MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATERIALS SCIENCE AND ENGINEERING
CAMBRIDGE, MASSACHUSETTS 02139

3.22 MECHANICAL PROPERTIES OF MATERIALS
PROBLEM SET 6 SOLUTIONS

1. *(Hertzberg 1.11) A platform is suspended by two parallel rods, as shown in the sketch, with each rod being 1.28 cm in diameter. Rod A is manufactured from 4340 steel [Q + T (650°C)] (E = 210 GPa, $\sigma_y$ = 885 MPa); rod B is made from 7075-T6 aluminum alloy (E = 70 GPa, $\sigma_y$ = 505 MPa).*
   a. *What uniform load can be applied to the platform before yielding can occur?*

If the strains are equal ( $\varepsilon_A = \varepsilon_B$ ), then the sum of the loads carried by each rod must equal the applied load.

$$P = P_A + P_B$$

One approach to solve the problem is to look at the stress each rod experiences compared to the other.

$$\frac{\sigma_A}{\sigma_B} = \frac{E_A \varepsilon_A}{E_B \varepsilon_B}$$

From the condition that the strains must be equal in each rod, we get the result that

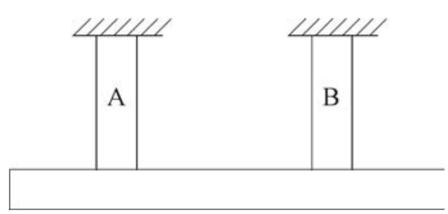
$$\frac{\sigma_A}{\sigma_B} = \frac{E_A}{E_B} = \frac{210\,\text{GPa}}{70\,\text{GPa}} = 3\,.$$

This indicates that the stress in the steel rod is three times higher than the stress in the aluminum rod. Because the yield strength of steel is not three times as high as the yield strength for aluminum, the steel is the limiting component that will yield first in this design. Thus, for the stress in steel rod, we use the yield strength of steel to find the largest uniform load to apply to the platform.

$$P = P_A + P_B = \sigma_A A_A + \sigma_B A_B = A_A(\sigma_A + \sigma_A/3) = \frac{\pi}{4}(1.28\times10^{-2}\text{ m})^2 \cdot (\frac{4}{3}885\times10^6\,\text{Pa}) = \boxed{151.8\text{ kN}}$$

   b. *Which rod will yield first?*
   *(Hint: Both rods experience the same elastic strain)*

The steel rod (rod A). See part a. for the explanation.



2. *(Hertzberg 1.27) A 40 cm diameter pipe is used to carry a pressure of 20 MPa without yielding. The alloy choices are Al 2024-T3 (* $\sigma_y = 345\text{ MPa}$ *,* $\rho = 2.7\text{ g/cm}^3$ *,* $\text{cost} = \$2/\text{kg}$ *) and 1015 steel (* $\sigma_y = 315\text{ MPa}$ *,* $\rho = 7.9\text{ g/cm}^3$ *,* $\text{cost} = \$0.5/\text{kg}$ *). With a safety factor of unity, compute*
   a. *the lightest, and*

The mass of the pipe per unit length is given by

$$m = 2\pi r t \rho .$$

The thickness of the pipe is determined by the yield strength of the material used. For a thin walled cylinder, the maximum and minimum principle stresses occur in the tangential and radial directions, respectively. Consequently, the pipe will yield when the tangential stress (or hoop stress) reaches the yield strength of the material, based on the Tresca yield criterion. From this conclusion, the thickness of the pipe for a given material is found to be

$$\sigma = \frac{Pr}{t}$$
$$\Rightarrow t = \frac{Pr}{\sigma_y} .$$

The thickness of the Al 2024-T3 pipe is

$$t = \frac{Pr}{\sigma_y} = \frac{(20\,\text{MPa})(20\text{ cm})}{345\text{ MPa}} = 1.16\text{ cm} .$$

The mass of this pipe is

$$m = 2\pi r t \rho = 2\pi(20\,\text{cm})(1.16\text{ cm})(2.7\text{ g/cm}^3) = 393.58\,\text{g/cm} = 39.35\,\text{kg/m} .$$

The thickness of the 1015 steel pipe is

$$t = \frac{Pr}{\sigma_y} = \frac{(20\,\text{MPa})(20\text{ cm})}{315\text{ MPa}} = 1.27\text{ cm} .$$

The mass of this pipe is

$$m = 2\pi r t \rho = 2\pi(20\,\text{cm})(1.27\text{ cm})(7.9\text{ g/cm}^3) = 1261\,\text{g/cm} = 126.1\,\text{kg/m} .$$

Clearly, the $\boxed{\text{aluminum pipe}}$ is lighter than the steel pipe.

   b. *the cheapest pipe per unit length, based on the two possible alloy choices.*

The price per unit length of the Al 2024-T3 pipe is

$$39.35\ kg/m\,(\$2/\text{kg}) = \$78.7/\text{m} .$$

The price per unit length of the 1015 steel pipe is

$$126.1\ kg/m\,(\$0.5/\text{kg}) = \$63.05/\text{m} .$$

From this calculation, the $\boxed{\text{steel pipe}}$ is cheaper.

3. *(Hertzberg 1.30) A 7178-T6 aluminum alloy (* $\sigma_y = 540\text{ MPa}$ *) is to be used to make a thin-walled cylindrical pressure vessel. If the diameter is 40 cm, what wall thickness is required to ensure that a pressure of 50 MPa will result in a maximum stress no greater than 50% of the alloy's yield strength?*

For a spherical pressure vessel, the maximum stress is the tangential (or hoop) stress. The design specifies that the tangential stress should not exceed half of the yield strength, which can be written by

$$\sigma = \frac{Pr}{t} = \sigma_y/2$$
$$\Rightarrow t = \frac{2Pr}{\sigma_y} = \frac{2(50\text{MPa})(20\,\text{cm})}{540\text{MPa}} = \boxed{3.704\text{ cm}} .$$

4. *Consider a metallic material with a yield strength* $\sigma_y = 250$ MPa *subjected at all points to the stress state*

$$\sigma_{ij} = \begin{bmatrix} 100 & 150 & 5 \\ 150 & 100 & 10 \\ 5 & 10 & 100 \end{bmatrix} \text{MPa} .$$

*Determine if yielding occurs using the Tresca and Von Mises yield criteria. Discuss the significance of the two yield criteria relative to each other*

The yield critera of the Tresca yield criterion is based on the difference between the maximum and minimum principle stresses. The principle values of the stress state are $\sigma_1 = 250.75\,\text{MPa}$ , $\sigma_2 = 99.34\,\text{MPa}$ , $\sigma_3 = -50.08\,\text{MPa}$ .

$$\sigma_1 - \sigma_3 = (250.75\text{ MPa}) - (-50.08\text{ MPa}) = 300.83\text{ MPa} > \sigma_y$$

The <u>Tresca yield criterion predicts yielding</u> in the material.

The equivalent stress determined by the Von Mises yield criterion can be evaluated by the expression

$$\sigma_{eq} = \sqrt{\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] + 3\left[\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2\right]} ,$$

or by the expression

$$\sigma_{eq} = \sqrt{\frac{3}{2} S_{ij} S_{ij}} .$$

Determining the deviatoric stress, $S_{ij}$, and using the latter equation, we can determine the equivalent stress.

$$S_{ij} = \begin{bmatrix} 0 & 150 & 5 \\ 150 & 0 & 10 \\ 5 & 10 & 0 \end{bmatrix} \text{MPa}$$

$$\sigma_{eq} = \sqrt{\frac{3}{2} S_{ij} S_{ij}} = \sqrt{\frac{3}{2}(2(150)(150) + 2(5)(5) + 2(10)(10)} \text{ MPa} = 260.53\,\text{MPa} > \sigma_y$$

The <u>Von Mises yield criterion also predicts yielding</u> in the material. The result using the first expression for the Von Mises equivalent stress would give the same result.

Considering the calculations based on both yield criteria, we note that the Tresca yield criterion is a more conservative estimate of whether or not yielding has occurred.

5. *A steel pipe ( $\sigma_y = 320$ MPa ) has a diameter of 10 cm and a thickness of 1 mm.*

   a. *What pressure is required to produce yielding in the wall of the pipe?*

The three principle stresses in the pipe are

$$\sigma_{1,\,\text{longitudinal}} = \frac{Pr}{2t}$$
$$\sigma_{2,\,\text{hoop}} = \frac{Pr}{t} \quad .$$
$$\sigma_{3,\,\text{radial}} = 0\,\text{MPa}$$

The Von Mises yield criterion predicts yielding when the equivalent stress equals the yield strength.

$$\sigma_{eq} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} = \sigma_y$$

Substituting the principle stresses into the yield criterion and solving for the pressure gives the result

$$\sigma_y = \sqrt{\frac{1}{2}\left[(\frac{Pr}{2t} - \frac{Pr}{t})^2 + (\frac{Pr}{t} - 0)^2 + (0 - \frac{Pr}{2t})^2\right]}$$
$$\Rightarrow P = \frac{2\sigma_y t}{\sqrt{3}\,r} = \frac{2(320\,\text{MPa})(0.1\,\text{cm})}{\sqrt{3}\,(5\,\text{cm})} = \boxed{7.390\,\text{MPa}}$$

Using the Tresca yield criterion gives a more conservative estimate of the pressure to induce yielding, 6.4 MPa. Either method is appropriate in solving these problems.

   b. *What is the equivalent plastic strain increment in terms of the increment of normal plastic strain in the hoop direction?*

The relationship between the plastic strain increments and applied stresses, given by the Levy Mises relationship, can be related to the equivalent stress and strain increment via the associated flow rule

$$d\varepsilon_{ij}^p = \frac{3}{2} S_{ij} \frac{d\varepsilon_{eq}^p}{\sigma_{eq}} .$$

Relating the increment of normal plastic strain in the (hoop) $x_2$ direction, we have

$$d\varepsilon_{22}^p = \frac{3}{2} S_{22} \frac{d\varepsilon_{eq}^p}{\sigma_{eq}} = \frac{3}{2}\left(\sigma_2 - \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3}\right)\frac{d\varepsilon_{eq}^p}{\sigma_{eq}} = \left(\sigma_2 - \frac{\sigma_1 + \sigma_3}{2}\right)\frac{d\varepsilon_{eq}^p}{\sigma_{eq}}$$
$$\Rightarrow d\varepsilon_{eq}^p = d\varepsilon_{22}^p \frac{\sigma_{eq}}{\sigma_2 - \dfrac{\sigma_1 + \sigma_3}{2}} \quad .$$

Solving for the equivalent stress, $\sigma_{eq}$, in terms of the applied stresses using the Von Mises yield criterion leads to the result that

$$\boxed{d\varepsilon_{eq}^p = \frac{2\sqrt{3}}{3} d\varepsilon_{22}^p} .$$

Solving for the equivalent stress by the Tresca yield criterion leads to the result that, $d\varepsilon_{eq}^p = \frac{4}{3} d\varepsilon_{22}^p$ .