BEH. 410 . May 2003

Final exam . SOLUTIONS

Problem 1

Change in total bending energy

Bending energy by unit area of lipid membrane (Kamm Ch 2.1 (2.44))

$$U_b^A = \frac{K_b}{2}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}\right)^2 = \frac{K_b}{2}\left(\frac{1}{R}+\frac{1}{R}\right)^2 \text{ for sphere}$$
$$= \frac{2K_b}{R^2}$$

Total bending energy of one sphere $U_b = 4\pi R^2 \times U_b^A$

$= 8\pi K_b$. The bending energy to form a small or large sphere is the same

From 1 to 2 spheres $\boxed{\Delta U_b = + 8\pi K_b} = 80\pi k_B T = \underline{10^{-18} \text{ J}}$

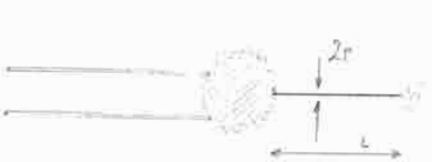
b) Ratio of small to large vesicles

If change in energy due to surface tension is negligible, then at thermal equilibrium

$$\frac{P_{2\ \text{spheres}}}{P_{1\ \text{sphere}}} = \exp\left(-\frac{\Delta U_b}{k_B T}\right) = \exp(-80\pi) = \underline{7\ 10^{-110}}$$ from Boltzmann distribution law

There will exist almost no small vesicles (of radius $R_1 = 1\,\mu m$) under these conditions

c)



When a tether is formed, there is no change in the spherical region of the neutrophil, thus no energy change associated with this part.

Consider a tether of area dA (length dL, radius r); change of energy due to tension $\boxed{dU = N\,dA}$
Because the surface tension N is here constant, this expression is simpler than the more general one $\frac{dU}{dA} = \frac{Eh}{2(1-\nu)}\left(\varepsilon_1^2 + \varepsilon_2^2\right)$ (Kamm Ch. 2.1 (2.33))

$FL \quad \frac{F}{L} \quad L^2$

d) Energy provided by force ($F\,dL$) ~ energy (dU) to create extra surface ($dA = 2\pi r\,dL$) + adhesion energy between the lipid bilayer and the cortex ($J\pi r^2$)

Thus $F\,dL = (2\pi r\,dL)\frac{K_b}{2r^2} + J\pi r^2$

$$\boxed{F\,dL = \pi K_b \frac{dL}{r} + \pi J r^2}$$

▷ for neutrophils:

$$\frac{\text{tension}}{\text{binding}} = \frac{2r^2 N}{K_b} = \frac{2 \times (5 \times 10^{-8})^2 \times 3.5 \times 10^{-5}}{4\ 10^{-18}} = 4\%$$

Binding is dominant over tension

Problem 2

a) The tissue is elastic, homogeneous, isotropic, and incompressible

Hooke's law
$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \qquad (1)$$
$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2G\varepsilon_{ij} \qquad (2)$$

Here we have $\varepsilon_{33} = 0$ and $\sigma_{22} = 0$, therefore, from (2), $\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{22}) + 2G\varepsilon_{22} = 0$

or $\varepsilon_{22} = \frac{-\lambda}{2G+\lambda}\varepsilon_{11} = \boxed{\frac{-\lambda\varepsilon_0}{2G+\lambda}\sin(\omega t) = \varepsilon_{22}(x_2, t)}$ independent of x_2

Problem 2 (continued)

b) Tissue is poroelastic ; Governing equations

- constitutive relation (from (2)) $\quad \sigma_{22} = (2G+\lambda)\,\varepsilon_{22} + \lambda\,\varepsilon_{11} - p \quad (3)$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \quad (4)$$

- fluid / structure interactions (same notations as in class) $\quad U_2 = -k\,\dfrac{\partial p}{\partial x_2} \quad (5)$
- conservation of mass (same notations as in class) $\quad U_2 = -\dfrac{\partial u_2}{\partial t} \quad (6)$
- conservation of momentum (force balance along axis 2) $\quad \dfrac{\partial \sigma_{22}}{\partial x_2} = 0 \quad (7)$

From (7) and (3)
$$\frac{\partial \sigma_{22}}{\partial x_2} = 0 = H\frac{\partial \varepsilon_{22}}{\partial x_2} + \lambda\frac{\partial \varepsilon_{11}}{\partial x_2} - \frac{\partial p}{\partial x_2} \overset{(4)(5)}{=} H\frac{\partial^2 u_2}{\partial x_2^2} + 0 - \frac{U_2}{-k} \overset{(6)}{=} H\frac{\partial^2 u_2}{\partial x_2^2} + \frac{1}{k}\left(-\frac{\partial u_2}{\partial t}\right)$$

$$\boxed{\frac{\partial u_2}{\partial t} = Hk\,\frac{\partial^2 u_2}{\partial x_2^2}} \quad (8)$$

c) High frequency limit

The material behaves as though it were incompressible, since the fluid has no time to escape

Therefore $\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0 \Rightarrow \varepsilon_{22} = -\varepsilon_{11} = -\varepsilon_0 \sin(\omega t) = -\dfrac{\partial u_2}{\partial x_2}$

Given that $u_2(x_2 = 0, t) = 0$, integrating yields

$$\boxed{u_2(x_2, t) = -\varepsilon_0\, x_2 \sin(\omega t)}$$

d) Low frequency limit

We're now considering a situation close to the steady state case $\quad \dfrac{\partial^2 u_2}{\partial x_2^2} = 0 \quad (9)$

Boundary conditions : $u_2(x_2 = 0, t) = 0$

from (3) $\quad \sigma_{22} = 0 = H\,\varepsilon_{22}(x_2 = h, t) + \lambda\,\varepsilon_{11}(x_2 = h, t) - p(x_2 = h)$

$$= H\left.\frac{\partial u_2}{\partial x_2}\right|_{x_2 = h} + \lambda\,\varepsilon_0 \sin(\omega t) - 0$$

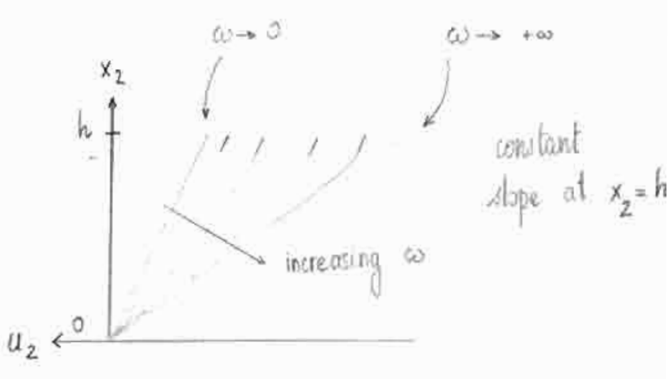
or $\quad \left.\dfrac{\partial u_2}{\partial x_2}\right|_{x = h} = \dfrac{-\lambda\,\varepsilon_0}{H}\sin(\omega t)$

Solve (9) :
$$\boxed{u_2(x_2, t) = \frac{-\lambda\,\varepsilon_0}{2G+\lambda}\, x_2 \sin(\omega t)}$$

e) General case

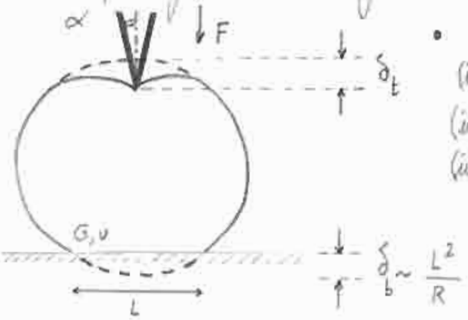
To solve (8) one need one initial condition
two boundary conditions

- $u_2(x_2 = 0, t) = 0$
- $\left.\dfrac{\partial u_2}{\partial x_2}\right|_{x_2 = h,\, t} = \dfrac{-\lambda\,\varepsilon_0}{H}\sin(\omega t)$



Problem

a) AFM probing of a homogeneous, isotropic, linear elastic nucleus:



- Dimensional analysis for deformation on the top of the nucleus

(i) $Q_0 = \delta_t$, $Q_1 = \alpha$ $\quad Q_2 = \nu$ $\quad Q_3 = G$ $\quad Q_4 = F$

(ii) units $\quad$ m $\quad$ – $\quad$ – $\quad$ Pa $\quad$ Pa.m^2

(iii) $\delta_t =$ function (α, ν, G, F)

(iv) $\Pi_0 = \dfrac{\delta}{\left(\frac{F}{G}\right)^{1/2}}$ $\quad \Pi_1 = \alpha$ $\quad \Pi_2 = \nu$

(v) $\delta \sqrt{\dfrac{G}{F}} =$ function $(\alpha, \nu) = \left(\dfrac{\pi}{4} \cdot \dfrac{1-\nu}{\tan\alpha}\right)^{1/2}$ from L # 21

- Force balance for deformation on the bottom

$F \sim \sigma L^2 \sim G \varepsilon L^2 \sim G \dfrac{\delta_b}{L} L^2 \sim G \delta_b L \sim G \delta_b (\delta_b R)^{1/2} \sim G \delta_b^{3/2} R^{1/2}$

or $\delta_b \sim \left(\dfrac{F}{G}\right)^{2/3} R^{-1/3}$

- Total deformation $\boxed{\delta = c_1 \delta_t + c_2 \delta_b \sim c_1 \left(\dfrac{F}{G}\right)^{1/2} \left(\dfrac{\pi}{4} \cdot \dfrac{1-\nu}{\tan\alpha}\right)^{1/2} + c_2 \left(\dfrac{F}{G}\right)^{2/3} R^{-1/3}}$ $\quad c_1, c_2$ constants

b) AFM probing of a shell deforming due to bending

- Energy considerations on the top of the nucleus : (Kamm, Ch. 2.1 (2.44))

$F \delta_t \sim$ (bending energy per unit area) * (characteristic distance over which binding occurs)^2

$\sim \dfrac{K_B}{2} \left(\dfrac{\partial^2 u_3}{\partial x_1^2} + \dfrac{\partial^2 u_3}{\partial x_2^2} \right)^2 R^2 \sim K_B \left(\dfrac{\delta_t}{R^2}\right)^2 R^2 \sim K_B \dfrac{\delta_t^2}{R^2}$

- Deformation on the bottom surface neglected
- Total deformation $\boxed{\delta \sim \delta_t \sim \dfrac{F R^2}{K_B}}$