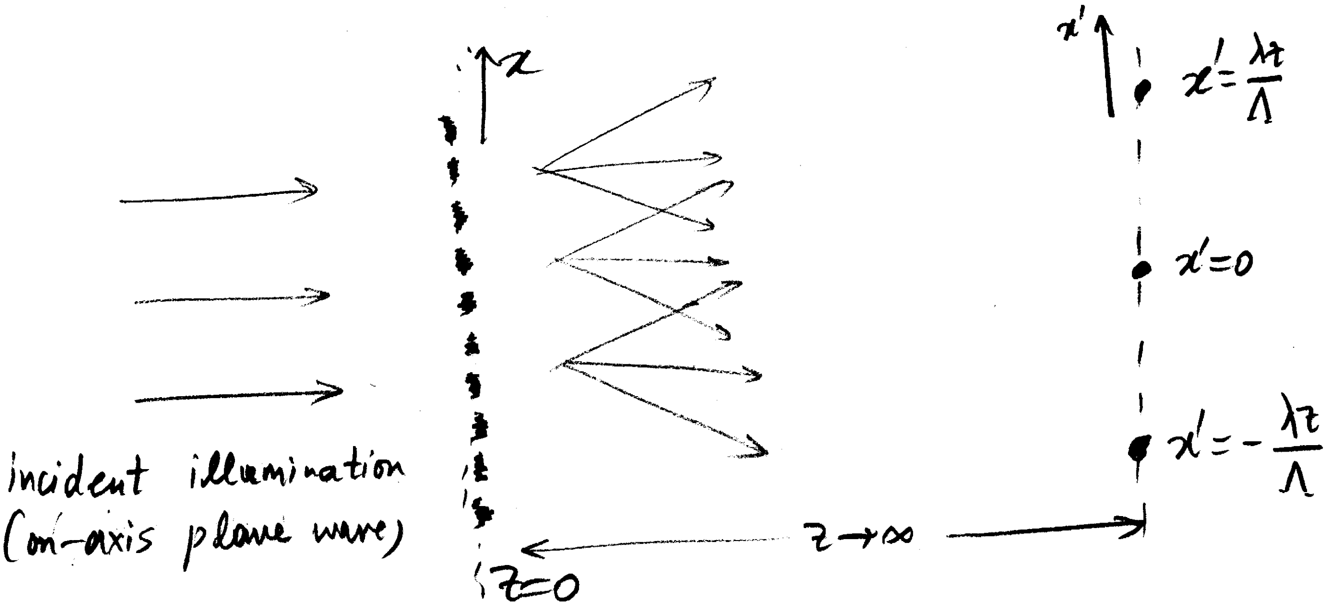
$$f(x) = \frac{1}{2} + \frac{1}{2}\cos\left(2\pi\frac{x}{\Lambda}\right)$$

$$= \underbrace{\overbrace{\frac{1}{2}}^{\text{plane wave } u=0}}_{\boxed{\text{DC-term or 0-th order}}} + \underbrace{\overbrace{\frac{1}{4}e^{i2\pi\frac{x}{\Lambda}}}^{\text{plane wave, } u=\frac{1}{\Lambda}} + \overbrace{\frac{1}{4}e^{-i2\pi\frac{x}{\Lambda}}}^{\text{plane wave, } u=-\frac{1}{\Lambda}}}_{\boxed{\text{diffracted orders}}}$$

$$\Rightarrow F(u) = \frac{1}{2}\delta(u) + \frac{1}{4}\delta\left(u-\frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(u+\frac{1}{\Lambda}\right)$$

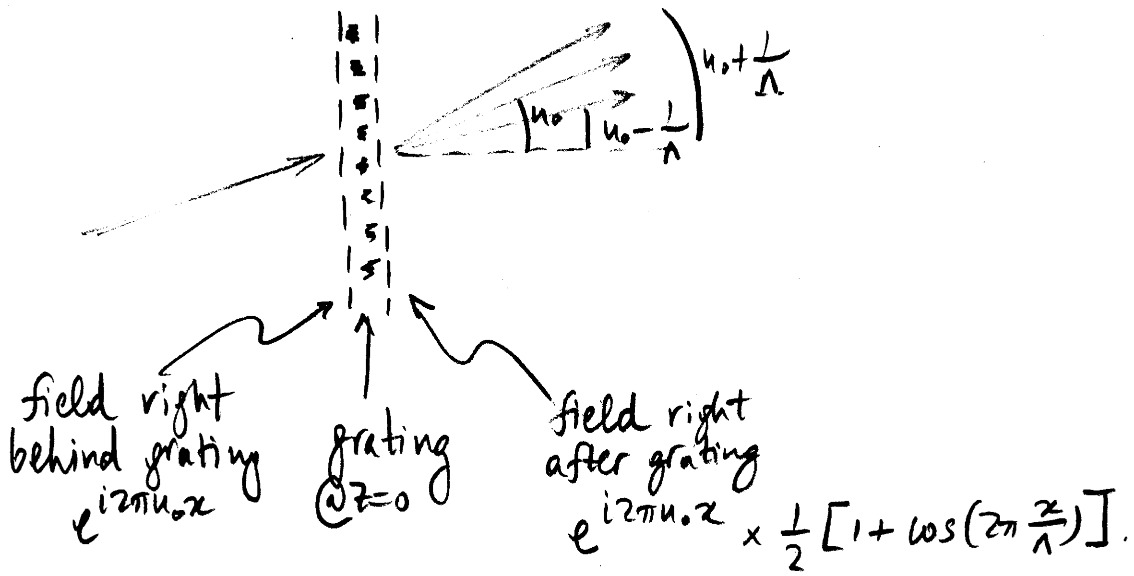
$$\Rightarrow g(x') = e^{i\pi\frac{x'^2+y'^2}{\lambda z}}\left[\frac{1}{2}\delta\left(\frac{x'}{\lambda z}\right) + \frac{1}{4}\delta\left(\frac{x'}{\lambda z}-\frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(\frac{x'}{\lambda z}+\frac{1}{\Lambda}\right)\right]$$

// <u>Note</u> without being too rigorous mathematically, we treat the intensity corresponding to the $\delta$-function field as a "very bright & sharp" spot. //



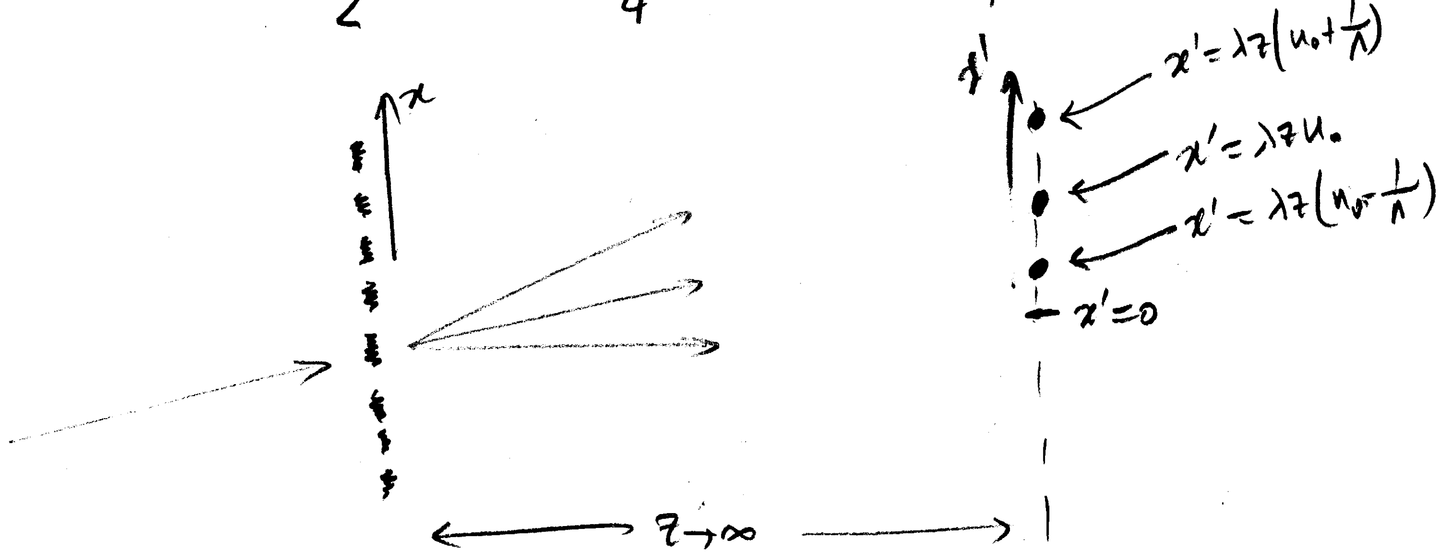
(3) How does the result of problem #2 change if the illumination is a plane wave incident at angle $\theta_0$ wrt the optical axis? $(\theta_0 \ll 1)$.

Soln. Let $u_0 = \frac{\sin\theta_0}{\lambda}$ $\rightarrow$ plane wave is $e^{i2\pi u_0 x}$ (@ $z=0$).




$$\mathcal{F}\left\{ e^{i2\pi u_0 x} \times \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{\Lambda}\right)\right]\right\} =$$

$$= \frac{1}{2}\delta(u-u_0) + \frac{1}{4}\delta\left(u-u_0-\frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(u-u_0+\frac{1}{\Lambda}\right)$$



(4) What is the Fraunhofer pattern of a truncated sinusoidal amplitude grating

$$f(x) = \frac{1}{2}\left[1+\cos\left(2\pi \frac{x}{\Lambda}\right)\right] \mathrm{rect}\left(\frac{x}{a}\right)$$

Assume $a \gg \Lambda$.

<u>Soln</u>

Let $f_1(x) = \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{\Lambda}\right)\right] \Rightarrow F_1(u) = \frac{1}{2}\delta(u) + \frac{1}{4}\delta\left(u-\frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(u+\frac{1}{\Lambda}\right)$

$f_2(x) = \mathrm{rect}\left(\frac{x}{a}\right) \Rightarrow F_2(u) = a\,\mathrm{sinc}(au)$.

According to the convolution theorem

$$\mathcal{F}\{f_1(x)\, f_2(x)\} = F_1(u) * F_2(u)$$

recall $\delta(u-u_0) * A(u) = \int_{-\infty}^{\infty} \delta(u-u_0)\, A(u'-u)\, du = A(u'-u_0)$

$$= \left[\frac{1}{2}\delta(u) + \frac{1}{4}\delta\left(u-\frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(u+\frac{1}{\Lambda}\right)\right] * a\,\mathrm{sinc}(au)$$

$$= \frac{1}{2} a\,\mathrm{sinc}(au) + \frac{1}{4} a\,\mathrm{sinc}\left(a\left(u-\frac{1}{\Lambda}\right)\right) + \frac{1}{4} a\,\mathrm{sinc}\left(a\left(u+\frac{1}{\Lambda}\right)\right)$$

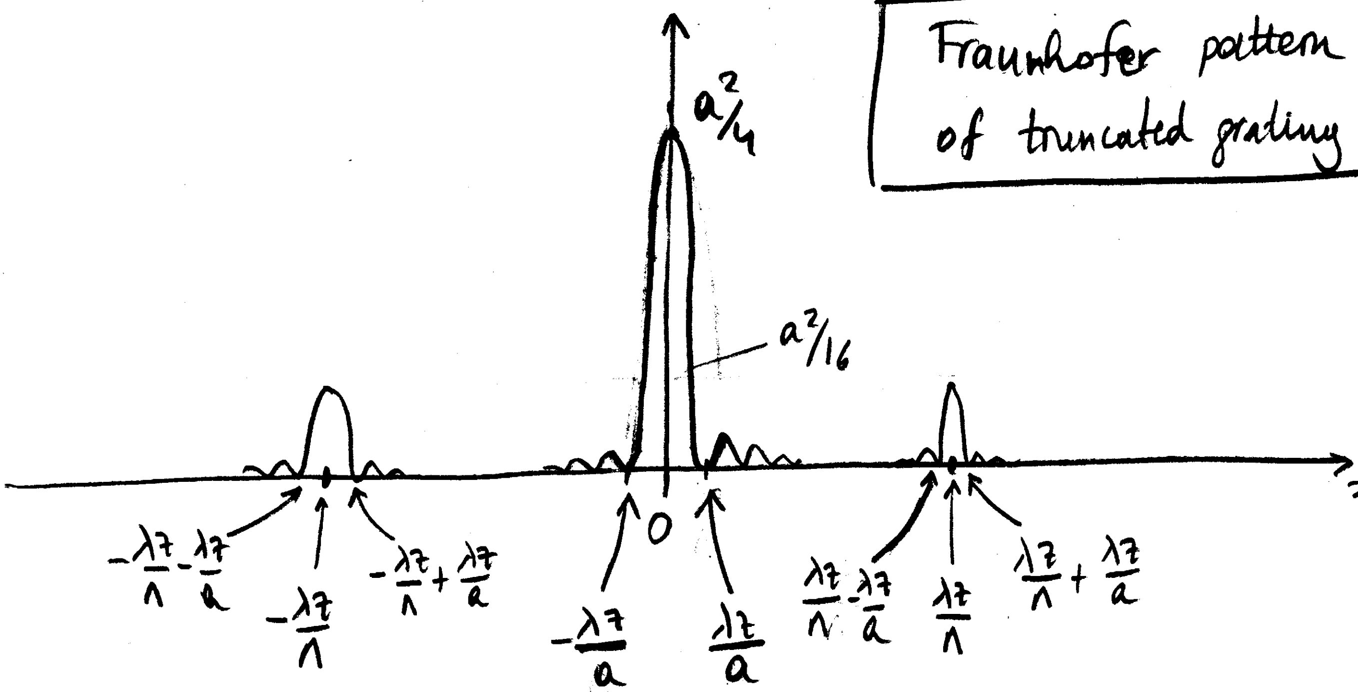
<u>Note</u>: When you take $|\,|^2$, cross-terms can be ignored. Why?

$$|y(x')|^2 \simeq \frac{a^2}{4}\mathrm{sinc}^2\left(\frac{ax'}{\lambda z}\right) + \frac{a^2}{16}\mathrm{sinc}^2\left(a\left(\frac{x'}{\lambda z}-\frac{1}{\Lambda}\right)\right) + \frac{a^2}{16}\mathrm{sinc}^2\left(a\left(\frac{x'}{\lambda z}+\frac{1}{\Lambda}\right)\right)$$

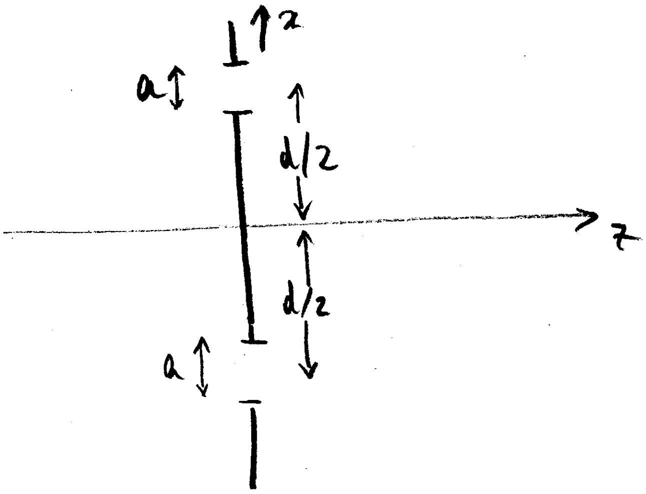
↑

Fraunhofer pattern (intensity)

Fraunhofer pattern of truncated grating




(5) What is the Fraunhofer diffraction pattern of two identical slits (width a) separated by a distance $d \gg a$?




$$f(x) = \text{rect}\left(\frac{x-\frac{d}{2}}{a}\right) + \text{rect}\left(\frac{x+\frac{d}{2}}{a}\right)$$

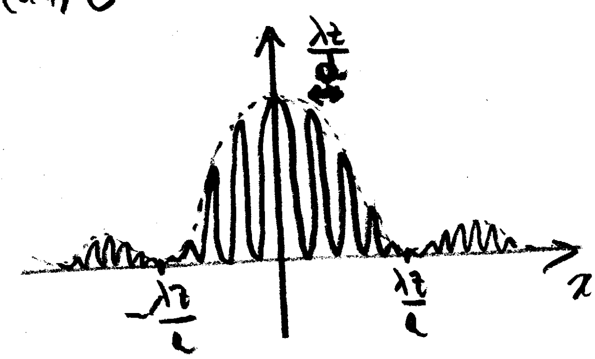
use scaling & shift thms + linearity

$$F(u) = a\,\text{sinc}(au)\,e^{-i2\pi u\frac{d}{2}} + a\,\text{sinc}(au)\,e^{i2\pi u\frac{d}{2}}$$

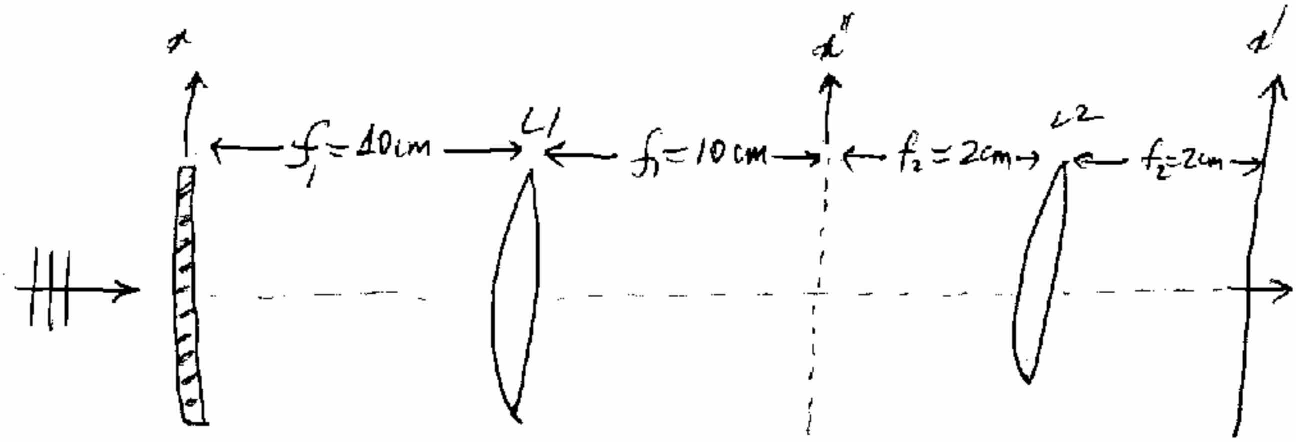
$$= 2a\,\text{sinc}(au)\cos(\pi u d)$$

$$|g(x')|^2 = 4a^2\,\text{sinc}^2\left(\frac{ax'}{\lambda z}\right)\cos\left(\frac{\pi x' d}{\lambda z}\right)$$

("modulated" sinc pattern)



(6)




$$t(x) = \frac{1}{2}\left[1 + \cos\left(2\pi \frac{x}{10\mu m}\right)\right]$$

In the 4F system shown above, the sinusoidal transparency $t(x)$ is illuminated by a monochromatic plane-wave on-axis, at wavelength $\lambda = 1\mu m$. Describe quantitatively the fields at the Fourier plane $(x'')$ and the output plane $(x')$.

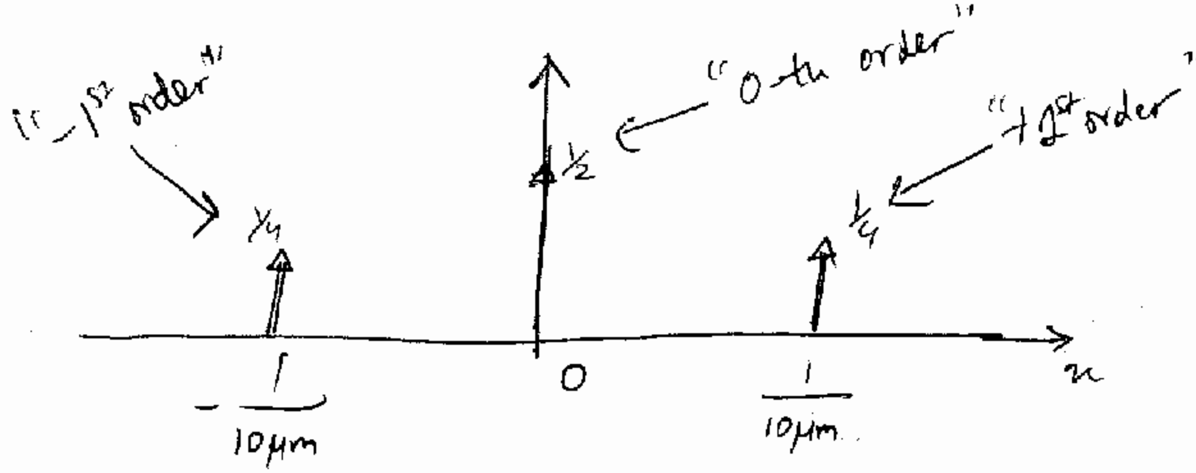
Answer.

The field immediately past the transparency is

$$g_{in}(x) = \underbrace{1}_{\text{plane-wave on-axis}} \cdot \underbrace{t(x)}_{\text{transmission function of transparency}}$$

The field at the Fourier plane is $G_{in}\left(\frac{x''}{\lambda f_1}\right)$ where $G_{in}(u)$ is the Fourier transform of $g_{in}(x)$, i.e.

$$G_{in}(u) = \frac{1}{2}\left[\delta(u) + \frac{1}{2}\delta\left(u - \frac{1}{10\mu m}\right) + \frac{1}{2}\delta\left(u + \frac{1}{10\mu m}\right)\right]$$



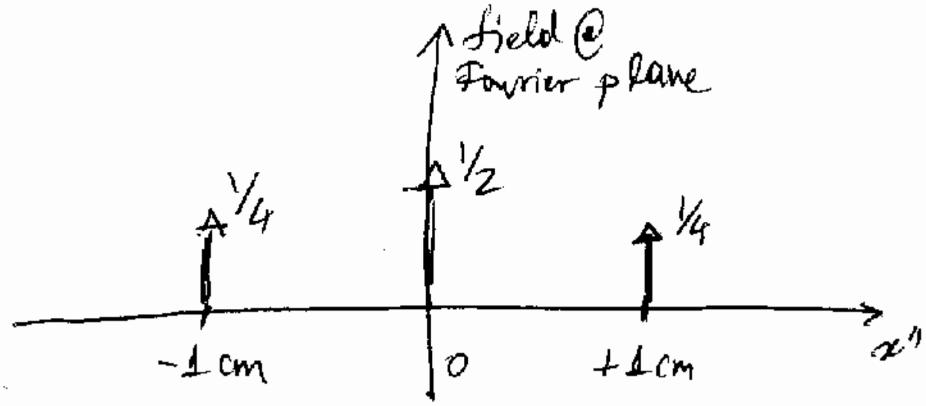

The field at the ~~output~~ Fourier plane will consist of three peaks corresponding to the three $\delta$-functions of the Fourier transform. The locations of these peaks are found as follows:

$+1^{st}$ order: $u = \frac{1}{10\mu m} \Rightarrow \frac{x''}{\lambda f_1} = \frac{1}{10\mu m} \Rightarrow$

$$\Rightarrow x'' = \frac{\lambda f_1}{10\mu m} = \frac{1\mu m \times 10 cm}{10\mu m} = 1\,cm$$

0 th order: $x'' = 0$

$-1^{st}$ order: $x'' = \ldots = -1\,cm$



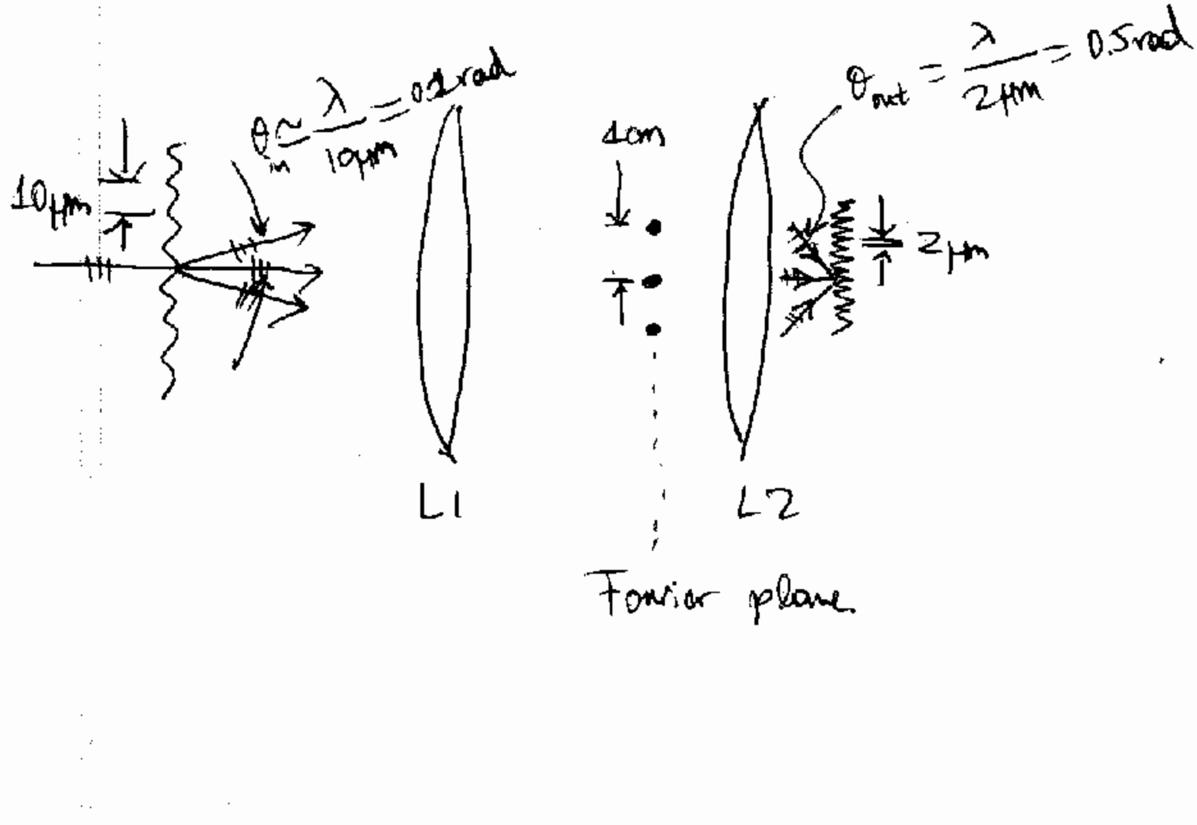
The field at the output plane is $g_{in}\left(-\frac{f_1}{f_2}x\right) =$

$$= g_{in}\left(-\frac{10\ \text{cm}}{2\ \text{cm}}x'\right) = g_{in}(-5x')$$

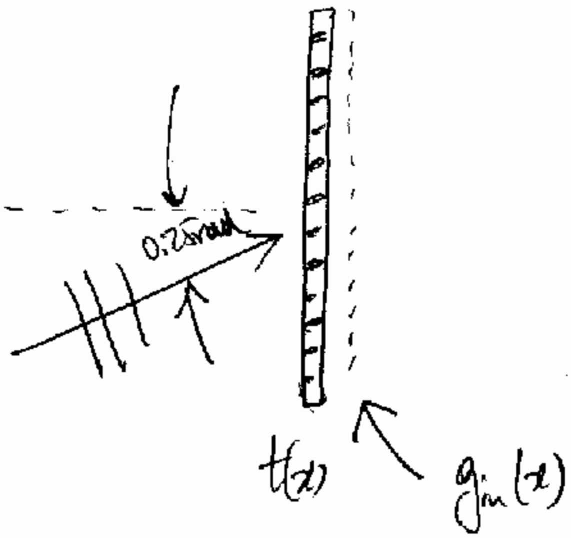
$$= \frac{1}{2}\left[1 + \cos\left(2\pi\frac{x'}{2\,\mu m}\right)\right]$$

So the field has been <u>laterally demagnified</u> by this imaging system.

Notice that lateral demagnification implies angular magnification according to the following diagram:



(7) Repeat the calculations of problem 6, except this time with illumination of a tilted plane wave incident at angle $\theta \approx 0.25$ rad w.r.t. the optical axis




Answer This time

$$g_{in}(x) = \underbrace{e^{i\frac{2\pi}{\lambda}\sin\theta \cdot x}}_{\text{plane-wave off-axis}} \cdot \underbrace{t(x)}_{\text{transmission function of transparency}}$$

where $\sin\theta \approx \theta = 0.25$ (paraxial approximation), Therefore

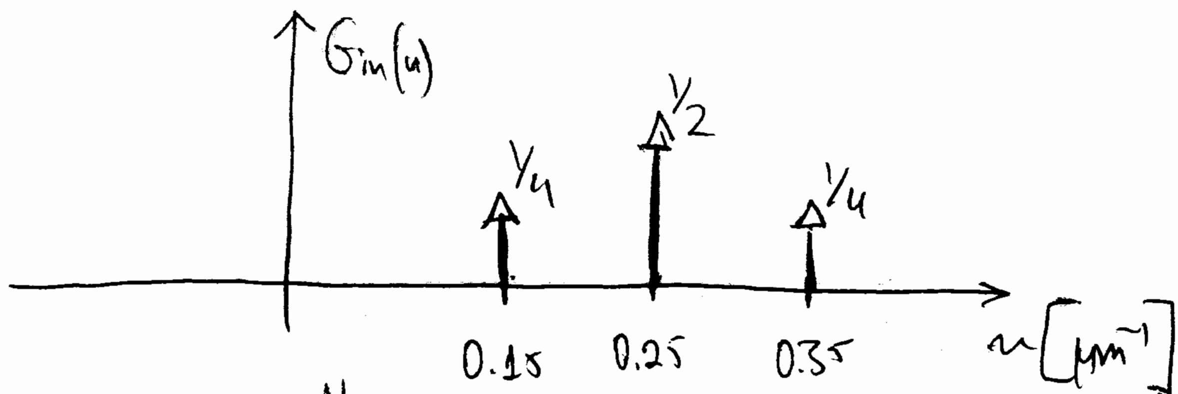
$$g_{in}(x) = e^{i2\pi\frac{0.25x}{1\mu m}} \times \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{10\,\mu m}\right)\right] =$$

$$= \frac{1}{2}e^{i2\pi\frac{0.25x}{1\mu m}}\left[1+\frac{1}{2}e^{i2\pi\frac{x}{10\mu m}}+\frac{1}{2}e^{-i2\pi\frac{x}{10\mu m}}\right]$$

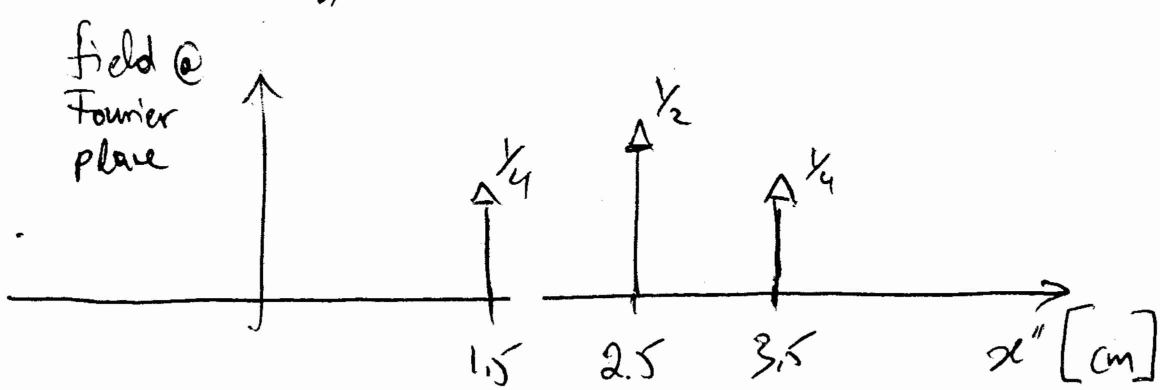
$$= \frac{1}{2} e^{i2\pi \frac{x}{4\mu m}} + \frac{1}{4} e^{i2\pi\left(\frac{1}{10}+\frac{1}{4}\right)\frac{x}{\mu m}} + \frac{1}{4} e^{i2\pi\left(-\frac{1}{10}+\frac{1}{4}\right)\frac{x}{\mu m}}$$

$$\hookrightarrow G_{in}(u) = \frac{1}{2}\delta\left(u - \frac{1}{4\mu m}\right) + \frac{1}{4}\delta\left(u - 0.35\,\mu m^{-1}\right) + \frac{1}{4}\delta\left(u - 0.15\,\mu m^{-1}\right)$$

$\hookrightarrow$




$\Downarrow$



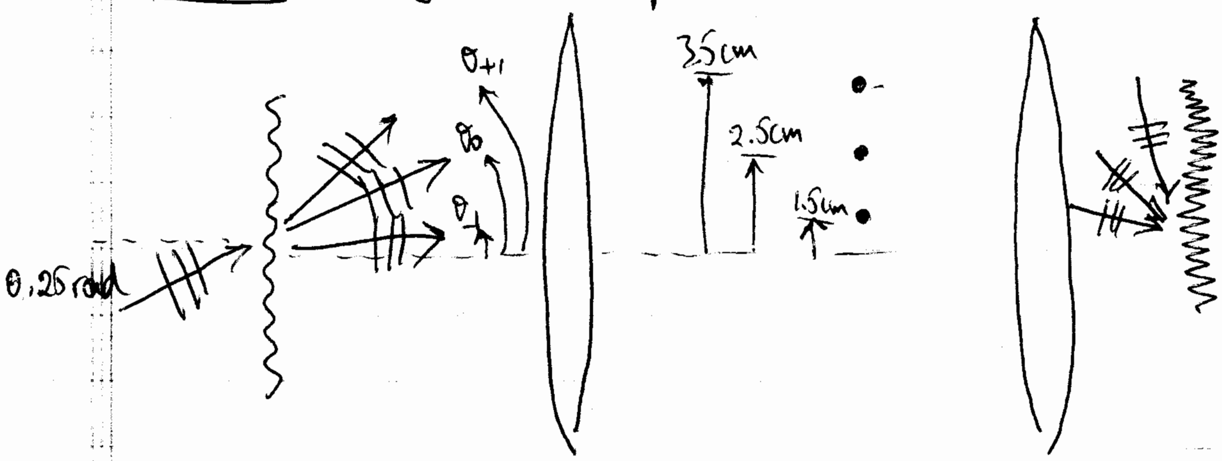

Note 1. We can get this result faster by use of the shift theorem of Fourier transforms:

$$g_{in}(x) = e^{-i2\pi\frac{x}{4\mu m}} \cdot t(x)$$

$$\Rightarrow G_{in}(u) = T\left(u - \frac{1}{4\mu m}\right)$$

So it is the result of Problem 6 shifted to the right by $0.25\,\mu m^{-1}$.

Note 2. Physical explanation




The diffracted order angles are

$$\theta_{-1} \simeq 0.25 - 0.1 = 0.15 \text{ rad}$$

$$\theta_0 \simeq 0.25 + 0 = 0.25 \text{ rad}$$

$$\theta_{+1} \simeq 0.25 + 0.1 = 0.35 \text{ rad}$$

Compare this with the diagram in the answer to Problem 6 !

Field @ output plane:

$$g_{out}(x') = g_{in}(-5x') = e^{-i2\pi 1.25 \frac{x}{1\mu m}} \left[1 + \cos\left(2\pi \frac{x}{2\mu m}\right)\right]$$

Notice the intensity $|g_{out}(x')|^2$ is the same as in problem 6. The extra phase factor indicates that the overall output field is propagating "downwards."

(8) Repeat Problem 7 with a truncated grating of size 1mm

Answer. Now the input field is

$$g_{in} = e^{i2\pi \frac{x}{4\mu m}} \times \frac{1}{2}\left[1+\cos\left(2\pi \frac{x}{10\mu m}\right)\right] \times \underbrace{\mathrm{rect}\left(\frac{x}{1mm}\right)}_{\text{truncates the grating to total size of 1mm}}$$

We need to compute $G_{in}(u)$. We will do it in two steps

1st) compute the FT of $\frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{10\mu m}\right)\right] \times \mathrm{rect}\left(\frac{x}{1mm}\right)$ by applying the convolution theorem

2nd) apply the shift theorem to take into account the $e^{i2\pi\frac{x}{4\mu m}}$ factor

1st step:

$$\frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{10\mu m}\right)\right] \quad \times \quad \mathrm{rect}\left(\frac{x}{1mm}\right)$$

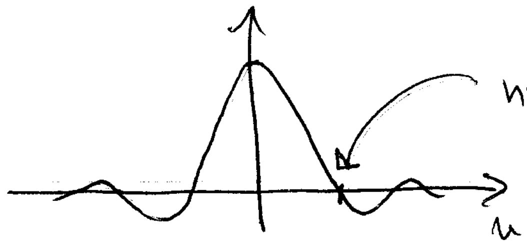
$$\Big\downarrow \mathcal{F} \qquad\qquad \Big\downarrow \circ \qquad\qquad \Big\downarrow \mathcal{F}$$

$$\left[\frac{1}{2}\delta(u)+\frac{1}{4}\delta\left(u-\frac{1}{10\mu m}\right)+\frac{1}{4}\delta\left(u+\frac{1}{10\mu m}\right)\right] * \underbrace{\cancel{(1mm)}}_{\text{neglect from now on}} \mathrm{sinc}(1mm \cdot u)$$

=

$$= \frac{1}{2}\,\mathrm{sinc}\left(1\mathrm{mm}\cdot u\right) + \frac{1}{4}\,\mathrm{sinc}\left(\left[1\mathrm{mm}\left(u-\frac{1}{10\mu m}\right)\right]\right) + \frac{1}{4}\,\mathrm{sinc}\left[1\mathrm{mm}\left(u+\frac{1}{10\mu m}\right)\right]$$

Let's plot this before continuing

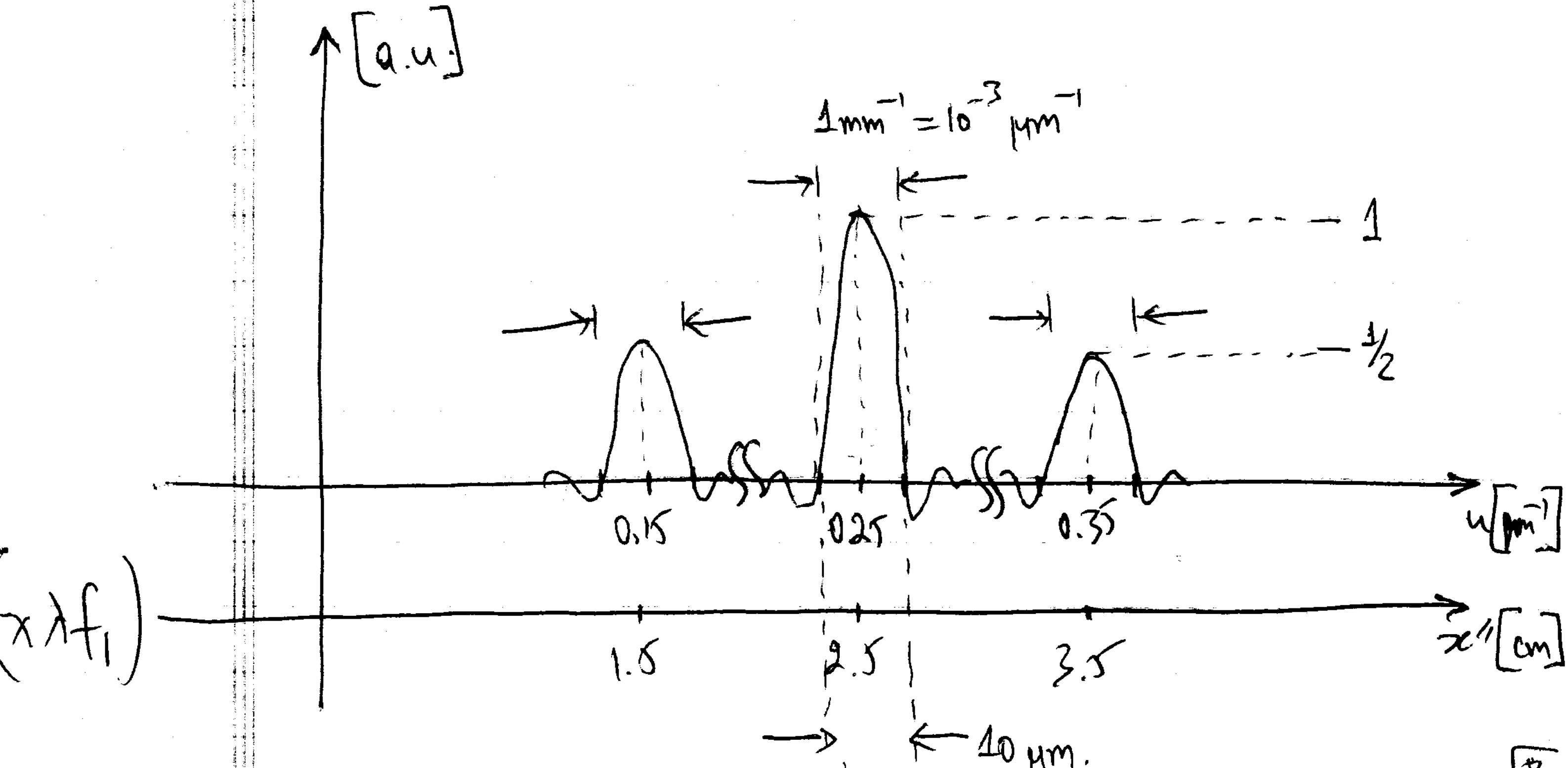



null when $1\mathrm{mm}\cdot u = 1 \Rightarrow$

$\Rightarrow u = 1\,\mathrm{mm}^{-1}$

2nd step. By direct application of the shift theorem (see also Pr. 7)

$$G_{in}(u) = \frac{1}{2}\,\mathrm{sinc}\left[1\mathrm{mm}\left(u-\frac{0.25}{\mu m}\right)\right] + \frac{1}{4}\,\mathrm{sinc}\left[1\,\mathrm{mm}\left(u-\frac{0.35}{\mu m}\right)\right] +$$
$$+ \frac{1}{4}\,\mathrm{sinc}\left[1\,\mathrm{mm}\left(0-\frac{0.15}{\mu m}\right)\right]$$



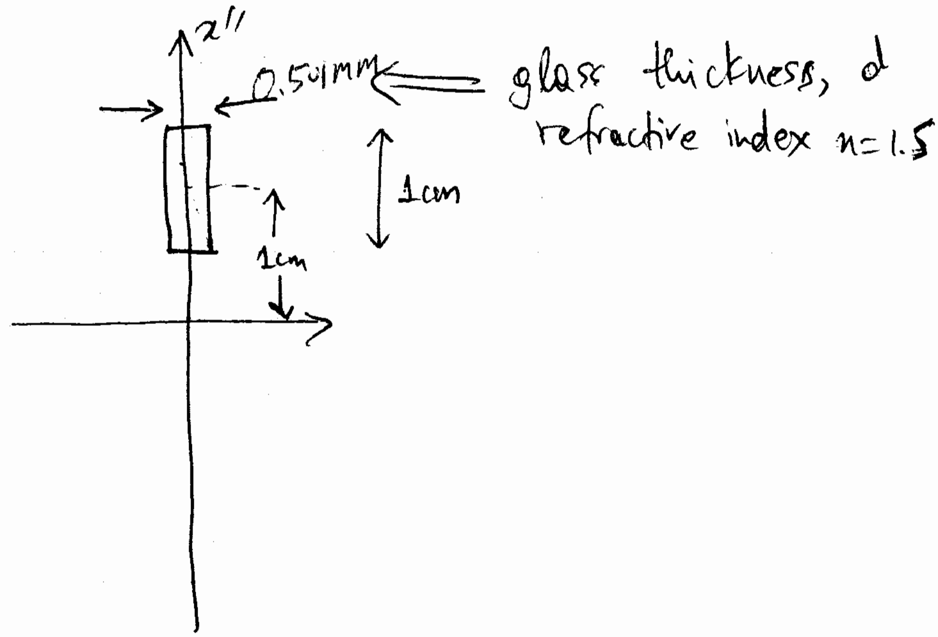
Field @ output plane:

$$g_{out}(x') = g_{in}(-5x') =$$

$$= e^{-i2\pi\, 1.25 \frac{x}{\mu m}} \left[1+\cos\left(2\pi \frac{x}{2\mu m}\right)\right] \mathrm{rect}\left(\frac{x}{0.2mm}\right)$$

(it is still a truncated grating, shrunk by a factor of 5 compared to the original grating).

(9) In the optical system of Problem 6 (infinitely large grating, on-axis plane wave illumination) we place ~~the~~ a small piece of glass at the Fourier plane as follows:




What is the output field? What is the output intensity?

Answer The piece of glass delays the $+1^{st}$ order field by a phase equal to

$$\phi = 2\pi \frac{(n-1)\, d}{\lambda} = 2\pi \frac{0.5 \times 501\,\mu m}{1\,\mu m} =$$

$$= 501\pi \iff \phi = \pi \quad \left(\begin{matrix}\text{phase is}\\ \text{mod } 2\pi\end{matrix}\right)$$

So, ~~after the~~ immediately after the Fourier plane, the field is

$$\frac{1}{2}\left[\delta(u) + \frac{1}{2}e^{i\phi}\,\delta\left(u - \frac{1}{10\,\mu m}\right) + \frac{1}{2}\delta\left(u + \frac{1}{10\,\mu m}\right)\right]$$

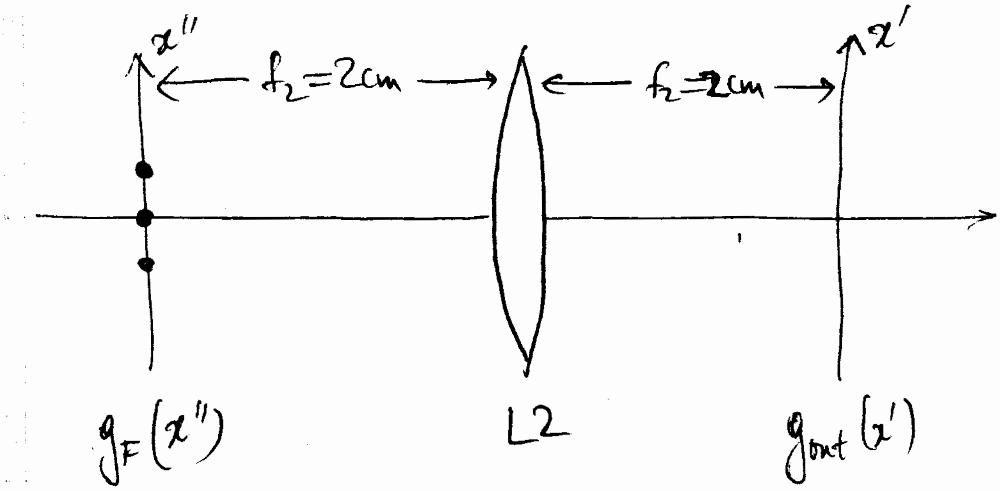
phase delay, $e^{i\phi} = e^{i\pi} = -1$

$$= \frac{1}{2}\left[\delta(u) - \frac{1}{2}\delta\left(u - \frac{1}{10\mu m}\right) + \frac{1}{2}\delta\left(u + \frac{1}{10\mu m}\right)\right]$$

$\Downarrow$ in coordinates $x'' = \lambda f_1 u$

$$\frac{1}{2}\left[\delta(x'') - \frac{1}{2}\delta(x'' - 1cm) + \frac{1}{2}\delta(x'' + 1cm)\right] \equiv g_F(x'')$$

Now consider the second half of the 4F system




Since L2 is acting as a Fourier-transforming lens,

$$g_{out}(x') = G_F\left(\frac{x''}{\lambda f_2}\right)$$

$$= \frac{1}{2}\left[1 - \frac{1}{2}e^{i2\pi \frac{x'}{2\mu m}} + \frac{1}{2}e^{-i2\pi \frac{x'}{2\mu m}}\right]$$

$$= \frac{1}{2}\left[1 - i\sin\left(2\pi \frac{x'}{2\mu m}\right)\right] \quad // \text{ field}$$

$$\left|g_{out}(x')\right|^2 = \frac{1}{2}\left[1 + \sin^2\left(2\pi \frac{x'}{2\mu m}\right)\right] \quad // \text{ intensity}.$$

**2.** **(20%)** Consider the 4F optical system shown in Figure B, where lenses L1, L2 are identical with focal length $f$. A thin transparency with arbitrary transmission function $t(x)$ is placed at the input plane of the system, and illuminated with a monochromatic, coherent plane wave at wavelength $\lambda$, incident on-axis. At the Fourier plane of the system we place the amplitude filter shown in Figure C. The filter is opaque everywhere except over two thin stripes of width $a$, located symmetrically around the $y''$ axis. The distance between the stripe centers is $x_0 > a$.

**2.a)** Which range of spatial frequencies must $t(x)$ contain for the system to transmit any light to its image plane?

**2.b)** Write an expression for the field at the image plane as convolution of $t(x)$ with the coherent impulse response of this system.

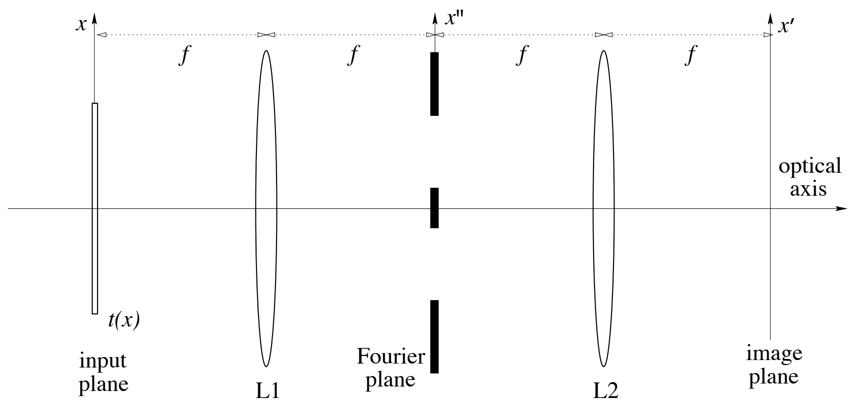



Figure B

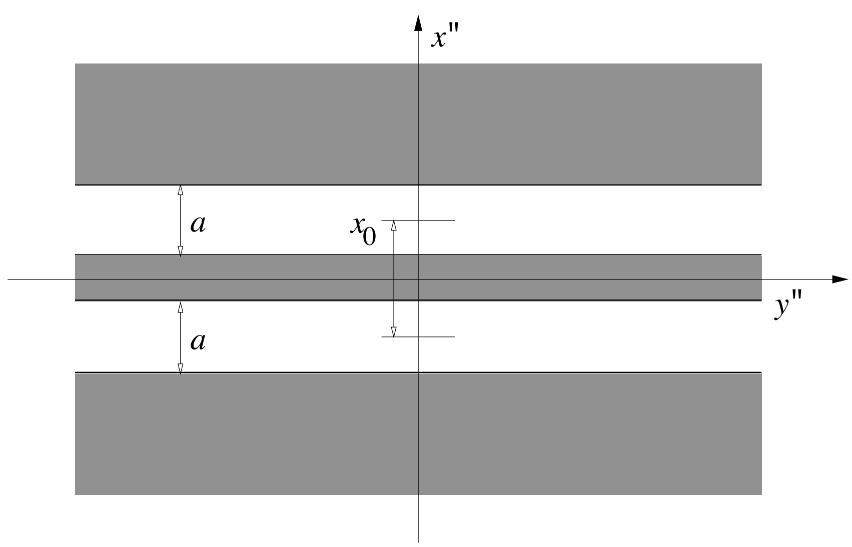



Figure C

Problem 2.

(2.a) The system admits frequencies

$$\frac{\frac{x_0}{2}-\frac{a}{2}}{\lambda f} \le u \le \frac{\frac{x_0}{2}+\frac{a}{2}}{\lambda f}$$

$$\frac{\frac{x_0}{2}+\frac{a}{2}}{\lambda f} \le u \le -\frac{\frac{x_0}{2}-\frac{a}{2}}{\lambda f}$$

(2.b) $$H(u) = \mathrm{rect}\left(\frac{u-\frac{x_0}{2\lambda f}}{\frac{a}{\lambda f}}\right) + \mathrm{rect}\left(\frac{u+\frac{x_0}{2\lambda f}}{\frac{a}{\lambda f}}\right)$$

$$\downarrow \mathcal{F} \qquad\qquad \downarrow \mathcal{F}$$

$$h(x) = \frac{a}{\lambda f}\,\mathrm{sinc}\left(\frac{ax'}{\lambda f}\right) e^{-i2\pi\frac{x_0 x'}{2\lambda f}} + \frac{a}{\lambda f}\,\mathrm{sinc}\left(\frac{ax'}{\lambda f}\right) e^{i2\pi\frac{x_0 x'}{2\lambda f}}$$

$$= \frac{2a}{\lambda f}\,\mathrm{sinc}\left(\frac{ax'}{\lambda f}\right)\cos\left(\frac{\pi x_0 x'}{\lambda f}\right)$$

output is

$$g(x') = \frac{2a}{\lambda f}\int_{-\infty}^{\infty} t(x'-x)\,\mathrm{sinc}\left(\frac{ax}{\lambda f}\right)\cos\left(\frac{\pi x_0 x}{\lambda f}\right)dx.$$

$$= t(x) * h(x)\Big|_{x'}$$