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3.22 MECHANICAL PROPERTIES OF MATERIALS
PROBLEM SET 2 SOLUTIONS

1. *(Nye 2.4) Transform the following tensors to their principle axes, using the Mohr circle construction. Also, determine rotation and direction cosines for each transformation.*

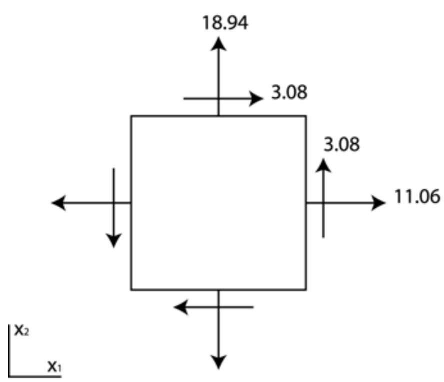
For the following problems, the following result from the Mohr's circle construction is used,

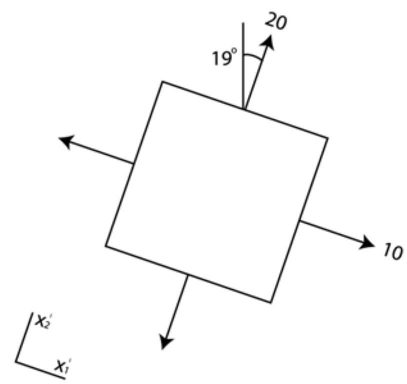
$$\sigma_1, \sigma_2 = a \pm r = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \ ,$$

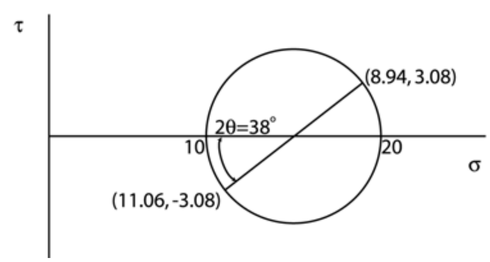
where a is the center and r is the radius of the circle representing the state in the x-y plane at various orientations in the x-y plane. The angle made between the orientation of the given state and the principle direction (counterclockwise) is given by

$$\tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \ .$$

a. $\begin{bmatrix} 11.06 & 3.08 & 0 \\ 3.08 & 18.94 & 0 \\ 0 & 0 & 43 \end{bmatrix}$







Notice that there are no cross terms on the x_3-plane, indicating that the value in the x_3-direction is a principle value. To find the other two principle values, a Mohr's circle construction is employed.

$$\sigma_1, \sigma_2 = a \pm r = \frac{11.06+18.94}{2} \pm \sqrt{\left(\frac{11.06-18.94}{2}\right)^2 + 3.08^2} = 10, 20$$

$$\theta_n = \frac{1}{2}\arctan\left(\frac{2(3.08)}{11.06-18.94}\right) = -19^\circ$$

From these results, the tensor oriented in the principle directions is given by

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 43 \end{bmatrix}.$$

The principle state is obtained by rotating the original axes 19° about the x_3-axis from the x_2-axis toward the x_1-axis.

The direction cosines, ℓ_{ij}, for this transformation are given by (using the notation in *class* that the first subscript refers to the old axes)

$$\ell_{ij} = \begin{bmatrix} \cos(19^\circ) & \cos(90^\circ - 19^\circ) & 0 \\ \cos(90^\circ + 19^\circ) & \cos(19^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(19^\circ) & \sin(19^\circ) & 0 \\ -\sin(19^\circ) & \cos(19^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The convention used in Nye is opposite (the first subscript refers to the new axes) and the transformation matrix becomes

$$\ell_{ij} = \begin{bmatrix} \cos(19^\circ) & -\sin(19^\circ) & 0 \\ \sin(19^\circ) & \cos(19^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The details above will be omitted in the following parts. For a thorough explanation of Mohr's circle construction, the student is directed to Dowling, *Mechanical Behavior of Materials* (Prentice Hall) 1999, 208-214, which is one of many texts that discuss the subject.

b. $\begin{bmatrix} -6 & -3\sqrt{3} & 0 \\ -3\sqrt{3} & 0 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

In the principle orientation

$$\begin{bmatrix} -9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 10 \end{bmatrix},$$

where the principle state is obtained by rotating the original axes 30° about the x_3-axis from the x_1-axis toward the x_2-axis. Direction cosines are (*class* convention)

$$\ell_{ij} = \begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) & 0 \\ \sin(30^\circ) & \cos(30^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c. $\begin{bmatrix} 8 & 0 & -4 \\ 0 & 12 & 0 \\ -4 & 0 & 2 \end{bmatrix}$

In the principle orientation

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where the principle state is obtained by rotating the original axes 26.57° about the x_2-axis from the x_3-axis toward the x_1-axis. Direction cosines are (*class* convention)

$$\ell_{ij} = \begin{bmatrix} \cos(26.57^\circ) & 0 & \sin(26.57^\circ) \\ 0 & 1 & 0 \\ -\sin(26.57^\circ) & 0 & \cos(26.57^\circ) \end{bmatrix}$$

2. *A copper single crystal (cubic) experiences the stress state*

$$\sigma_{ij} = \begin{bmatrix} 10 & -8 & 4 \\ -8 & 20 & 0 \\ 4 & 0 & 5 \end{bmatrix} (\text{MPa}).$$

a. Determine the compliance matrix. ($E_{11} = 66.7\,\text{GPa}$ *,* $G_{12} = 75.2\,\text{GPa}$ *,* $\nu_{12} = 0.42$ *) (Note* $\nu_{ij} = -\dfrac{\varepsilon_j}{\varepsilon_i}$ *.)*

The compliance matrix for a cubic material is defined by three independent elastic constants S_{11}, S_{12}, and S_{44}. S_{11} and S_{12} are found by assuming uniaxial tension where the only stress is σ_1.

$$\varepsilon_1 = S_{11}\sigma_1$$
$$\varepsilon_2 = S_{21}\sigma_1$$

From these relations and the given data, the elastic constants are calculated.

$$\varepsilon_1 = S_{11}\sigma_1 = \sigma_1/E_{11} \rightarrow S_{11} = 1/E_{11} = 1.50\times10^{-11}\ \text{Pa}^{-1}$$

$$\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{21}\sigma_1}{S_{11}\sigma_1} = -\frac{S_{21}}{S_{11}} \rightarrow S_{12} = S_{21} = -\nu_{12}\cdot S_{11} = -\frac{\nu_{12}}{E_{11}} = -0.63\times10^{-11}\,\text{Pa}^{-1}$$

S_{44} is found by assuming only a shear stress is present, σ_4.

$$\varepsilon_4 = S_{44}\sigma_4 = \sigma_4/G \rightarrow S_{44} = 1/G = 1.33\times10^{-11}\,\text{Pa}^{-1}$$

The compliance tensor for copper is given by

$$[S] = \begin{bmatrix} 1.50 & -0.63 & -0.63 & 0 & 0 & 0 \\ -0.63 & 1.50 & -0.63 & 0 & 0 & 0 \\ -0.63 & -0.63 & 1.50 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.33 \end{bmatrix} \times10^{-11}\ \text{Pa}^{-1}$$

b. *Determine the engineering strain in the sample of copper assuming that all deformation is linear elastic.*

To find strain, use the anisotropic form of Hooke's law. The stress tensor is given in the question and must be converted into matrix form.

$$[\varepsilon]=[S][\sigma]=\begin{bmatrix} 1.50 & -0.63 & -0.63 & 0 & 0 & 0 \\ -0.63 & 1.50 & -0.63 & 0 & 0 & 0 \\ -0.63 & -0.63 & 1.50 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.33 \end{bmatrix}\times 10^{-11}\ \text{Pa}^{-1}\cdot\begin{bmatrix} 10 \\ 20 \\ 5 \\ 0 \\ 4 \\ -8 \end{bmatrix}\times 10^{6}\ \text{Pa} = \begin{bmatrix} -0.075 \\ 2.055 \\ -1.14 \\ 0 \\ 0.532 \\ -1.064 \end{bmatrix}\times 10^{-4}$$

Note that in the reduced notation, the strain matrix is engineering strain, and not tensor strain. To find the tensor strain from the matrix strain, one must divide the shear strains by two. The normal strains remain the same.

c. *Determine the strain energy in the sample if the deformation is done isothermally and reversibly.*

$$\begin{aligned} U &= \frac{1}{2}[\sigma][\varepsilon] = \frac{1}{2}(\sigma_1\varepsilon_1+\sigma_2\varepsilon_2+\sigma_3\varepsilon_3+\sigma_4\varepsilon_4+\sigma_5\varepsilon_5+\sigma_6\varepsilon_6) \\ &= \frac{1}{2}[(10\text{MPa})(-0.075\times10^{-4})+(20\text{MPa})(2.055\times10^{-4})+(5\text{MPa})(-1.14\times10^{-4})\cdots \\ &\quad \cdots(0\text{MPa})(0)+(4\text{MPa})(0.532\times10^{-4})+(-8\text{MPa})(-1.064\times10^{-4})] \\ &= 2264.5\text{Pa} = \boxed{2.2645\,\text{kJ/m}^3} \end{aligned}$$

d. *If a hydrostatic pressure of 5 MPa is superimposed on the stressed copper sample, determine the resulting strain energy in the sample.*

`The resulting stress state, $\sigma^{\bullet}_{ij}$, imposed on the copper single crystal can be found by superimposing the stress state, σ_{ij}, and hydrostatic compressive stress, $-P\delta_{ij}$, because we are in the linear elastic regime.

$$\sigma^{\bullet}_{ij} = \sigma_{ij} - P\delta_{ij} = \begin{bmatrix} 5 & -8 & 4 \\ -8 & 15 & 0 \\ 4 & 0 & 0 \end{bmatrix}\text{MPa}$$

To find the strain energy, the same calculations in (b) and (c) can be made with the stress state $\sigma^{\bullet}_{ij}$. The resulting strain energy density in the sample after application of the hydrostatic compressive stress is $\boxed{1.9345\ \text{kJ/m}^3}$.

3. *(Hertzberg 1.18) For three BCC metals, tungsten, molybdenum, and iron, compute the elastic moduli in the <100> and <111> directions. Compare the anisotropy in these three metals.*

For the case of cubic crystals, the modulus of elasticity in any direction is given by the equation

$$\frac{1}{E} = s_{11} - 2\left[(s_{11} - s_{12}) - \tfrac{1}{2} s_{44}\right]\left(l_1^2 l_2^2 + l_2^2 l_3^2 + l_1^2 l_3^2\right). \quad \text{(Hertzberg 1-14)}$$

Thus, the modulus in a certain direction for a cubic material is defined by its three stiffness coefficients and the particular direction, given by the direction cosine values. The stiffness coefficients for tungsten, molybdenum, and iron single crystals can be found in the literature.

Tungsten: $s_{11} = 0.26\times10^{-11}\ \text{Pa}^{-1}$, $s_{12} = -0.07\times10^{-11}\ \text{Pa}^{-1}$, $s_{44} = 0.66\times10^{-11}\ \text{Pa}^{-1}$

Molybdenum: $s_{11} = 0.28\times10^{-11}\ \text{Pa}^{-1}$, $s_{12} = -0.08\times10^{-11}\ \text{Pa}^{-1}$, $s_{44} = 0.91\times10^{-11}\ \text{Pa}^{-1}$

Iron: $s_{11} = 0.80\times10^{-11}\ \text{Pa}^{-1}$, $s_{12} = -0.28\times10^{-11}\ \text{Pa}^{-1}$, $s_{44} = 0.86\times10^{-11}\ \text{Pa}^{-1}$

The direction cosines for the two directions are:

<100>: $l_1 = 1$, $l_2 = 0$, $l_3 = 0$

<111>: $l_1 = 1/\sqrt{3}$, $l_2 = 1/\sqrt{3}$, $l_3 = 1/\sqrt{3}$

Now we can calculate the modulus of elasticity in the desired direction for tungsten, molybdenum and iron using equation (1-14). Anisotropy can be evaluated by the ratio $E_{<100>}/E_{<111>}$.

	$E_{<100>}$ (GPa)	$E_{<111>}$ (GPa)	$E_{<100>}/E_{<111>}$
Tungsten	384.6	384.6	1
Molybdenum	357.1	291	1.227
Iron	125	272.7	0.458

The maximum and minimum elastic moduli are always in the <100> and <111> directions. (This idea is discussed in the text in depth.) From the calculations above, molybdenum and iron are both anisotropic as single crystals. A single crystal of tungsten, however, is isotropic.

4. *(Hertzberg 1.25) A cylindrical rod of steel is deformed elastically in tension to a load of 49 kN. If the original rod length and diameter are 25 cm and 15 mm, respectively, determine the rod length and diameter under load, assuming that the material possesses the following properties: E = 205 GPa, ν = 0.25.*

First we can determine the rod length under load from Hooke's Law. The engineering stress, a good approximation of the true stress within elastic deformations, is calculated to be

$$\sigma = \frac{P}{A} = \frac{P}{\frac{\pi}{4}d^2} = \frac{(49\times10^3\ \text{N})}{\frac{\pi}{4}(15\times10^3\ \text{m})^2} = 277.283\,\text{MPa}$$

We can solve for the change in length of the rod by Hooke's Law.

$$\sigma = E\varepsilon$$

$$\sigma = E\frac{\Delta\ell}{\ell_0}$$

$$\Rightarrow \Delta\ell = \frac{\sigma\ell_0}{E} = \frac{(277.283\ \text{MPa})(25\times10^{-2}\ \text{m})}{205\times10^9\,\text{Pa}} = 3.38\times10^{-4}\ \text{m} = 0.0338\ \text{cm}$$

$$\Rightarrow \ell = \ell_0 + \Delta\ell = 25\ \text{cm} + 0.0338\ \text{cm} = \boxed{25.0338\ \text{cm}}$$

The contraction of the rod diameter under load is due to the Poisson effect given by

$$\nu = -\frac{\varepsilon_d}{\varepsilon_\ell} = -\frac{\Delta d/d_0}{\Delta \ell/\ell_0}$$

$$\Rightarrow \Delta d = -\nu\left(\Delta \ell/\ell_0\right)d_0 = -(0.25)\left(0.0338 \text{ cm}/25 \text{ cm}\right)(15 \text{ mm}) = -0.005 \text{ mm}$$

$$\Rightarrow d = d_0 + \Delta d = 15 \text{ mm} + (-0.005 \text{ mm}) = \boxed{14.995 \text{ mm}}$$

5. In problem set 1, the principle stresses were determined for the given stress tensor.

$$\sigma_{ij} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 7 & -3\sqrt{3} \\ 0 & -3\sqrt{3} & 13 \end{bmatrix} \text{MPa}$$

a. *Find the unit normals for each principle stress.*

The principle values for the above tensor were determined to be $\lambda^{(1)} = 25, \lambda^{(2)} = 16, \lambda^{(3)} = 4$, corresponding to the principle stresses $\sigma_1 = 25 \text{ MPa}, \sigma_2 = 16 \text{ MPa}, \sigma_3 = 4 \text{ MPa}$. In order to find the directions of the principle stresses, we solve the equation $\left(\sigma_{ij} - \lambda\delta_{ij}\right)n_j = 0$ for each value of λ. For $\lambda^{(1)} = 25$, we get the following linear equations:

$$\left(\sigma_{ij} - 25\delta_{ij}\right)n_j = 0$$

$$(25-25)n_1 + 0n_2 + 0n_3 = 0$$
$$0n_1 + (7-25)n_2 + -3\sqrt{3}n_3 = 0$$
$$0n_1 + -3\sqrt{3}n_2 + (13-25)n_3 = 0$$

$$0n_1 + 0n_2 + 0n_3 = 0$$
$$0n_1 + (-18)n_2 + -3\sqrt{3}n_3 = 0$$
$$0n_1 + -3\sqrt{3}n_2 + (-12)n_3 = 0$$

$$n_2 = \frac{-3\sqrt{3}}{18}n_3$$
$$n_2 = \frac{-12}{3\sqrt{3}}n_3$$

$$n_2 = n_3 = 0$$

The principle direction is a unit vector, so the length must equal 1. This is expressed by

$$n_1^2 + n_2^2 + n_3^2 = 1.$$

Thus, we find that for $\lambda^{(1)} = 25$, $\boxed{\vec{n}^{(1)} = 1x_1}$.

For $\lambda^{(2)} = 16$, we get the following linear equations:

$$\left(\sigma_{ij} - 16\delta_{ij}\right) n_j = 0$$

$$\begin{gathered}(25-16)n_1 + 0n_2 + 0n_3 = 0 \\ 0n_1 + (7-16)n_2 + -3\sqrt{3}n_3 = 0 \\ 0n_1 + -3\sqrt{3}n_2 + (13-16)n_3 = 0\end{gathered}$$

$$\begin{gathered}9n_1 + 0n_2 + 0n_3 = 0 \\ 0n_1 + (9)n_2 + -3\sqrt{3}n_3 = 0 \\ 0n_1 + -3\sqrt{3}n_2 + (-3)n_3 = 0\end{gathered}$$

$$\begin{gathered}n_1 = 0 \\ n_2 = \frac{-3\sqrt{3}}{9} n_3 \\ n_2 = \frac{-3}{3\sqrt{3}} n_3\end{gathered}$$

$$n_2 = \frac{-\sqrt{3}}{3} n_3$$

The principle direction is a unit vector, so the length must equal 1. This is expressed by

$$\begin{gathered}n_1^2 + n_2^2 + n_3^2 = 1 \\ \left(\frac{-\sqrt{3}}{3} n_3\right)^2 + n_3^2 = 1 \quad . \\ \Rightarrow n_3 = \frac{\sqrt{3}}{2}, \; n_2 = -\frac{1}{2}\end{gathered}$$

Thus, we find that for $\lambda^{(2)} = 16$, $\boxed{\vec{n}^{(2)} = -\frac{1}{2} x_2 + \frac{\sqrt{3}}{2} x_3}$.

For $\lambda^{(2)} = 4$, we get the following linear equations:

$$\left(\sigma_{ij} - 4\delta_{ij}\right) n_j = 0$$

$$\begin{gathered}(25-4)n_1 + 0n_2 + 0n_3 = 0 \\ 0n_1 + (7-4)n_2 + -3\sqrt{3}n_3 = 0 \\ 0n_1 + -3\sqrt{3}n_2 + (13-4)n_3 = 0\end{gathered}$$

$$\begin{gathered}21n_1 + 0n_2 + 0n_3 = 0 \\ 0n_1 + (3)n_2 + -3\sqrt{3}n_3 = 0 \\ 0n_1 + -3\sqrt{3}n_2 + (9)n_3 = 0\end{gathered}$$

$$n_1 = 0$$
$$n_2 = \frac{3\sqrt{3}}{3} n_3$$
$$n_2 = \frac{9}{3\sqrt{3}} n_3$$

$$n_2 = \sqrt{3} n_3$$

The principle direction is a unit vector, so the length must equal 1. This is expressed by

$$n_1^2 + n_2^2 + n_3^2 = 1$$
$$\left(\sqrt{3} n_3\right)^2 + n_3^2 = 1 \quad .$$
$$\Rightarrow n_3 = \frac{1}{2}, n_2 = \frac{\sqrt{3}}{2}$$

Thus, we find that for $\lambda^{(3)} = 4$, $\boxed{\vec{n}^{(3)} = \frac{\sqrt{3}}{2} x_2 + \frac{1}{2} x_3}$.

b. Use the unit normals to find the direction cosine matrix for the transformation from the original axes to the principle axes.

To find the cosine of the angle between two axes, we can use the definition of the dot product of two unit vectors.

$$a \cdot b = \cos(\measuredangle a, b) \quad \forall a, b \text{ unit vectors}$$

Let us call the original axes $x^{(1)} = 1x_1, x^{(2)} = 1x_2, x^{(3)} = 1x_3$ and the new axes (defined by the normals above) $n^{(1)}, n^{(2)}, n^{(3)}$. The components of the direction cosines are defined as (class convention)

$$\ell_{ij} = \cos(\measuredangle x^{(i)}, n^{(j)}) = x^{(i)} \cdot n^{(j)}$$

$$\ell_{ij} = \begin{bmatrix} x^{(1)} \cdot n^{(1)} & x^{(1)} \cdot n^{(2)} & x^{(1)} \cdot n^{(3)} \\ x^{(2)} \cdot n^{(1)} & x^{(2)} \cdot n^{(2)} & x^{(2)} \cdot n^{(3)} \\ x^{(3)} \cdot n^{(1)} & x^{(3)} \cdot n^{(2)} & x^{(3)} \cdot n^{(3)} \end{bmatrix}$$

$$\ell_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

c. Perform the tensor transformation using the direction cosines on the stress state in the problem? Explain the significance of the transformed tensor.

The tensor transformation rule for a second order tensor is

$$\sigma'_{ij} = \ell_{ki} \ell_{lj} \sigma_{kl}$$

From the stress tensor given and the direction cosines defined in part (b), we obtain the nine tensor components of the stress tensor in the axes aligned with the principle directions. Note that we have previously determined that the stress tensor is symmetric, so we actually only have to solve for six of the stresses. (Terms with direction cosine values of zero have been omitted.

$$\sigma'_{11} = \ell_{k1}\ell_{l1}\sigma_{kl} = \ell_{11}\ell_{11}\sigma_{11} = (1)(25) = 25$$

$$\begin{aligned}\sigma'_{22} = \ell_{k2}\ell_{l2}\sigma_{kl} &= \ell_{22}\ell_{22}\sigma_{22} + \ell_{32}\ell_{22}\sigma_{32} + \ell_{22}\ell_{32}\sigma_{23} + \ell_{32}\ell_{32}\sigma_{33} \\ &= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)(7) + \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)(-3\sqrt{3}) + \cdots \\ &\quad \left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)(-3\sqrt{3}) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)(13) \\ &= 16\end{aligned}$$

$$\begin{aligned}\sigma'_{33} = \ell_{k3}\ell_{l3}\sigma_{kl} &= \ell_{23}\ell_{23}\sigma_{22} + \ell_{33}\ell_{23}\sigma_{32} + \ell_{23}\ell_{33}\sigma_{23} + \ell_{33}\ell_{33}\sigma_{33} \\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)(7) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)(-3\sqrt{3}) + \cdots \\ &\quad \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)(-3\sqrt{3}) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(13) \\ &= 4\end{aligned}$$

$$\begin{aligned}\sigma'_{12} = \sigma'_{21} = \ell_{k1}\ell_{l2}\sigma_{kl} = \ell_{11}\ell_{l2}\sigma_{1l} &= \ell_{11}\ell_{12}\sigma_{11} + \ell_{11}\ell_{22}\sigma_{12} + \ell_{11}\ell_{32}\sigma_{13} \\ &= (1)(0)(25) + (1)\left(-\frac{1}{2}\right)(0) + (1)\left(\frac{\sqrt{3}}{2}\right)(0) = 0\end{aligned}$$

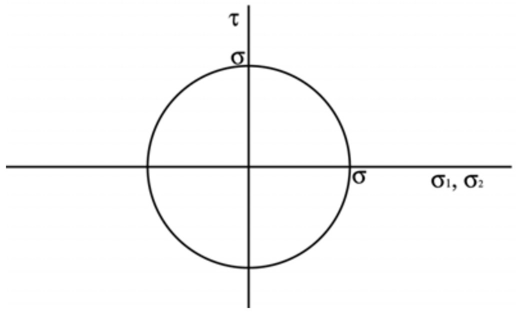
$$\begin{aligned}\sigma'_{13} = \sigma'_{31} = \ell_{k1}\ell_{l3}\sigma_{kl} = \ell_{11}\ell_{l3}\sigma_{1l} &= \ell_{11}\ell_{13}\sigma_{11} + \ell_{11}\ell_{23}\sigma_{12} + \ell_{11}\ell_{33}\sigma_{13} \\ &= (1)(0)(25) + (1)\left(\frac{\sqrt{3}}{2}\right)(0) + (1)\left(\frac{1}{2}\right)(0) = 0\end{aligned}$$

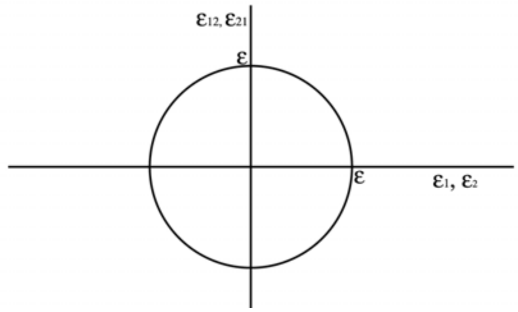
$$\begin{aligned}\sigma'_{23} = \sigma'_{32} = \ell_{k2}\ell_{l3}\sigma_{kl} = \ell_{22}\ell_{23}\sigma_{22} &= \ell_{22}\ell_{33}\sigma_{23} + \ell_{32}\ell_{23}\sigma_{32} + \ell_{32}\ell_{33}\sigma_{33} \\ &= \left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)(7) + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)(3\sqrt{3}) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)(3\sqrt{3}) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)(13) = 0\end{aligned}$$

$$\sigma'_{ij} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{MPa}$$

The transformed tensor after transformation consists of only the principle stresses (only normal stresses). This is expected because the transformation was designed to transform the axes to the principle directions. (When the tensor is oriented with the principle directions as the axes, only the principle stresses remain.)

6. *For an isotropic material, the compliance matrix constants are related by the expression* $s_{66} = 2(s_{11} - s_{12})$. *Derive this result using Mohr's circle and Hooke's Law.*





From Mohr's circle, a state of pure shear ($\sigma_6 = \sigma$) is equally expressed by an equi-biaxial stress ($\sigma_1 = -\sigma_2 = \sigma$) state for an isotropic material in any orientation. Using matrix notation, the strains for each of these stress states are represented by

$$\varepsilon_6 = S_{66}\sigma_6 = S_{66}\sigma \text{ (pure shear)}$$

$$\varepsilon_1 = S_{11}\sigma_1 + S_{12}\sigma_2 = (S_{11} - S_{12})\sigma \text{ (equi-biaxial stress)}$$

The Mohr's circle for strain can be expressed in the same manner as stress with the stress tensor components represented by strain tensor components. From the Mohr's circle construction for strain, the ε_{11} component is equal to the ε_{12} component. For engineering strains, we find the relationship

$$\varepsilon_1 \equiv \varepsilon_{11} = \varepsilon_{12} \equiv \frac{\varepsilon_6}{2}.$$

Combining this equation with the equations above, we find

$$\varepsilon_1 = \frac{\varepsilon_6}{2}$$

$$(S_{11} - S_{12})\sigma = \frac{S_{66}\sigma}{2}$$

$$\therefore \boxed{S_{66} = 2(S_{11} - S_{12})}$$

7. *A single crystal titanium carbide component is aligned such that the loading direction is parallel to the <100> direction. The component design requires that the modulus be at least 470 GPa in the loading direction. The modulus in the <100> direction is 476.2 GPa, which satisfies the requirement. However, careful inspection reveals that the single crystal is misaligned 10° toward the <010> direction about the axis parallel to the <001> direction. Because realignment of the component is expensive, determine if the misaligned component can be used and still meet the design requirement.*

From the literature, the compliance in matrix notation for titanium carbide single crystals is

$$[S]=\begin{bmatrix} 0.21 & -0.036 & -0.036 & 0 & 0 & 0 \\ -0.036 & 0.21 & -0.036 & 0 & 0 & 0 \\ -0.036 & -0.036 & 0.21 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.561 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.561 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.561 \end{bmatrix} \times 10^{-11}\ \mathrm{Pa}^{-1}.$$

We can verify that the modulus in the <100> direction is

$$E_{<100>}=\frac{1}{S_{11}}=\frac{1}{0.21\times10^{-11}\,\mathrm{Pa}^{-1}}=476.2\ \mathrm{GPa}.$$

To evaluate the modulus of the misaligned component in the loading direction, an instructive approach is to find the S_{11} component of the compliance tensor for the rotation of 10° and use the equation above to determine the modulus in this new direction.

The transformation rule for fourth order tensors is (*class* convention)

$$S'_{ijkl}=\ell_{mi}\ell_{nj}\ell_{ok}\ell_{pl}S_{mnop}.$$

Note that tensor transformations are valid for tensors only; reduced tensor notation matrices do not transform by tensor transformation rules. In our case, we are only interested in the compliance in one direction, so we can write

$$S'_{1111}=\ell_{m1}\ell_{n1}\ell_{o1}\ell_{p1}S_{mnop}.$$

At this point we must find the direction cosines (*class* convention) for the misalignment so that the loading direction is parallel to the new x_1-axis,

$$\ell_{ij}=\begin{bmatrix} \cos(10^\circ) & -\sin(10^\circ) & 0 \\ \sin(10^\circ) & \cos(10^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Although the tensor transformation has 81 terms, ℓ_{11} and ℓ_{21} are the only nonzero direction cosine terms that appear in $S'_{1111}=\ell_{m1}\ell_{n1}\ell_{o1}\ell_{p1}S_{mnop}$. The resulting expanded form is

$$\begin{aligned} S'_{1111}=\ &\ell_{11}\ell_{11}\ell_{11}\ell_{11}S_{1111}+\cdots \\ &\ell_{11}\ell_{11}\ell_{11}\ell_{21}S_{1112}+\ell_{11}\ell_{11}\ell_{21}\ell_{11}S_{1121}+\ell_{11}\ell_{21}\ell_{11}\ell_{11}S_{1211}+\ell_{21}\ell_{11}\ell_{11}\ell_{11}S_{2111}+\cdots \\ &\ell_{11}\ell_{11}\ell_{21}\ell_{21}S_{1122}+\ell_{11}\ell_{21}\ell_{21}\ell_{11}S_{1221}+\ell_{11}\ell_{21}\ell_{11}\ell_{21}S_{1212}+\ell_{21}\ell_{11}\ell_{11}\ell_{21}S_{2112}+\ell_{21}\ell_{11}\ell_{21}\ell_{11}S_{2121}+\ell_{21}\ell_{21}\ell_{11}\ell_{11}S_{2211}+\cdots \\ &\ell_{21}\ell_{21}\ell_{21}\ell_{11}S_{2221}+\ell_{21}\ell_{21}\ell_{11}\ell_{21}S_{2212}+\ell_{21}\ell_{11}\ell_{21}\ell_{21}S_{2122}+\ell_{11}\ell_{21}\ell_{21}\ell_{21}S_{1222}+\cdots \\ &\ell_{21}\ell_{21}\ell_{21}\ell_{21}S_{2222} \end{aligned}$$

From the symmetry of the strain tensor and stress tensors that are related by the compliance tensor, we can simplify the expression. (Note that S_{16} and S_{26} are zero)

$$S'_{1111}=(\ell_{11})^4S_{1111}+\cancel{4(\ell_{11})^3(\ell_{21})S_{1112}}+2(\ell_{11})^2(\ell_{21})^2S_{1122}+4(\ell_{11})^2(\ell_{21})^2S_{1212}+\cancel{4(\ell_{21})^3(\ell_{11})S_{2221}}+(\ell_{21})^4S_{2222}.$$

In reduced tensor notation, the expression becomes

$$\begin{aligned} S'_{11} &= (\ell_{11})^4 S_{11} + 2(\ell_{11})^2(\ell_{21})^2 S_{12} + 4(\ell_{11})^2(\ell_{21})^2 S_{66}/4 + (\ell_{21})^4 S_{22} \\ &= [(0.9848)^4(0.21) + 2(0.9848)^2(0.1736)^2(-0.036) + 4(0.9848)^2(0.1736)^2(0.561)/4 + (0.1736)^4(0.21)]\times 10^{-11}\ \text{Pa}^{-1} \\ &= 2.12\times 10^{-12}\ \text{Pa}^{-1} \end{aligned}$$

The modulus in the direction of loading considering the misalignment is given by

$$E'_{misaligned} = \frac{1}{S'_{11}} = \frac{1}{2.12\times 10^{-12}\,\text{Pa}^{-1}} = 471.7\ \text{GPa}\ .$$

We conclude that the $\boxed{\text{component will satisfy the design requirement}}$ despite the misalignment, obviating the need for expensive realignment.

This result could have also been obtained by using equation (1-14) in Hertzberg because titanium carbide is cubic. The unit normal in the misaligned direction is $\vec{\ell} = (\cos 10)\hat{x}_1 + (\sin 10)\hat{x}_2 + 0\hat{x}_3$. Thus, from equation (1-14) we obtain the modulus in the misaligned direction.

$$\begin{aligned} \frac{1}{E_{misaligned}} &= s_{11} - 2\left[(s_{11} - s_{12}) - \tfrac{1}{2} s_{44}\right](\ell_1^2\ell_2^2 + \ell_2^2\ell_3^2 + \ell_1^2\ell_3^2) \\ &= (0.21\times 10^{-11}\ \text{Pa})\cdots \\ &\quad -2\left[(0.21\times 10^{-11}\ \text{Pa} - (-0.036\times 10^{-11}\ \text{Pa})) - \tfrac{1}{2}(0.561\times 10^{-11}\ \text{Pa})\right](\cos^2(10)\sin^2(10)) \end{aligned}$$

$$\Rightarrow E_{misaligned} = 471.1\,\text{GPa}$$