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3.22 MECHANICAL PROPERTIES OF MATERIALS
PROBLEM SET 1 SOLUTIONS

Reading
Ashby, M.F., 1982, *Tensors: Notes and Problems* (Course Notes). Sections 1 – 4.
Nye, J.F., 2000, *Physical Properties of Crystals* (Clarendon Press). Chapter I, section 1 and Chapter V.

1.For each of the following expressions, identify the free indices and the dummy index pairs, and determine the number of expanded equations represented and how many terms each expanded equation will have.

a. $j_i = \sigma_{ij} E_j$

Free indices are repeated in each term in the equation. *i* is the only free index. Dummy index pairs occur as a pair of indices in the same term. *j* is the only dummy index. If there are *n* free indices in an indicial notation equation, there are 3^n corresponding expanded equations. Trivially, the number of equations represented by this expression is 3. A term in an indicial notation equation with *n* dummy index pairs expands to 3^n terms. The number of terms in each expanded equation is 3.

b. $P_i = d_{ijk} \sigma_{jk}$

The free index is *i*. The dummy indexes are *j* and *k*. There are 3 equations represented with 9 terms in each equation.

c. $T'_{ij} = a_{ik} a_{jl} T_{kl}$

The free indices are *i* and *j*. The dummy indices are *k* and *l*. There are 9 equations represented with 9 terms in each equation.

d. $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$

The free indices are *i* and *j*. The dummy index is *k*. There are 9 equations represented with 4 terms in each equation.

e. $p_i = 5$

The free index is *i*. There are 3 equations represented with 1 term in each equation.

2. Evaluate the following expressions involving the Kronecker delta δ_{ij}*, the alternating tensor* e_{ijk} *and an arbitrary second rank tensor* T_{ij}*.*

a. δ_{ii}

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

b. $\delta_{ij}\delta_{ik}\delta_{jk}$

$$\delta_{ij}\delta_{ik}\delta_{jk} = \delta_{1j}\delta_{1k}\delta_{jk} + \delta_{2j}\delta_{2k}\delta_{jk} + \delta_{3j}\delta_{3k}\delta_{jk} = 3$$

c. $\delta_{ik}\delta_{kj}$

$\delta_{ik}\delta_{kj} = \delta_{i1}\delta_{1j} + \delta_{i2}\delta_{2j} + \delta_{i3}\delta_{3j} = \delta_{ij}$

(Only dummy index pairs are expanded.)

d. $\delta_{ik}T_{kj}$

$\delta_{ik}T_{kj} = \delta_{i1}T_{1j} + \delta_{i2}T_{2j} + \delta_{i3}T_{3j} = T_{ij}$

e. $\delta_{jk}e_{ijk}$

$\delta_{jk}e_{ijk} = e_{ijj} = 0$

3. Write the expanded form of the following expressions.

a. T_{ii}

$T_{ii} = T_{11} + T_{22} + T_{33}$ (a scalar)

b. ε_{ij}

$\varepsilon_{ij} = \varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}, \varepsilon_{31}, \varepsilon_{32}, \varepsilon_{33}$ (9 components of ε_{ij})

c. $\sigma_{ij,j}$

$$\sigma_{ij,j} = \begin{cases} \sigma_{1j,j} = \dfrac{\partial\sigma_{11}}{\partial x_1} + \dfrac{\partial\sigma_{12}}{\partial x_2} + \dfrac{\partial\sigma_{13}}{\partial x_3} \\ \sigma_{2j,j} = \dfrac{\partial\sigma_{21}}{\partial x_1} + \dfrac{\partial\sigma_{22}}{\partial x_2} + \dfrac{\partial\sigma_{23}}{\partial x_3} \\ \sigma_{3j,j} = \dfrac{\partial\sigma_{31}}{\partial x_1} + \dfrac{\partial\sigma_{32}}{\partial x_2} + \dfrac{\partial\sigma_{33}}{\partial x_3} \end{cases}$$

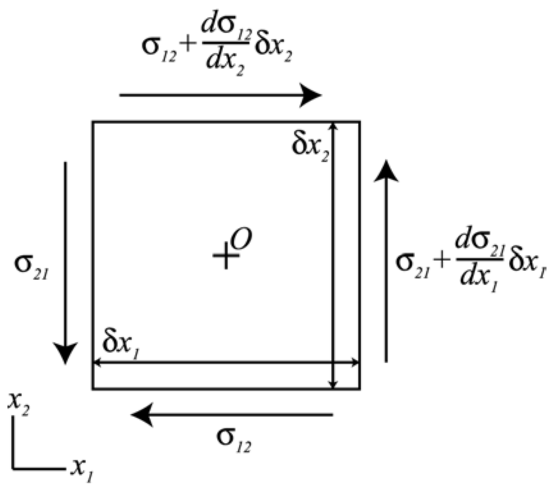
4. Define matter tensors and field tensors and explain how they each depend on crystal structure. Give one example of each type of tensor.

Matter tensors measure physical crystal properties. Physical properties in a given crystal must also have any extra symmetry elements that are possessed by the crystal (Neumann's Principle). Therefore, matter tensors are connected to crystal structure by symmetry. Examples of matter tensors are compliance/stiffness, permittivity, and magnetic susceptibility tensors.

Field tensors measure a state imposed on a material, such as an applied field or a generalized displacement. Because these states can be applied in any arbitrary orientation with respect to a crystal, there is no relation between field tensors and crystal structure. Examples of field tensors are stresses, strains, electric fields, and magnetic fields.

5. Show that the stress tensor is symmetric ($\sigma_{ij} = \sigma_{ji}$) in the absence of body forces and accelerations. Consider a three-dimensional infinitesimally small material element and assume that stress can vary linearly along the element. (This is sometimes referred to as the theorem of conjugate shear stresses.)

Static equilibrium requires that no net forces and no body-torques act of an infinitesimal volume element. No net forces on an element leads to the result $\sigma_{ij,j} = 0$. To show that the stress tensor is symmetric, the net moments on an element must be equal to zero. Consider an infinitesimal volume element with sides of equal length δx_1, δx_2, δx_3. If we take the moment parallel to the x_3 direction through the center of the volume element, point O, the only forces that contribute to the moment are a result of the four shear stresses shown in the figure. All other stresses will not induce a moment about the chosen axis through point O. The shear stresses are shown to vary linearly across the element.



Using the convention that a force inducing a counterclockwise rotation is positive, the following expression is obtained.

$$\sum_{x_3} M = \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_1}\delta x_1\right)\delta x_2 \delta x_3 \frac{\delta x_1}{2} + \sigma_{21}\delta x_2 \delta x_3 \frac{\delta x_1}{2} - \left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_2}\delta x_2\right)\delta x_1 \delta x_3 \frac{\delta x_2}{2} - \sigma_{12}\delta x_1 \delta x_3 \frac{\delta x_2}{2} = 0$$

Divide through by the volume of the element.

$$\left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_1}\delta x_1\right) + \sigma_{21} - \left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_2}\delta x_2\right) - \sigma_{12} = 0$$

In the limit of an infinitesimal limit, $\delta x_1 \to 0$, $\delta x_2 \to 0$, $\delta x_3 \to 0$, the derivatives become small compared to the shear terms.

$$\sigma_{21} + \sigma_{21} - \sigma_{12} - \sigma_{12} = 0$$
$$\sigma_{12} = \sigma_{21}$$

By similar derivations taking the moment parallel to the x_1 and x_2 directions through point O, it is found in general that

$$\boxed{\sigma_{ij} = \sigma_{ji}}\,.$$

6. Given the stress tensor below, answer the following questions.

$$\sigma_{ij} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 7 & -3\sqrt{3} \\ 0 & -3\sqrt{3} & 13 \end{bmatrix} \text{ MPa}$$

a. *Calculate the normal traction on the plane defined by the normal* $n_i = 1/\sqrt{2}\,\hat{x}_1 + 1/\sqrt{2}\,\hat{x}_2$.

$$\begin{aligned} T_{normal} &= \sigma_{ij} n_j n_i \\ &= \sigma_{11}n_1n_1 + \sigma_{12}n_2n_1 + \sigma_{13}n_3n_1 + \sigma_{21}n_1n_2 + \sigma_{22}n_2n_2 + \sigma_{23}n_3n_2 + \sigma_{31}n_1n_3 + \sigma_{32}n_2n_3 + \sigma_{33}n_3n_3 \\ &= (25\,\text{MPa})(1/\sqrt{2})(1/\sqrt{2}) + 0 + 0 + 0 + (7\,\text{MPa})(1/\sqrt{2})(1/\sqrt{2}) + 0 + 0 + 0 + 0 \\ &= \boxed{16\,\text{MPa}} \end{aligned}$$

b. *Calculate the three invariants of stress* (I_1, I_2, I_3).

$$I_1 = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 25 + 7 + 13 = \boxed{45}$$

$$\begin{aligned} I_2 &= -1/2\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right) = -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2 \\ &= -\left(25\cdot 7 + 7\cdot 13 + 13\cdot 25\right) + (-3\sqrt{3})^2 + 0 + 0 = \boxed{-564} \end{aligned}$$

$$I_3 = \det(\sigma_{ij}) = \sigma_{11}(\sigma_{22}\sigma_{33} - \sigma_{23}\sigma_{32}) + 0 + 0 = 25\left(7\cdot 13 - 3\sqrt{3}\cdot 3\sqrt{3}\right) = \boxed{1600}$$

c. *Calculate the principle stresses* $(\sigma_1, \sigma_2, \sigma_3)$. *Show work.*

For a given stress tensor σ_{ij}, the principle stresses are given by the λ (eigenvalues) that satisfy the equation

$$\left(\sigma_{ij} - \lambda\delta_{ij}\right)n_j = 0\,.$$

The vectors n_j (eigenvectors) define the direction in which the principle stresses act. For the equation to have a nontrivial solution $(n_j \neq 0)$, the determinant of the matrix $\sigma_{ij} - \lambda\delta_{ij}$ must have a determinant equal to zero. Solving the cubic equation

$$\det\left(\sigma_{ij} - \lambda\delta_{ij}\right) = 0$$

yields three values for λ.

$$\det\left(\sigma_{ij} - \lambda\delta_{ij}\right) = 0$$

$$\begin{vmatrix} 25-\lambda & 0 & 0 \\ 0 & 7-\lambda & -3\sqrt{3} \\ 0 & -3\sqrt{3} & 13-\lambda \end{vmatrix} = 0$$

$$(25-\lambda)\left((7-\lambda)(13-\lambda) - (-3\sqrt{3})(-3\sqrt{3})\right) = 0$$

$$(25-\lambda)\left(\lambda^2 - 20\lambda + 64\right) = 0$$

$$\Rightarrow \lambda = 25,\ 16,\ 4$$

Thus, the principle stresses are $\boxed{\sigma_1 = 25\text{ MPa}}$, $\boxed{\sigma_2 = 16\text{ MPa}}$, $\boxed{\sigma_3 = 4\text{ MPa}}$.

d. Write a new stress tensor σ'_{ij} from the principle stresses (part b) in the form given below, using the convention that $\sigma_1 > \sigma_2 > \sigma_3$. Calculate the invariants of stress tensor σ'_{ij}.

$$\sigma'_{ij} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{MPa}$$

$$I_1 = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 25 + 16 + 4 = \boxed{45}$$

$$I_2 = -1/2\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right) = -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2$$

$$= -\left(25 \cdot 16 + 16 \cdot 4 + 4 \cdot 25\right) + 0 + 0 + 0 = \boxed{-564}$$

$$I_3 = \det(\sigma_{ij}) = \sigma_{11}(\sigma_{22}\sigma_{33} - \sigma_{23}\sigma_{32}) + 0 + 0 = 25\left(16 \cdot 4 - 0 \cdot 0\right) = \boxed{1600}$$

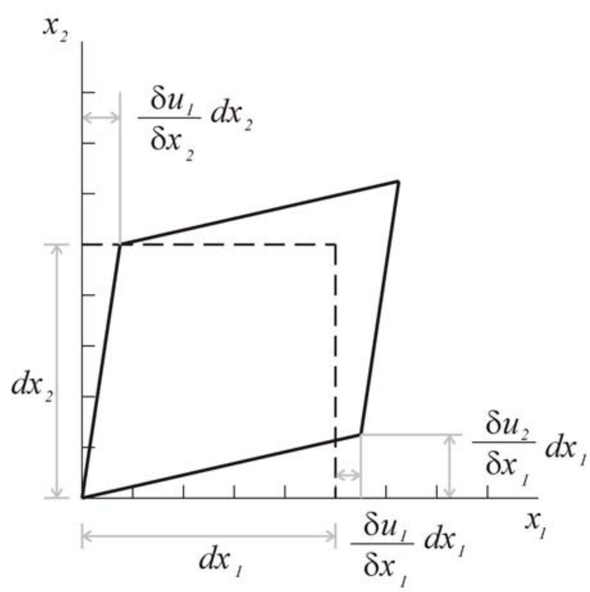
This example illustrates that the invariants do not change for a given stress state for different orientations of axes.

7. Define the terms homogeneous and isotropic in a continuous medium at the continuum level. Does one necessarily imply the other? Explain.

Homogeneous means that properties are identical at all points in the material (with respect to one orientation). *Isotropic* means that properties do not vary with direction or orientation. Thus, all isotropic materials are necessarily homogeneous. However, the reverse is not true, as in anisotropic materials that are homogeneous.

8. Use the given deformed element to answer the questions.

a. Write an expression for displacement ($u_1(x_1,x_2)$ and $u_2\ (x_1,x_2)$) in both directions, x_1 and x_2.



From the deformed element, we can measure the components of deformation, $\frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{x}_1}, \frac{\partial \boldsymbol{u}_2}{\partial \boldsymbol{x}_1}, \frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{x}_2}$, and $\frac{\partial \boldsymbol{u}_2}{\partial \boldsymbol{x}_2}$.

$\frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{x}_1} = \frac{\frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{x}_1} \boldsymbol{dx}_1}{\boldsymbol{dx}_1} \approx \frac{0.4}{3.8} \approx 0.105263$ The ratio of the two lengths is important; the scale is arbitrary. (Strain is a dimensionless quantity.)

$\frac{\partial \boldsymbol{u}_2}{\partial \boldsymbol{x}_1} \approx 0.250$

$\frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{x}_2} \approx 0.157895$

$\frac{\partial \boldsymbol{u}_2}{\partial \boldsymbol{x}_2} \approx 0.0$

From these values we can find expressions for displacement.

$$\boldsymbol{u}_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{x}_1} \boldsymbol{x}_1 + \frac{\partial \boldsymbol{u}_1}{\partial \boldsymbol{x}_2} \boldsymbol{x}_2 = \boxed{0.105263 \boldsymbol{x}_1 + 0.157895 \boldsymbol{x}_2}$$

$$\boldsymbol{u}_2(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{\partial \boldsymbol{u}_2}{\partial \boldsymbol{x}_1} \boldsymbol{x}_1 + \frac{\partial \boldsymbol{u}_2}{\partial \boldsymbol{x}_2} \boldsymbol{x}_2 = 0.250 \boldsymbol{x}_1 + 0.0 \boldsymbol{x}_2 = \boxed{0.250 \boldsymbol{x}_1}$$

b. Determine the strain matrix for this 2D example.

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix}$$

The strain tensor is defined as $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. From the above values for the derivatives of displacement, we get

$$\varepsilon_{ij} = \begin{bmatrix} 0.105263 & 0.2039475 \\ 0.2039475 & 0 \end{bmatrix}.$$

c. Determine the rotational part of the deformation matrix.

$$\begin{bmatrix} 0 & \varpi_{12} \\ \varpi_{21} & 0 \end{bmatrix}$$

The rotation is defined by $\varpi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$. From the above values for the derivatives of displacement, we get

$$\varpi_{ij} = \begin{bmatrix} 0 & -0.0460525 \\ 0.0460525 & 0 \end{bmatrix}.$$