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3.22 MECHANICAL PROPERTIES OF MATERIALS
PROBLEM SET 4 SOLUTIONS

1. *Consider a 500 nm thick aluminum film on a 500 μm thick silicon wafer, 200 mm in diameter. After deposition at 60°C, the wafer is cooled to room temperature. What is the radius of curvature of the film? (E_{Si} = 150 GPa, ν_{Si} = 0.17, $\alpha_{Si} = 3\times10^{-6}\ ^{\circ}C^{-1}$, E_{Al} = 69 GPa, ν_{Al} = 0.33, $\alpha_{Al} = 23\times10^{-6}\ ^{\circ}C^{-1}$)*

The use of the Stoney formula is appropriate for this system because the aluminum film is very thin compared to the relatively thick substrate. The Stoney formula is given by

$$\rho = \frac{\bar{E}_s h_s^2}{6\sigma_m h_f},$$

where ρ is the radius of curvature, $\bar{E}_s$ is the biaxial modulus of the substrate, h_s and h_f are the substrate and film thickness, and σ_m is the stress at the interface.

The biaxial modulus of the substrate is calculated by

$$\bar{E}_s = \frac{E_s}{1-\nu_s} = \frac{150\,\text{GPa}}{1\text{-}0.17} = 180.7\,\text{GPa}.$$

The stress at the interface in the aluminum film is given by

$$\sigma_m = \varepsilon_m \bar{E}_f = \varepsilon_m \frac{E_f}{1-\nu_f}.$$

To find the strain at the interface, we must calculate the mismatch strain due to the thermal expansion coefficient mismatch.

$$\varepsilon_m = (\alpha_s - \alpha_f)\cdot\Delta T = (3\times10^{-6}\ ^{\circ}\text{C}^{-1} - 23\times10^{-6}\ ^{\circ}\text{C}^{-1})\cdot(20\,^{\circ}\text{C} - 60^{\circ}\,\text{C}) = 0.0008$$

Solving for the mismatch stress,

$$\sigma_m = \varepsilon_m \bar{E}_f = \varepsilon_m \frac{E_f}{1-\nu_f} = 0.0008\frac{69\,\text{GPa}}{1-0.33} = 82.39\,\text{MPa}$$

The radius of curvature of the film-on-substrate system can be calculated from the Stoney formula.

$$\rho = \frac{\bar{E}_s h_s^2}{6\sigma_m h_f} = \frac{(180.7\,\text{GPa})(500\times10^{-6}\,\text{m})^2}{6(0.08239\,\text{GPa})(500\times10^{-9}\,\text{m})} = \boxed{182.8\,\text{m}}$$

The positive curvature (concave) indicates that the film is in tension after cooling to room temperature. Because the thermal expansion coefficient for aluminum is larger than silicon, the aluminum attempts to contract more than the silicon during the cooling. However, at the interface, the aluminum cannot contract fully due to the constraining silicon, and as a result the aluminum is in tension.

2. *A bilyer is composed of a 1 mm thick coating (* $E = 350$ GPa, $\alpha = 9\times10^{-6}\ {}^{\circ}\mathrm{C}^{-1}$ *) on a 5 mm substrate (* $E = 70$ GPa, $\alpha = 23\times10^{-6}\ {}^{\circ}\mathrm{C}^{-1}$ *) and has a width of 5 mm. The bilayer has zero curvature at the initial temperature of 20°C. The bilayer is then heated to 100°C. Calculate the resulting curvature and the maximum tensile and compressive stresses. Where do the maximum internal stresses occur?*

To find the curvature of the bilayer, we can use the equation for the most general bilayer derived in class

$$\kappa = \frac{(\alpha_2 - \alpha_1)(T - T_0)}{\dfrac{h}{2} + \dfrac{2(E_1 I_1 + E_2 I_1)}{h}\left(\dfrac{1}{E_1 a_1 b} + \dfrac{1}{E_2 a_2 b}\right)}$$

where the coating properties are denoted by the subscript 1 and the substrate properties by 2. To calculate the curvature, we need the moments of inertia for each layer.

$$I_1 = \frac{ba_1^3}{12} = \frac{(0.005\text{ m})(0.001\text{ m})^3}{12} = 4.1667\times10^{-13}\ \mathrm{m}^2$$

$$I_2 = \frac{ba_2^3}{12} = \frac{(0.005\text{ m})(0.005\text{ m})^3}{12} = 5.20833\times10^{-11}\ \mathrm{m}^4$$

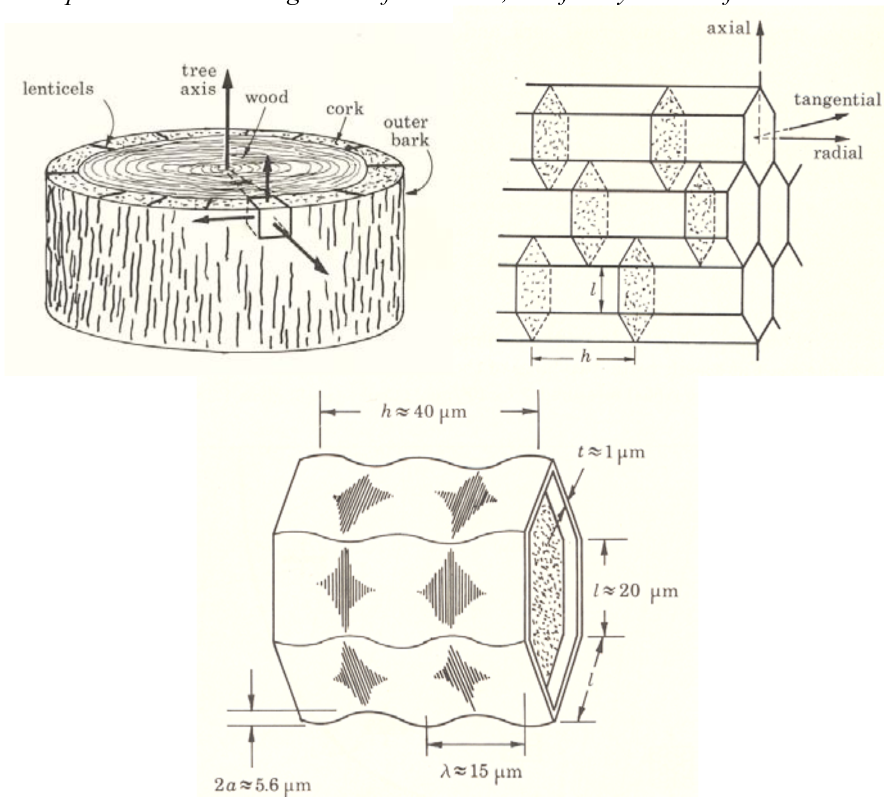
Entering the parameters of the bilayer system into the given equation, we find that the curvature of the bilayer is $\boxed{\kappa = 0.25\ \mathrm{m}^{-1}}$.

The maximum stresses for a bilayer were derived in class and exist at the interface of the two layers.

$$\sigma_{\max,1} = \kappa\left(\frac{2(E_1 I_1 + E_2 I_2)}{ha_1 b} + \frac{a_1 E_1}{2}\right) = \boxed{107\text{ MPa}}$$

$$\sigma_{\max,2} = -\kappa\left(\frac{2(E_1 I_1 + E_2 I_2)}{ha_2 b} + \frac{a_2 E_2}{2}\right) = \boxed{-56.4\text{ MPa}}$$

3. *The structure and mechanics of cork: The structure and properties of cork are approximately axisymmetric. In the schematics on the next page, we see a macroscopic view of cork as it exists on a tree, a microscopic look at the arrangement of cork cells, and finally a view of an individual cork cell.*



Ignoring the natural variations in cell structure, we can model this cellular material as a linear-elastic solid by

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \,.$$

Axisymmetry reduces the number of independent compliances to five (note that the axis of symmetry is the x_1 axis).

$$\left.\begin{aligned}
\varepsilon_{11} &= S_{1111}\sigma_{11} + S_{1122}\sigma_{22} + S_{1122}\sigma_{33} \\
\varepsilon_{22} &= S_{1122}\sigma_{11} + S_{2222}\sigma_{22} + S_{2233}\sigma_{33} \\
\varepsilon_{33} &= S_{1122}\sigma_{11} + S_{2233}\sigma_{22} + S_{2222}\sigma_{33} \\
\varepsilon_{23} &= (S_{2222} - S_{2233})\sigma_{23} \\
\varepsilon_{31} &= 2S_{1212}\sigma_{31} \\
\varepsilon_{12} &= 2S_{1212}\sigma_{12}
\end{aligned}\right\}$$

Measurement of four of the five compliances is straightforward. Simple tensile or compressive tests are done on the cork along the axis of symmetry (x_1 axis) and orthogonal to this direction (x_2 and x_3 axes) to measure the Young's modulus and Poisson's ratio in these directions.

$$\left.\begin{aligned}
&1/E_1 = S_{1111} \\
&1/E_2 = 1/E_3 = S_{2222} \\
&\nu_{12}/E_1 = \nu_{13}/E_1 = \nu_{21}/E_2 = \nu_{31}/E_2 = -S_{1122} \\
&\nu_{23}/E_2 = \nu_{32}/E_2 = -S_{2233}
\end{aligned}\right\}$$

The modulus G_{23} in the x_2, x_3-plane is obtained from these measurements by

$$1/G_{23} = 2(1+\nu_{23})/E_2 = 2(S_{2222} - S_{2233}) \,.$$

To determine the fifth compliance value, S_{1212}, which is related to the shear modulus, G_{12}, we rotate the cork through 45° about the x_3-axis and cut a cube with one face normal to the x_3-axis, and the other two at 45° to the x_2-axis and x_1-axis. A simple compression test in the new x_1 direction then gives a new Young's modulus, E'.

a. *What is the relationship between the shear modulus, G_{12}, and S_{1212}?*

The shear modulus G_{12} relates the engineering shear strain and shear stress. Using matrix notation, we have

$$\varepsilon_6 = S_{66}\sigma_6 = \frac{1}{G_{12}}\sigma_6 .$$

To convert between matrix and tensor notation, we have the relation that

$$S_{66} = 4S_{1212} .$$

Thus we find that

$$\boxed{G_{12} = 1/4S_{1212}} .$$

b. *By rotating the S_{ijkl} through 45° about the x_3-axis, obtain an expression for S'_{1111} in terms of the compliances in the original orientation.*

The direction cosines for this change in orientation by rotation about the x_3-axis is

$$\ell_{ij} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The tensor equation for the transformation of a fourth order tensor is $S'_{ijkl} = \ell_{mi}\ell_{nj}\ell_{ok}\ell_{pl}S_{mnop}$. To find the S'_{1111} component, we use the transformation rule.

$$S'_{1111} = \ell_{m1}\ell_{n1}\ell_{o1}\ell_{p1}S_{mnop}$$

Although the expanded form of the tensor transformation has 81 terms per equation, we note that ℓ_{11} and ℓ_{21} are the only nonzero direction cosine terms that appear in the tensor transformation. The resulting expanded form is

$$\begin{aligned} S'_{1111} = {} & \ell_{11}\ell_{11}\ell_{11}\ell_{11}S_{1111} + \cdots \\ & \ell_{11}\ell_{11}\ell_{11}\ell_{21}S_{1112} + \ell_{11}\ell_{11}\ell_{21}\ell_{11}S_{1121} + \ell_{11}\ell_{21}\ell_{11}\ell_{11}S_{1211} + \ell_{21}\ell_{11}\ell_{11}\ell_{11}S_{2111} + \cdots \\ & \ell_{11}\ell_{11}\ell_{21}\ell_{21}S_{1122} + \ell_{11}\ell_{21}\ell_{21}\ell_{11}S_{1221} + \ell_{11}\ell_{21}\ell_{11}\ell_{21}S_{1212} + \cdots \\ & \ell_{21}\ell_{11}\ell_{11}\ell_{21}S_{2112} + \ell_{21}\ell_{11}\ell_{21}\ell_{11}S_{2121} + \ell_{21}\ell_{21}\ell_{11}\ell_{11}S_{2211} + \cdots \\ & \ell_{21}\ell_{21}\ell_{21}\ell_{11}S_{2221} + \ell_{21}\ell_{21}\ell_{11}\ell_{21}S_{2212} + \ell_{21}\ell_{11}\ell_{21}\ell_{21}S_{2122} + \ell_{11}\ell_{21}\ell_{21}\ell_{21}S_{1222} + \cdots \\ & \ell_{21}\ell_{21}\ell_{21}\ell_{21}S_{2222} \end{aligned}$$

From the symmetry of the strain tensor and stress tensor that are related by the compliance tensor, we can simplify the expression. (Note that S_{1112} and S_{2212} are zero)

$$S'_{1111} = (\ell_{11})^4 S_{1111} + \cancel{4(\ell_{11})^3(\ell_{21})S_{1112}} + 2(\ell_{11})^2(\ell_{21})^2 S_{1122} + 4(\ell_{11})^2(\ell_{21})^2 S_{1212} + \cancel{4(\ell_{21})^3(\ell_{11})S_{2221}} + (\ell_{21})^4 S_{2222}$$

In tensor notation, the expression for the compliance in the new direction is

$$\boxed{S'_{1111} = \frac{1}{4}S_{1111} + \frac{1}{2}S_{1122} + S_{1212} + \frac{1}{4}S_{2222}}$$

c. *Use your answers from parts (a) and (b) to find an expression for the shear modulus G_{12}, in terms of experimental parameters* E_1, E_2, ν_{12}, and E'.

From part (a) we have

$$\frac{1}{G_{12}} = 4S_{1212} .$$

From the answer to part (b) we can solve for S_{1212} and substitute this expression into the equation above.

$$\frac{1}{G_{12}} = 4\left[S'_{1111} - \frac{1}{4}S_{1111} - \frac{1}{2}S_{1122} - \frac{1}{4}S_{2222} \right]$$

Replacing the compliance values with the corresponding experimental values leads to the result (Note that the value of the compliance in the new direction is related to the new Young's modulus by $S'_{1111} = 1/E'$.)

$$\frac{1}{G_{12}} = 4\left[\frac{1}{E'} - \frac{1}{4}\frac{1}{E_1} - \frac{-1}{2}\frac{\nu_{12}}{E_1} - \frac{1}{4}\frac{1}{E_2} \right].$$

$$\boxed{\frac{1}{G_{12}} = \frac{4}{E'} - \frac{1-2\nu_{12}}{E_1} - \frac{1}{E_2}}$$

d. *For the isotropic case, what does the equation derived in part (c) become?*

For an isotropic material, which is more symmetric than the transversely isotropic symmetry assumed above, the moduli and Poisson's ratios do not vary with direction. Hence, we can rewrite the answer in part (c) as

$$\frac{1}{G} = \frac{4}{E} - \frac{1-2\nu}{E} - \frac{1}{E} .$$

Simple algebra leads to the result

$$\frac{1}{G} = \frac{4-1+2\nu-1}{E}$$

$$\frac{1}{G} = \frac{2+2\nu}{E}$$

$$\Rightarrow \boxed{G = \frac{E}{2(1+\nu)}}$$

Thus, the expression in part (c) correctly reduces to the relationship between the elastic modulus, shear modulus and Poisson's ratio for the isotropic case.

4. *An isotropic sample of material subjected to a compressive stress $\sigma_z = -p$ is confined so that it cannot deform in either the x- or y-directions.*
   a. *Do the stresses occur in the material in the x- and y-directions? If so, how are they related to σ_z?*

If the sample were not confined in the x- and y-directions, we would expect the material to expand in the transverse direction upon compressive loading. The fact that the material is constrained in the transverse direction indicates that a stress must evolve to maintain zero strain in the transverse direction. Looking at Hooke's law for an isotropic material in three dimensions, we find

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) = 0 .$$

Due to the symmetry of the problem, we know $\sigma_x = \sigma_y$. Thus, we can find a relationship between the transverse stress and the applied compressive stress

$$\boxed{\sigma_x = \sigma_y = \frac{\nu}{1-\nu}\sigma_z} .$$

   b. *Determine the stiffness $E' = \sigma_z / \varepsilon_z$ in the direction of the applied stress in terms of the isotropic elastic constants E and ν for the material. Is E' equal to the elastic modulus from uniaxial loading? Why or why not?*

Again using Hooke's law to find the strain in the z-direction,

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) = \frac{\sigma_z}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right) .$$

Using this expression for the strain in the z-direction, we can calculate the stiffness in the z-direction.

$$E' = \frac{\sigma_z}{\varepsilon_z} = \frac{\sigma_z}{\frac{\sigma_z}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right)} = \boxed{\frac{E}{\left(1 - \frac{2\nu^2}{1-\nu}\right)}}$$

The stiffness calculated here is not equal to the elastic modulus in uniaxial loading. In uniaxial loading, the transverse stresses are zero and do not contribute to the uniaxial deformation. However, in this case the transverse stresses do affect the strain in the direction of loading.

   c. *What happens if the Poisson's ratio for the material approaches 0.5?*

As Poisson's ratio approaches 0.5, the strain in the z-direction goes to zero for any applied compressive stress. This can be understood without derivation. We know that materials with a Poisson's ratio of 0.5 are incompressible ($\varepsilon_x + \varepsilon_y + \varepsilon_z = 0$). Because the transverse directions are constrained (zero strain), the strain in the loading direction must be zero to satisfy the incompressibility condition, regardless of the applied compressive stress.

5. *Consider a flat plate of isotropic material that lies in the x-y plane and which is subjected to applied loading in this plane only. Such a plate is under plane stress, so that* $\sigma_z = \tau_{yz} = \tau_{xz} = 0 \text{ MPa}$.

   a. *Does the thickness of the plate usually change when the plate is loaded?*

   For all non-zero Poisson's ratio materials, there will be some change in the thickness of the plate due to the Poisson effect. By inspecting Hooke's law for a three dimensional isotropic material

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

   we notice that even in a state of plane stress, the stresses in the orthogonal directions contribute to the strain in the out-of-plane direction.

   b. *Under what conditions does the thickness not change? That is, when is this state of plane stress also a state of plane strain?*

   Looking at Hooke's law again, it is clear that if the stresses in the orthogonal directions to the out-of-plane direction are equal and opposite, the out-of-plane strain is necessarily zero, indicating no thickness change.