Topic one: Production line profit maximization subject to a production rate constraint

The profit maximization problem

$$\max_{\mathbf{N}} \quad J(\mathbf{N}) = AP(\mathbf{N}) - \sum_{i=1}^{k-1} b_i N_i - \sum_{i=1}^{k-1} c_i \bar{n}_i(\mathbf{N})$$

$$\text{s.t.} \quad P(\mathbf{N}) \geq \hat{P},$$

$$N_i \geq N_{\min}, \forall i = 1, \cdots, k-1.$$

where
- $P(\mathbf{N})$ = production rate, parts/time unit
- $\hat{P}$ = required production rate, parts/time unit
- A = profit coefficient, \$/part
- $\bar{n}_i(\mathbf{N})$ = average inventory of buffer $i, i = 1, \cdots, k-1$
- b_i = buffer cost coefficient, \$/part/time unit
- c_i = inventory cost coefficient, \$/part/time unit

An example about the research goal

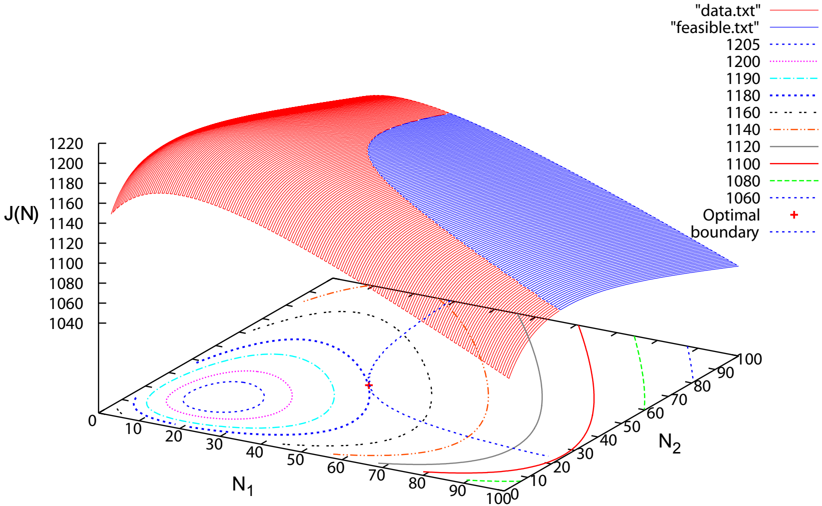


Figure 2: $J(\mathbf{N})$ vs. N_1 and N_2

Two problems

Original constrained problem

$$\begin{aligned}
\max_{\mathbf{N}} \quad J(\mathbf{N}) &= AP(\mathbf{N}) - \sum_{i=1}^{k-1} b_i N_i - \sum_{i=1}^{k-1} c_i \bar{n}_i(\mathbf{N}) \\
\text{s.t.} \quad P(\mathbf{N}) &\geq \hat{P}, \\
N_i &\geq N_{\min}, \forall i = 1, \cdots, k-1.
\end{aligned}$$

Simpler unconstrained problem (Schor's problem)

$$\begin{aligned}
\max_{\mathbf{N}} \quad J(\mathbf{N}) &= AP(\mathbf{N}) - \sum_{i=1}^{k-1} b_i N_i - \sum_{i=1}^{k-1} c_i \bar{n}_i(\mathbf{N}) \\
\text{s.t.} \quad N_i &\geq N_{\min}, \forall i = 1, \cdots, k-1.
\end{aligned}$$

Data

$r_1 = .1, p_1 = .01, r_2 = .11, p_2 = .01, r_3 = .1, p_3 = .009, \hat{P} = .88$

Cost function

$J(\mathbf{N}) = 2000P(\mathbf{N}) - N_1 - N_2 - \bar{n}_1(\mathbf{N}) - \bar{n}_2(\mathbf{N})$

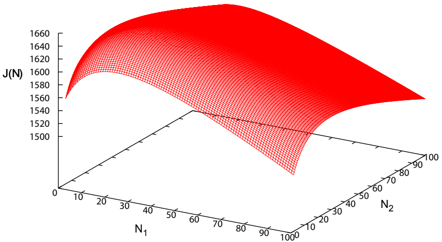


Figure 3: $J(\mathbf{N})$ vs. N_1 and N_2

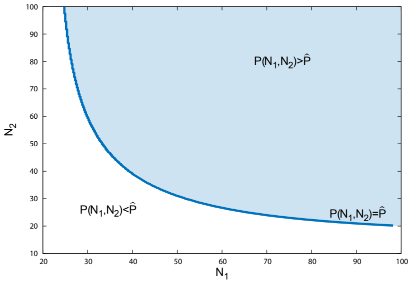


Figure 4: $P(\mathbf{N})$

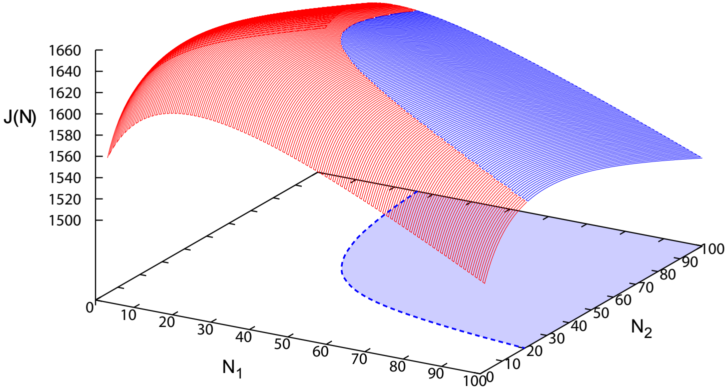


Figure 5: $J(\mathbf{N})$ vs. N_1 and N_2

Two cases

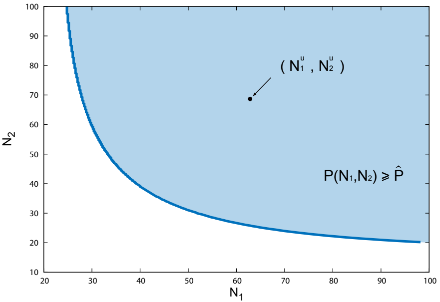
Case 1

The solution of the unconstrained problem is $\mathbf{N}^u$ s.t. $P(\mathbf{N}^u) \geq \hat{P}$. In this case, the solution of the constrained problem is the same as the solution of the unconstrained problem. We are done.

Unconstrained problem

$$\max_{\mathbf{N}} J(\mathbf{N}) = AP(\mathbf{N}) - \sum_{i=1}^{k-1} b_i N_i - \sum_{i=1}^{k-1} c_i \bar{n}_i(\mathbf{N})$$

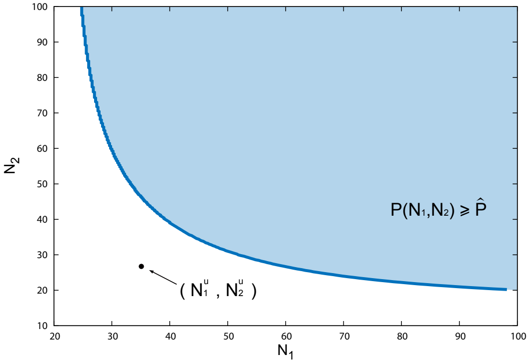
$$\text{s.t. } N_i \geq N_{\min}, \forall i = 1, \cdots, k-1.$$



Two cases (continued)

Case 2

$\mathbf{N}^u$ satisfies $P(\mathbf{N}^u) < \hat{P}$. This is not the solution of the constrained problem.



Two cases (continued)

Case 2 (continued)

In this case, we consider the following unconstrained problem:

$$\max_{\mathbf{N}} \quad J(\mathbf{N}) \quad = \quad A'P(\mathbf{N}) - \sum_{i=1}^{k-1} b_i N_i - \sum_{i=1}^{k-1} c_i \bar{n}_i(\mathbf{N})$$

$$\text{s.t.} \qquad N_i \quad \geq \quad N_{\min}, \forall i = 1, \cdots, k-1.$$

in which A is replaced by A'. Let $\mathbf{N}^\star(A')$ be the solution to this problem and $P^\star(A') = P(\mathbf{N}^\star(A'))$.

The constrained problem

$$\max_{\mathbf{N}} \quad J(\mathbf{N}) \quad = \quad A'P(\mathbf{N}) - \sum_{i=1}^{k-1} b_i N_i - \sum_{i=1}^{k-1} c_i \bar{n}_i(\mathbf{N})$$

$$\text{s.t.} \quad P(\mathbf{N}) \quad \geq \quad \hat{P},$$

$$N_i \quad \geq \quad N_{\min}, \forall i = 1, \cdots, k-1.$$

has the same solution for all A' in which the solution of the corresponding unconstrained problem

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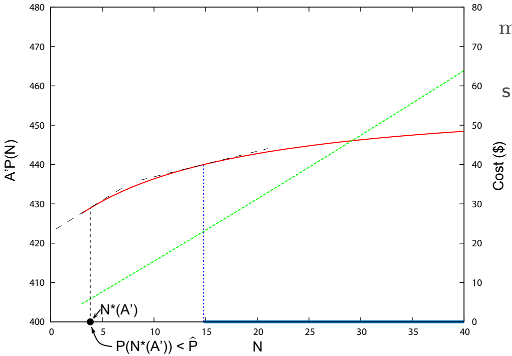
$$\text{s.t.} \quad N_i \quad \geq \quad N_{\min}, \forall i = 1, \cdots, k-1.$$

has $P^{\star}(A') \leq \hat{P}$.

We claim

If the optimal solution of the unconstrained problem is not that of the constrained problem, then the solution of the constrained problem, $(N_1^\star, \cdots, N_{k-1}^\star)$, satisfies $P(N_1^\star, \cdots, N_{k-1}^\star) = \hat{P}$.



$$\max_{\mathbf{N}} J(N) = 500P(N) - N - \bar{n}(N)$$

$$\text{s.t. } P(N) \geq \hat{P}$$
$$N \geq N_{\min}$$

$$\max_{\mathbf{N}} J(\mathbf{N}) = 500\hat{P} - N - \bar{n}(N)$$

$$\text{s.t. } P(N) \geq \hat{P} \Rightarrow P(N) = \hat{P}$$
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We formally prove this by the Karush-Kuhn-Tucker (KKT) conditions of nonlinear programming.

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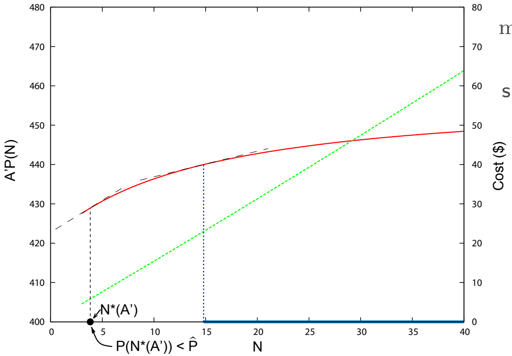

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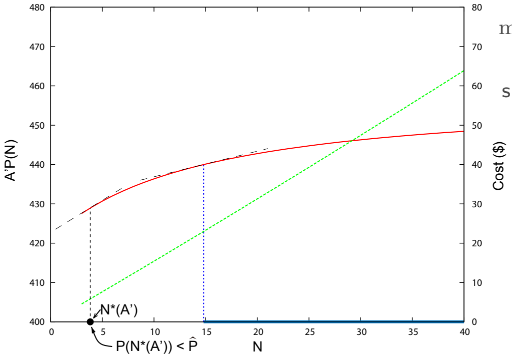

$$\begin{aligned} \max_{\mathbf{N}} J(N) &= 500P(N) - N - \bar{n}(N) \\ \text{s.t. } P(N) &\geq \hat{P} \\ N &\geq N_{\min} \end{aligned}$$

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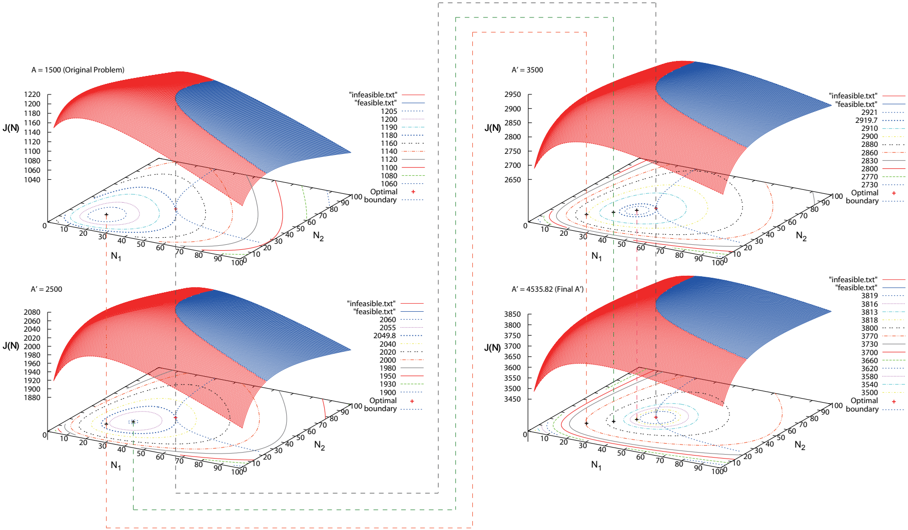
We formally prove this by the Karush-Kuhn-Tucker (KKT) conditions of nonlinear programming.

Interpretation of the assertion



Let $x^\star$ be a local minimum of the problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & h_1(x) = 0, \cdots, h_m(x) = 0, \\ & g_1(x) \le 0, \cdots, g_r(x) \le 0, \end{aligned}$$

where f, h_i, and g_j are continuously differentiable functions from $\Re^n$ to $\Re$. Then there exist unique Lagrange multipliers $\lambda_1^\star, \cdots, \lambda_m^\star$ and $\mu_1^\star, \cdots, \mu_r^\star$, satisfying the following conditions:

$$\nabla_x L(x^\star, \lambda^\star, \mu^\star) = 0,$$

$$\mu_j^\star \ge 0, j = 1, \cdots, r,$$

$$\mu_j^\star g_j(x^\star) = 0, j = 1, \cdots, r.$$

where $L(x, \lambda, \mu) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^r \mu_j g_j(x)$ is called the Lagrangian function.

Minimization form

The constrained problem

$$\begin{aligned}\min_{\mathbf{N}} \quad -J(\mathbf{N}) &= -AP(\mathbf{N}) + \sum_{i=1}^{k-1} b_i N_i + \sum_{i=1}^{k-1} c_i \bar{n}_i(\mathbf{N}) \\ \text{s.t.} \quad \hat{P} - P(\mathbf{N}) &\leq 0 \\ N_{\min} - N_i &\leq 0, \forall i = 1, \cdots, k-1\end{aligned}$$

We have argued that we treat N_i as continuous variables, and $P(N)$ and $J(N)$ as continuously differentiable functions.

The Slater constraint qualification for convex inequalities guarantees the existence of Lagrange multipliers for our problem. So, there exist unique Lagrange multipliers $\mu_i^\star, i = 0, \cdots, k-1$ for the constrained problem to satisfy the KKT conditions:

$$-\nabla J(\mathbf{N}^\star) + \mu_0^\star \nabla(\hat{P} - P(\mathbf{N}^\star)) + \sum_{i=1}^{k-1} \mu_i^\star \nabla(N_{\min} - N_i) = 0 \qquad (1)$$

or

$$-\begin{pmatrix} \frac{\partial J(\mathbf{N}^\star)}{\partial N_1} \\ \frac{\partial J(\mathbf{N}^\star)}{\partial N_2} \\ \vdots \\ \frac{\partial J(\mathbf{N}^\star)}{\partial N_{k-1}} \end{pmatrix} - \mu_0^\star \begin{pmatrix} \frac{\partial P(\mathbf{N}^\star)}{\partial N_1} \\ \frac{\partial P(\mathbf{N}^\star)}{\partial N_2} \\ \vdots \\ \frac{\partial P(\mathbf{N}^\star)}{\partial N_{k-1}} \end{pmatrix} - \mu_1^\star \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \cdots - \mu_{k-1}^\star \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad (2)$$

and

$$\mu_i^\star \geq 0, \forall i = 0, \cdots, k-1, \quad (3)$$

$$\mu_0^\star(\hat{P} - P(\mathbf{N}^\star)) = 0, \quad (4)$$

$$\mu_i^\star(N_{\min} - N_i^\star) = 0, \forall i = 1, \cdots, k-1, \quad (5)$$

where $\mathbf{N}^\star$ is the optimal solution of the constrained problem. Assume that $N_i^\star > N_{\min}$ for all i. In this case, by equation (5), we know that $\mu_i^\star = 0, \forall i = 1, \cdots, k-1$.

The KKT conditions are simplified to

$$-\begin{pmatrix} \frac{\partial J(\mathbf{N}^\star)}{\partial N_1} \\ \frac{\partial J(\mathbf{N}^\star)}{\partial N_2} \\ \vdots \\ \frac{\partial J(\mathbf{N}^\star)}{\partial N_{k-1}} \end{pmatrix} - \mu_0^\star \begin{pmatrix} \frac{\partial P(\mathbf{N}^\star)}{\partial N_1} \\ \frac{\partial P(\mathbf{N}^\star)}{\partial N_2} \\ \vdots \\ \frac{\partial P(\mathbf{N}^\star)}{\partial N_{k-1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{6}$$

$$\mu_0^\star(\hat{P} - P(\mathbf{N}^\star)) = 0, \tag{7}$$

where $\mu_0^\star \geq 0$. Since $\mathbf{N}^\star$ is not the optimal solution of the unconstrained problem, $\nabla J(\mathbf{N}^\star) \neq 0$. Thus, $\mu_0^\star \neq 0$ since otherwise condition (6) would be violated. By condition (7), the optimal solution $\mathbf{N}^\star$ satisfies $P(\mathbf{N}^\star) = \hat{P}$.

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$$\mu_0^\star(\hat{P} - P(\mathbf{N}^\star)) = 0,$$

In addition, conditions (6) and (7) reveal how we could find $\mu_0^\star$ and $\mathbf{N}^\star$. For every $\mu_0^\star$, condition (6) determines $\mathbf{N}^\star$ since there are $k-1$ equations and $k-1$ unknowns. Therefore, we can think of $\mathbf{N}^\star = \mathbf{N}^\star(\mu_0^\star)$. We search for a value of $\mu_0^\star$ such that $P(\mathbf{N}^\star(\mu_0^\star)) = \hat{P}$. As we indicate in the following, this is exactly what the algorithm does.

Replacing $\mu_0^\star$ by $\mu_0 > 0$ in constraint (6) gives

$$-\begin{pmatrix} \frac{\partial J(\mathbf{N}^c)}{\partial N_1} \\ \frac{\partial J(\mathbf{N}^c)}{\partial N_2} \\ \vdots \\ \frac{\partial J(\mathbf{N}^c)}{\partial N_{k-1}} \end{pmatrix} - \mu_0 \begin{pmatrix} \frac{\partial P(\mathbf{N}^c)}{\partial N_1} \\ \frac{\partial P(\mathbf{N}^c)}{\partial N_2} \\ \vdots \\ \frac{\partial P(\mathbf{N}^c)}{\partial N_{k-1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{8}$$

where $\mathbf{N}^c$ is the unique solution of (8). Note that $\mathbf{N}^c$ is the solution of the following optimization problem:

$$\begin{aligned} \min_{\mathbf{N}} \quad & -\bar{J}(\mathbf{N}) = -J(\mathbf{N}) + \mu_0(\hat{P} - P(\mathbf{N})) \\ \text{s.t.} \quad & N_{\min} - N_i \le 0, \forall i = 1, \cdots, k-1. \end{aligned} \tag{9}$$

The problem above is equivalent to

$$\max_{\mathbf{N}} \quad \bar{J}(\mathbf{N}) = J(\mathbf{N}) - \mu_0(\hat{P} - P(\mathbf{N}))$$
$$\text{s.t.} \quad N_{\min} - N_i \leq 0, \forall i = 1, \cdots, k-1. \tag{10}$$

or

$$\max_{\mathbf{N}} \quad \bar{J}(\mathbf{N}) = AP(\mathbf{N}) - \sum_{i=1}^{k-1} b_i N_i - \sum_{i=1}^{k-1} c_i \bar{n}_i - \mu_0(\hat{P} - P(\mathbf{N}))$$
$$\text{s.t.} \quad N_{\min} - N_i \leq 0, \forall i = 1, \cdots, k-1. \tag{11}$$

or

$$\max_{\mathbf{N}} \quad \bar{J}(\mathbf{N}) = (A + \mu_0)P(\mathbf{N}) - \sum_{i=1}^{k-1} b_i N_i - \sum_{i=1}^{k-1} c_i \bar{n}_i$$
$$\text{s.t.} \quad N_i \geq N_{\min}, \forall i = 1, \cdots, k-1. \tag{12}$$

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or, finally,

$$\max_{\mathbf{N}} \quad \bar{J}(\mathbf{N}) = A'P(\mathbf{N}) - \sum_{i=1}^{k-1} b_i N_i - \sum_{i=1}^{k-1} c_i \bar{n}_i \tag{13}$$

$$\text{s.t.} \quad N_i \geq N_{\min}, \forall i = 1, \cdots, k-1.$$

where $A' = A + \mu_0$. This is exactly the unconstrained problem, and $\mathbf{N}^c$ is its optimal solution. Note that $\mu_0 > 0$ indicates that $A' > A$.

In addition, the KKT conditions indicate that the optimal solution of the constrained problem $\mathbf{N}^\star$ satisfies $P(\mathbf{N}^\star) = \hat{P}$. This means that, for every $A' > A$ (or $\mu_0 > 0$), we can find the corresponding optimal solution $\mathbf{N}^c$ satisfying condition (8) by solving problem (13). We need to find the A' such that the solution to problem (13), denoted as $\mathbf{N}^\star(A')$, satisfies $P(\mathbf{N}^\star(A')) = \hat{P}$.

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$$\max_{\mathbf{N}} \quad \bar{J}(\mathbf{N}) = A' P(\mathbf{N}) - \sum_{i=1}^{k-1} b_i N_i - \sum_{i=1}^{k-1} c_i \bar{n}_i \tag{13}$$

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Then, $\mu_0 = A' - A$ and $\mathbf{N}^\star(A')$ satisfy conditions (6) and (7):

$$-\nabla J(\mathbf{N}^\star(A')) + \mu_0^\star \nabla(\hat{P} - P(\mathbf{N}^\star(A'))) = 0,$$

$$\mu_0^\star(\hat{P} - P(\mathbf{N}^\star(A'))) = 0.$$

Hence, $\mu_0^\star = A' - A$ is exactly the Lagrange multiplier satisfying the KKT conditions of the constrained problem, and $\mathbf{N}^\star = \mathbf{N}^\star(A')$ is the optimal solution of the constrained problem.

Consequently, solving the constrained problem through our algorithm is essentially finding the unique Lagrange multipliers and optimal solution of the problem.

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Consequently, solving the constrained problem through our algorithm is essentially finding the unique Lagrange multipliers and optimal solution of the problem.

Solve unconstrained problem

Solve, by a gradient method, the unconstrained problem for fixed A'

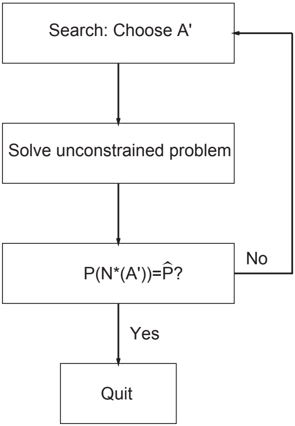
$$\max_{\mathbf{N}} \quad J(\mathbf{N}) \quad = \quad A'P(\mathbf{N}) - \sum_{i=1}^{k-1} b_i N_i - \sum_{i=1}^{k-1} c_i \bar{n}_i(\mathbf{N})$$

$$\text{s.t.} \qquad N_i \quad \geq \quad N_{\min}, \forall i = 1, \cdots, k-1.$$

Search

Do a one-dimensional search on $A' > A$ to find A' such that the solution of the unconstrained problem, $\mathbf{N}^\star(A')$, satisfies

$$P(\mathbf{N}^\star(A')) = \hat{P}.$$



Numerical experiment outline

- Experiments on short lines.
- Experiments on long lines.
- Computation speed.

Method we use to check the algorithm

$\hat{P}$ surface search in $(N_1, \cdots, N_{k-1})$ space. All buffer size allocations, $\mathbf{N}$, such that $P(\mathbf{N}) = \hat{P}$ compose the $\hat{P}$ surface.

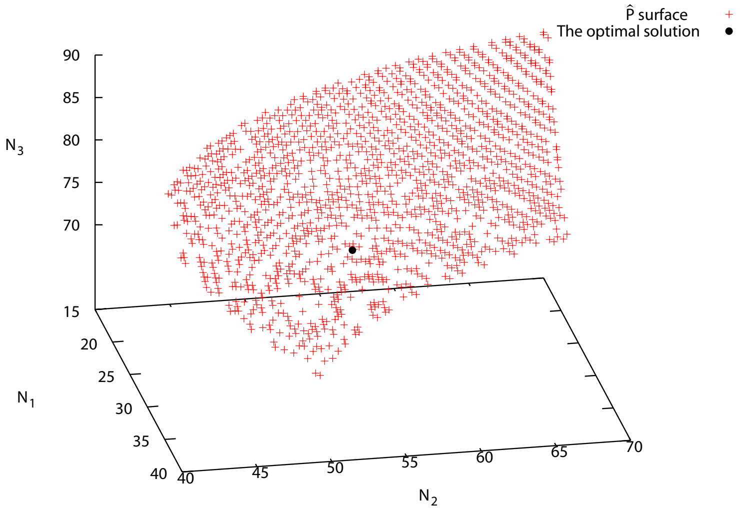


Figure 6: $\hat{P}$ Surface search

- Line parameters: $\hat{P} = .88$

machine	M_1	M_2	M_3	M_4	M_5
r	.11	.12	.10	.09	.10
p	.008	.01	.01	.01	.01

- Machine 4 is the least reliable machine (bottleneck) of the line.
- Cost function

$$J(\mathbf{N}) = 2500P(\mathbf{N}) - \sum_{i=1}^{4} N_i - \sum_{i=1}^{4} \bar{n}_i(\mathbf{N})$$

Results

- Optimal solutions

	$\hat{P}$ Surface Search	The algorithm	Error	Rounded $N^{\star}$
Prod. rate	.8800	.8800		.8800
$N_1^{\star}$	28.85	28.8570	0.02%	29.0000
$N_2^{\star}$	58.46	58.5694	0.19%	59.0000
$N_3^{\star}$	92.98	92.9068	0.08%	93.0000
$N_4^{\star}$	87.39	87.4415	0.06%	87.0000
$\bar{n}_1$	19.0682	19.0726	0.02%	19.1791
$\bar{n}_2$	34.3084	34.3835	0.23%	34.7289
$\bar{n}_3$	48.7200	48.6981	0.04%	48.9123
$\bar{n}_4$	31.9894	32.0063	0.05%	31.9485
Profit ($)	1798.2	1798.1	0.006%	1797.4000

- The maximal error is 0.23% and appears in $\bar{n}_2$.
- Computer time for this experiment is 2.69 seconds.

- Line parameters: $\hat{P} = .88$

machine	M_1	M_2	M_3	M_4	M_5	M_6
r	.11	.12	.10	.09	.10	.11
p	.008	.01	.01	.01	.01	.01

machine	M_7	M_8	M_9	M_{10}	M_{11}	M_{12}
r	.10	.11	.12	.10	.12	.09
p	.009	.01	.009	.008	.01	.009

- Cost function

$$J(\mathbf{N}) = 6000P(\mathbf{N}) - \sum_{i=1}^{11} N_i - \sum_{i=1}^{11} \bar{n}_i(\mathbf{N})$$

Results

- Optimal solutions, buffer sizes:

	$\hat{P}$ Surface Search	The algorithm	Error	Rounded $N^\star$
Prod. rate	.8800	.8800		.8799
$N_1^\star$	29.10	29.1769	0.26%	29.0000
$N_2^\star$	59.20	59.2830	0.14%	59.0000
$N_3^\star$	97.80	97.7980	0.002%	98.0000
$N_4^\star$	107.50	107.4176	0.08%	107.0000
$N_5^\star$	84.50	84.4804	0.02%	84.0000
$N_6^\star$	70.80	70.6892	0.17%	71.0000
$N_7^\star$	63.10	63.1893	0.14%	63.0000
$N_8^\star$	53.10	52.9274	0.33%	53.0000
$N_9^\star$	47.20	47.2232	0.05%	47.0000
$N_{10}^\star$	47.90	47.7967	0.22%	48.0000
$N_{11}^\star$	48.80	48.7716	0.06%	49.0000

Results (continued)

- Optimal solutions, average inventories:

	$\hat{P}$ Surface Search	The algorithm	Error	Rounded $N^\star$
$\bar{n}_1$	19.2388	19.2986	0.31%	19.1979
$\bar{n}_2$	34.9561	35.0423	0.25%	34.8194
$\bar{n}_3$	52.5423	52.6032	0.12%	52.6833
$\bar{n}_4$	45.1528	45.1840	0.07%	45.0835
$\bar{n}_5$	34.4289	34.4770	0.14%	34.2790
$\bar{n}_6$	30.7073	30.7048	0.01%	30.8229
$\bar{n}_7$	28.0446	28.1299	0.30%	28.0902
$\bar{n}_8$	21.5666	21.5438	0.11%	21.5932
$\bar{n}_9$	21.5059	21.5442	0.18%	21.4299
$\bar{n}_{10}$	22.6756	22.6496	0.11%	22.7303
$\bar{n}_{11}$	20.8692	20.8615	0.04%	20.9613
Profit ($)	4239.3	4239.2	0.002%	4239.5000

- Computer time is 91.47 seconds.

Consider a 4-machine 3-buffer line with constraints $\hat{P} = .87$. In addition, $A = 2000$ and all b_i and c_i are 1.

machine	M_1	M_2	M_3	M_4
r_{i1}	.10	.12	.10	.20
p_{i1}	.01	.008	.01	.007
r_{i2}	–	.20	–	.16
p_{i2}	–	.005	–	.004

	$\hat{P}$ Surf. Search	The algorithm	Error
$P(\mathbf{N}^\star)$	.8699	.8699	
$N_1^\star$	29.8600	29.9930	0.45%
$N_2^\star$	38.2200	38.0206	0.52%
$N_3^\star$	20.6800	20.7616	0.39%
$\bar{n}_1$	17.2779	17.3674	0.52%
$\bar{n}_2$	17.2602	17.1792	0.47%
$\bar{n}_3$	6.1996	6.2121	0.20%
Profit ($)	1610.3000	1610.3000	0.00%

Consider a 4-machine 3-buffer line with constraints $\hat{P} = .87$. In addition, $A = 2000$ and all b_i and c_i are 1.

machine	M_1	M_2	M_3	M_4
μ_i	1.0	1.02	1.0	1.0
r_{i1}	.10	.12	.10	.20
p_{i1}	.01	.008	.01	.012
r_{i2}	–	.20	–	.16
p_{i2}	–	.005	–	.006

	$P^\star$ Surf. Search	The algorithm	Error
$P(\mathbf{N}^\star)$	.8699	.8700	
$N_1^\star$	27.7200	27.9042	0.66%
$N_2^\star$	38.7900	38.9281	0.34%
$N_3^\star$	34.0700	34.1574	0.26%
$\bar{n}_1$	15.4288	15.5313	0.66%
$\bar{n}_2$	19.8787	19.9711	0.46%
$\bar{n}_3$	13.8937	13.9426	0.35%
Profit ($)	1590.0000	1589.7000	0.02%

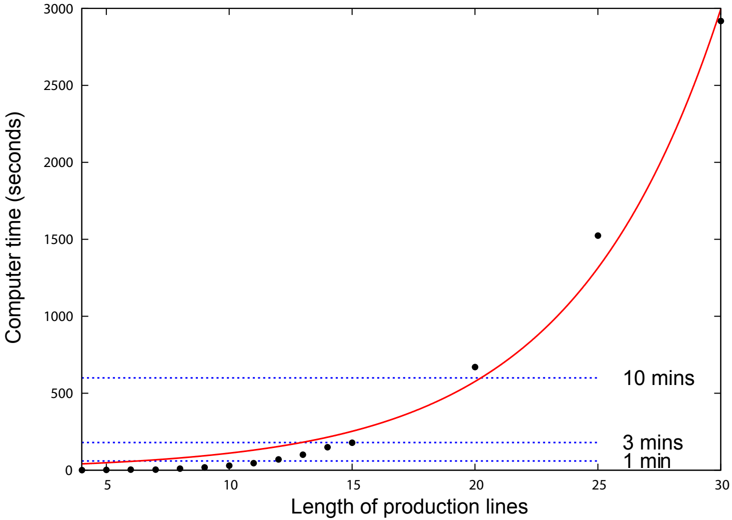
Experiment

- Run the algorithm for a series of experiments for lines having identical machines to see how fast the algorithm could optimize longer lines.
- Length of the line varies from 4 machines to 30 machines.
- Machine parameters are $p = .01$ and $r = .1$.
- In all cases, the feasible production rate is $\hat{P} = .88$.
- The objective function is

$$J(\mathbf{N}) = AP(\mathbf{N}) - \sum_{i=1}^{k-1} N_i - \sum_{i=1}^{k-1} \bar{n}_i(\mathbf{N}).$$

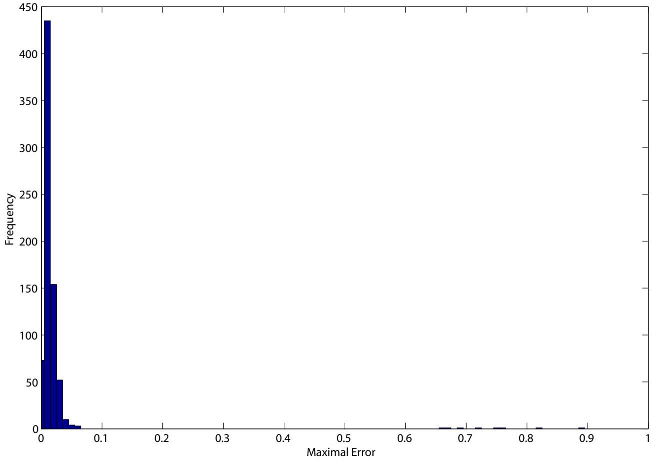
where $A = 500k$ for the line of length k.

Computation speed

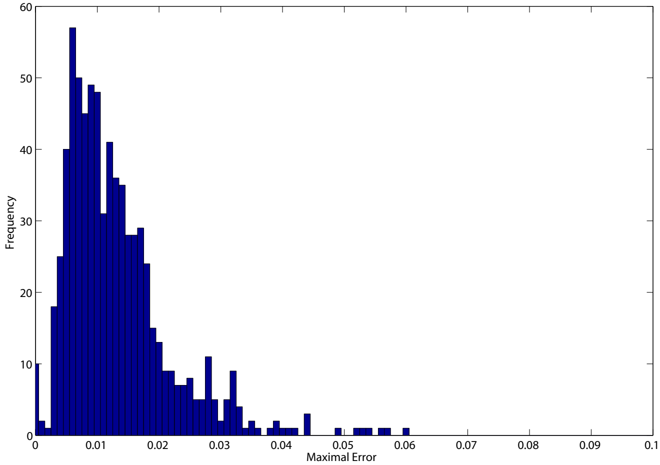


Algorithm reliability

We run the algorithm on 739 randomly generated 4-machine 3-buffer lines. 98.92% of these experiments have a maximal error less than 6%.



Taking a closer look at those 98.92% experiments, we find a more accurate distribution of the maximal error. We find that, out of the total 739 experiments, 83.90% of them have a maximal error less than 2%.



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