The Physical Basis of

DIMENSIONAL ANALYSIS

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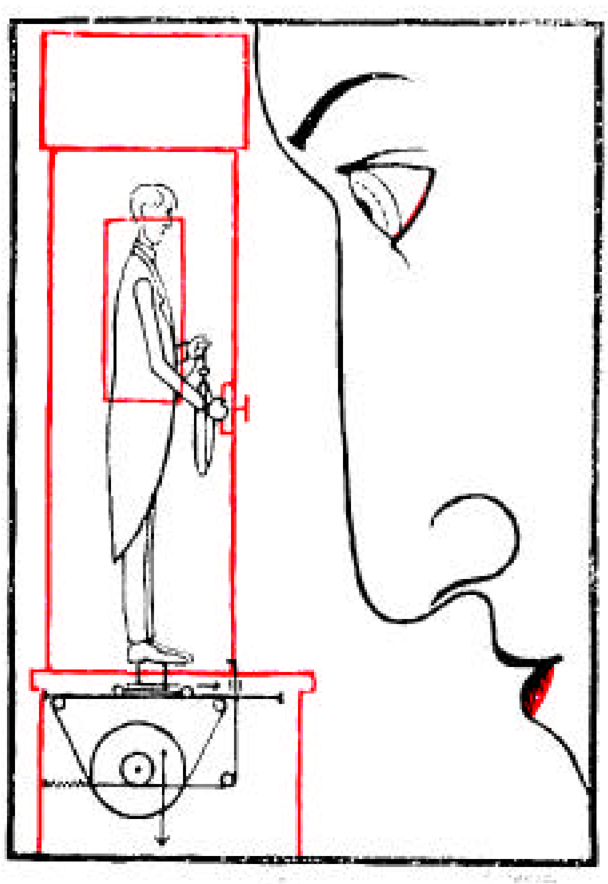

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Francis Bacon (1561-1628)[1]*:*

> "I found that I was fitted for nothing so well as the study of Truth; as having a nimble mind and versatile enough to catch the resemblance of things (which is the chief point), and at the same time steady enough to fix and distinguish their subtle differences..."
>
> "Think things, not words."

Albert Einstein (1879-1955)[2]*:*

> *"… all knowledge starts from experience and ends in it. Propositions arrived at by purely logical means are completely empty as regards reality."*

Percy W. Bridgman (1882-1961)[3]*:*

> "...what a man means by a term is to be found by observing what he does with it, not by what he says about it."

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[1] Catherine Drinker Bowen, 1963
[2] Einstein, 1933
[3] Bridgman, 1950

1. Introduction

Dimensional analysis offers a method for reducing complex physical problems to the simplest (that is, most economical) form prior to obtaining a quantitative answer. Bridgman (1969) explains it thus: "The principal use of dimensional analysis is to deduce from a study of the dimensions of the variables in any physical system certain limitations on the form of any possible relationship between those variables. The method is of great generality and mathematical simplicity".

At the heart of dimensional analysis is the concept of similarity. In physical terms, similarity refers to some equivalence between two things or phenomena that are actually different. For example, under some very particular conditions there is a direct relationship between the forces acting on a full-size aircraft and those on a small-scale model of it. The question is, what are those conditions, and what is the relationship between the forces? Mathematically, similarity refers to a transformation of variables that leads to a reduction in the number of independent variables that specify the problem. Here the question is, what kind of transformation works? Dimensional analysis addresses both these questions. Its main utility derives from its ability to contract, or make more succinct, the functional form of physical relationships. A problem that at first looks formidable may sometimes be solved with little effort after dimensional analysis.

In problems so well understood that one can write down in mathematical form all the governing laws and boundary conditions, and only the solution is lacking, similarity can also be inferred by normalizing all the equations and boundary conditions in terms of quantities that specify the problem and identifying the dimensionless groups that appear in the resulting dimensionless equations. This is an inspectional form of similarity analysis. Since inspectional analysis can take advantage of the problem's full mathematical specification, it may reveal a higher degree of similarity than a “blind” (less informed) dimensional analysis and in that sense prove more powerful. Dimensional analysis is, however, the only option in problems where the equations and boundary conditions are not completely articulated, and always useful because it is simple to apply and quick to give insight.

Some of the basic ideas of similarity and dimensional analysis had already surfaced in Fourier's work in the nineteenth century's first quarter, but the subject received more methodical attention only toward the close of that century, notably in the works of Lord Rayleigh, Reynolds, Maxwell, and Froude in England, and Carvallo, Vaschy and a number of other scientists and engineers in France (Macagno, 1971)[4]. By the 1920's the principles were essentially in place: Buckingham's now ubiquitous π–theorem had appeared (Buckingham, 1914), and Bridgman had published the monograph which still remains the classic in the field (Bridgman, 1922, 1931). Since then, the literature has grown prodigiously. Applications now include aerodynamics, hydraulics, ship design, propulsion, heat and mass transfer, combustion, mechanics of elastic and plastic structures, fluid-structure interactions, electromagnetic theory, radiation, astrophysics, underwater and underground explosions, nuclear blasts, impact dynamics, and chemical reactions and processing (see for example Sedov, 1959, Baker et al, 1973, Kurth, 1972, Lokarnik, 1991), and also biology (McMahon & Bonner, 1983) and even economics (de Jong, 1967).

Most applications of dimensional analysis are not in question, no doubt because they are well supported by experimental facts. The debate over the method's theoretical-philosophical underpinnings, on the other hand, has never quite stopped festering (e.g. Palacios, 1964). Mathematicians tend to find in the basic arguments a lack of rigor and are tempted to redefine the subject in their own terms (e.g. Brand, 1957), while physicists and engineers often find themselves uncertain about the physical meanings of the words in terms of which the analysis cast. The problem is that dimensional analysis is based on ideas that originate at such a substratal point in science that most scientists and engineers have lost touch with them. To understand its principles, we must return to some of the very fundamental concepts in science.

Dimensional analysis is rooted in the nature of the artifices we construct in order to describe the physical world and explain its functioning in quantitative terms. Einstein (1933) has said, "Pure logical thinking cannot yield us any knowledge of the empirical world; all knowledge starts from experience and ends in it. Propositions arrived at by purely logical means are completely empty as regards reality."

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[4] See, for example, the very first page of the first volume of James Clerk Maxwell's *A treatise on Electricity and Magnetism* (third edition, Clarendon Press, Cambridge, 1891; republished by Dover, New York, 1954)

This treatise is an attempt to explain dimensional analysis by tracing it back to its physical foundations. We will clarify the terms used in dimensional analysis, explain why and how it works, remark on its utility, and discuss some of the difficulties and questions that typically arise in its application. One single (unremarkable) application in mechanics will be used to illustrate the procedure and its pitfalls. The procedure is the same in all applications, a great variety of which may be found in the references and in the scientific literature at large.

2. Physical Quantities and Equations

2.1 Physical properties

Science begins with the observation and precise description of things and events. It is at this very first step that we face the fact upon which dimensional analysis rests: Description in absolute terms is impossible. We can do no more than compare one thing with another, to "catch the resemblance of things". When we say that something "is" a tree, we mean simply that it has a set of attributes that are in some way shared by certain familiar objects we have agreed to call trees.

Our brains have evolved to the point where we can recognize trees almost instantly, but describing something like a tree is actually a very complex business. Physics starts by breaking the descriptive process down into simpler terms. An object or event is described in terms of basic properties like length, mass, color, shape, speed, and time. None of these properties can be defined in absolute terms, but only by reference to something else: an object has the length of a meter stick, we say, the color of an orange, the weight of a certain familiar lump of material, or the shape of a sphere. The references may be made more precise, but in essence "description" is simply a noting of the similarities between one thing and a set of others that are known to us. We can do no more than compare one thing with another.

A physical property first arises as a concept based on experience, and is formalized by defining a comparison operation for determining whether two samples of it are equal ($\boldsymbol{A}=\boldsymbol{B}$) or unequal ($\boldsymbol{A}\neq\boldsymbol{B}$). (We shall use bold symbols when we are referring not to numerical values, but to actual physical attributes.) This operation, which is an entirely physical procedure, *defines* the property. Properties of the same kind (or simply, the same properties) are compared by means of the same comparison operation. Properties of different kinds cannot be compared because there exists no operation that defines equality. Asking whether a particular mass is physically equal to a particular length is meaningless: no procedure exists for making the comparison.

If a property is defined only in terms of a comparison operation, we have a procedure for establishing whether two samples of it are equal or unequal, but no concept of what it means for one to be larger or smaller than the other. Shape and color are examples. We have procedures for determining if two objects have the same shape, or the same color. But asking whether a square shape is smaller or larger than a circular shape, or whether green is smaller or larger than white, makes no sense. Properties like shape and color are useful for describing things, but cannot play a role in any quantitative analysis, which deals with relative magnitudes.

2.2 Physical quantities and base quantities

Science begins with observation and description, but its ultimate goal is to infer from those observations laws that express the phenomena of the physical world in the simplest and most general (that is, most economical) terms. That the language of mathematics is ideally suited for expressing those laws is not accidental, but follows from the constraints we put on the types of physical properties that are allowed to appear in quantitative analysis. The allowed types of properties are called "physical quantities".

Physical quantities are of two types: base quantities and derived quantities. The base quantities, which are defined in entirely physical terms, form a complete set of basic building blocks for an open-ended system of “derived” quantities that may be introduced as necessary. The base and derived quantities together provide a rational basis for describing and analyzing the physical world in quantitative terms.

A base quantity is defined by specifying *two physical operations*:

a comparison operation for determining whether two samples $\boldsymbol{A}$ and $\boldsymbol{B}$ of the property are equal ($\boldsymbol{A}=\boldsymbol{B}$) or unequal ($\boldsymbol{A} \neq \boldsymbol{B}$), and

an addition operation that defines what is meant by the sum $\boldsymbol{C}=\boldsymbol{A}+\boldsymbol{B}$ of two samples of the property.

Base quantities with the same comparison and addition operations are of the same kind (that is, different examples of the same quantity). The addition operation $\boldsymbol{A}+\boldsymbol{B}$ defines a physical quantity $\boldsymbol{C}$ of the same kind as the quantities being added. Quantities with different comparison and addition operations cannot be compared or added; no procedures exist for executing such operations. All physical quantities are *properties* of physical things or events. They are not themselves physical things or events. The comparison and addition operations involve physical manipulations of objects or events that *possess* the property under consideration (see the examples below).

The comparison and addition operations are physical, but they are required to have certain properties that mimic those of the corresponding mathematical operations for pure numbers:

(1) The comparison operation must obey the identity law (if $\boldsymbol{A}=\boldsymbol{B}$ and $\boldsymbol{B}=\boldsymbol{C}$, then $\boldsymbol{A}=\boldsymbol{C}$), and
(2) the addition operation must be commutative ($\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$), associative [$\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}$], and unique (if $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{C}$, there exists no finite $\boldsymbol{D}$ such that $\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{D}=\boldsymbol{C}$).

The two operations together define, in entirely physical terms,

(1) the concept of larger and smaller for like quantities (if there exists a finite $\boldsymbol{B}$ such that $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{C}$, then $\boldsymbol{C}>\boldsymbol{A}$),
(2) subtraction of like quantities (if $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{C}$, then $\boldsymbol{A}\equiv\boldsymbol{C}-\boldsymbol{B}$),
(3) multiplication of a physical quantity by a *pure number* (if $\boldsymbol{B}=\boldsymbol{A}+\boldsymbol{A}+\boldsymbol{A}$, then $\boldsymbol{B}\equiv 3\boldsymbol{A}$), and
(4) division of a physical quantity by a *pure number* (if $\boldsymbol{A}=\boldsymbol{B}+\boldsymbol{B}+\boldsymbol{B}$, then $\boldsymbol{B}\equiv\boldsymbol{A}/3$).

A base quantity is thus a property for which the following mathematical operations are defined in physical terms: comparison, addition, subtraction, multiplication by a pure number, and division by a pure number. Each of these operations is performed on physical properties of the same kind and yields a physical property of that kind, and each physical operation obeys the same rules as the corresponding mathematical operation for pure numbers.

This sets the stage for not only "catching the resemblance of things", but also expressing that resemblance in the language of mathematics.

It is important to note that mathematical operations other than the ones listed above are *not* defined in physical terms. No defining operation exists for forming a tangible entity that represents the product of a mass and a time, for example, or, for that matter, the product of one length and another (more on this later). Nor can we point to some tangible thing that "is" the cube root of a length, say, or the natural logarithm of a time. Products, ratios, powers, and exponential and other functions such as trigonometric functions and logarithms are defined for *numbers*, but have no physical correspondence in operations involving actual physical quantities.

Figure 2.1 illustrates the comparison and addition operations of some well-known physical quantities that can be chosen as base quantities. The figure shows them in simplistic, cartoon-like terms, but we are of course aware that each operation is actually associated with a carefully articulated procedure and a set of concepts that are often quite complex. For our present purpose we take these for granted, much like Dr. Samuel Johnson who, when queried about how he knew that the physical world really existed, satisfied himself by stamping his foot on the pavement.

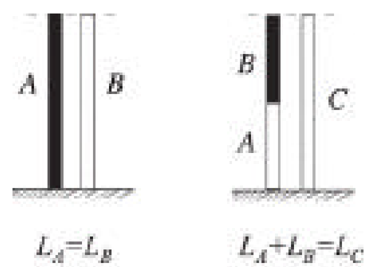

Figure 2.1a: The comparison and addition operations of length.

Figure 2.1b: The comparison and addition operations of mass.

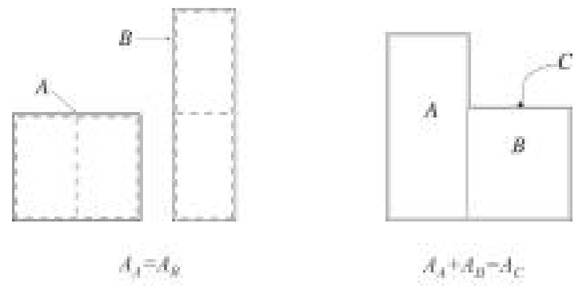

Figure 2.1c: The comparison and addition operations of area.

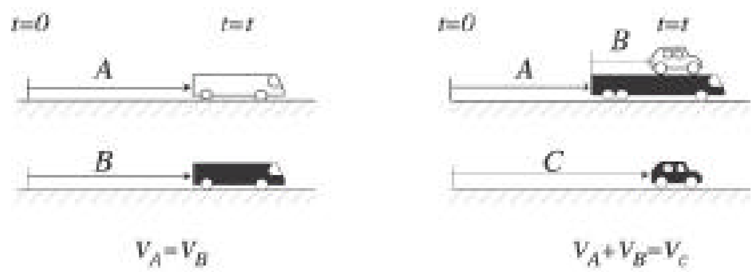

Figure 2.1d: The comparison and addition operations of velocity.

Figure 2.1e: The comparison and addition operations of force.

The comparison and addition operations for length and mass are familiar. Only one comment is necessary: the mass referred to in the figure is the "gravitational mass”, for which equality is defined by the statement that two masses are equal if a third mass exerts the same gravitational force on each of them separately at the same relative position.

Time is omitted from the figure, largely because its defining operations defy illustration in such simplistic terms. The concept of time is deeply rooted in our biological beings. It is, after all, the warp of our existence, the stuff, as some wit pointed out, that keeps everything from happening at once. Time characterizes an event, not a thing. Aristotle referred to time as a "dimension of motion", which pleases the poet in us, but leaves the scientist unmoved. We do best if we adopt a pragmatic notion of time and think of it as being defined in terms of comparison and addition operations involving idealized clocks or stopwatches, much like Einstein did in his popular exposition of the theory of relativity (Einstein, 1952). Whether those clocks are hourglasses or atomic clocks will affect the precision of the operations, but not their intrinsic character. What time "is" has no relevance in the present context, only the defining operations matter[5].

The concept of force arises in primitive terms from muscular effort, and is formalized based on the observation that a *net* force on an object (the vector sum of all the forces acting on the object) causes a rate of change in its velocity. Directionality is important in the definition: force is a vector quantity.

A reader accustomed to considering speed as "distance divided by time" and area as "length squared" may be surprised to see them included in figure 2.1 as (possible) base quantities with their own comparison and addition operations. We include them to show that the set of base quantities is very much a matter of choice. Velocity (or speed, if direction is presumed) is a certain property of motion. A self-propelled toy car running across a tabletop has a speed, and we can define acceptable procedures for establishing whether two speeds are equal or unequal and for adding two speeds, as in figure 2.1d. Speed can therefore be taken as a base quantity, should we choose to do so. The same goes for area. Note that two areas may be equal without being congruent, provided one of

[5] Sir Arthur Eddinton (1939): "It has come to be accepted practice in introducing new physical quantities that they shall be regarded as *defined* by the series of measuring operations and calculations of which they are the result. Those who associate with the result a mental picture of some entity disporting itself in a metaphysical realm of existence do so at their own risk; physics can accept no responsibility for this embellishment".

them can be "cut up" and reassembled (added back together) into a form which is congruent with the other: the addition operation is invoked in making a comparison, and the comparison operation in addition.

We have said that shape and color, though acceptable physical properties, cannot be base quantities because they lack acceptable addition operations. That shape is disqualified is obvious: what is the sum of a square and a circle? But why is color disqualified? We know that color in the form of light can be added according to well-defined rules, as when red light added to green produces yellow. Is this not an acceptable definition of physical addition? The answer is no. Blue equals blue. But according to this addition operation, blue plus blue also equals blue. The addition operation is *not unique:* $n\boldsymbol{A}=\boldsymbol{A}$ for any color $\boldsymbol{A}$, where n is any number. This disqualifies color from the ranks of physical quantities (and makes measurement of color in terms of a unit impossible). But, the persistent reader may argue, the color of an object can be identified by the wavelength of light reflected from it, and wavelength can be added. Is this not acceptable for an addition rule? The answer is again no. This is a rule for adding lengths, not for adding the property we perceive as color.

2.3 Unit and numerical value

The two operations that define a base quantity make it possible to express any such quantity as a multiple of a standard sample of its own kind, that is, to "measure it in terms of a unit". The standard sample—the unit—may be chosen arbitrarily. The comparison operation allows the replication of the unit, and the addition operation allows the identification and replication of fractions of the unit. The measuring process consists of *physically* adding replicas of the unit and fractions thereof until the sum equals the quantity being measured (figure 2.2). A count of the number of whole and fractional units required yields the numerical value of the quantity being measured. If $\boldsymbol{a}$ is the unit chosen for quantities of type $\boldsymbol{A}$, the process of measurement yields a numerical value A (a number) such that

$$\boldsymbol{A} = A\boldsymbol{a} \tag{2.1}$$

The measurement process is entirely physical. The only mathematics involved is the counting of the number of whole and fractional units once physical equality has been established between the quantity being measured and a sum of replicas of the unit and fractions thereof.

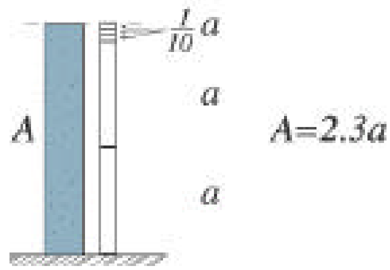

Figure 2.2: Measurement in terms of a unit and numerical value

It should be emphasized that numbers can be ascribed to properties in many arbitrary ways, but such numbers will *not* represent numerical values of physical quantities unless they are assigned by a procedure consistent with the one defined above.

The numerical value of a base quantity depends on the choice of unit. A physical quantity exists, independent of the choice of unit. My forefinger has the same length, regardless of whether I measure it in centimeters or inches. A quantity $\boldsymbol{A}$ can be measured in terms of a unit $\boldsymbol{a}$ or in terms of another unit $\boldsymbol{a}'$, but the quantity itself remains physically the same, that is,

$$\boldsymbol{A} = A\boldsymbol{a} = A'\boldsymbol{a}'. \tag{2.2}$$

If the unit $\boldsymbol{a}'$ is n times larger than $\boldsymbol{a}$,

$$\boldsymbol{a}' = n\boldsymbol{a}, \tag{2.3}$$

it follows from equation (2.2) that

$$A' = n^{-1}A. \tag{2.4}$$

If the size of a base quantity's unit is changed by a factor n, the quantity's numerical value changes by a factor n^{-1}.

By convention, all base quantities of the same kind are always measured in terms of the same unit. All base quantities of the same kind thus change by the same factor when the size of that quantity's unit is changed. That is, the ratio of the numerical values of any two quantities of the same kind is independent of base unit size.

Note also that when base quantities of the same kind are added physically ($\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{C}$), the numerical values satisfy an equation of the same form as the physical quantity equation ($A+B=C$), regardless of the size of the chosen unit. In other words, the numerical value equation mimics the physical equation, and its form is independent of the unit's size.

2.4 Derived quantities, dimensions, and dimensionless quantities

In describing physical things and events quantitatively, we refer to *numerical values* of base quantities, and also introduce numbers derived by inserting these values into certain mathematical formulas. We determine the distance L that an object moves in time t, for example, and calculate its speed $V=Lt^{-1}$; or we measure a body's mass m and speed V and calculate its kinetic energy $K=mV^2/2$. These numbers are *derived quantities of the first kind.*

Not all numbers obtained by inserting base quantities into formulas can be considered physical quantities[6]. Base quantities have a transparently physical origin, which gives rise to the fact that the ratio of any two samples of a base quantity remains constant when the base unit size is changed; an arbitrary choice cannot affect a relative physical magnitude. Bridgman (1931) postulated that this is in fact a defining attribute of *all* physical quantities. This is

Bridgman's principle of absolute significance of relative magnitude:

> A number Q obtained by inserting the numerical values of base quantities into a formula is a physical quantity if the ratio of any two samples of it remains constant when base unit sizes are changed.

Bridgman went on to show (Bridgman, 1931; see also the proof by Barenblatt, 1996) that a monomial formula satisfies the principle of

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[6] From this point on, the term quantity will be used for the *numerical value* of a physical quantity, unless otherwise noted.

absolute significance of relative magnitude *only* if it has the power-law form

$$Q = \alpha A^a B^b C^c \ldots \tag{2.5}$$

where A, B, C, etc. are numerical values of base quantities and the coefficient α and exponents a, b, c, etc. are real numbers whose values distinguish one type of derived quantity from another. *All monomial derived quantities have this power-law form*; no other form represents a physical quantity.

A derived quantity of the first kind is defined in terms of a numerical value, which depends on the choice of base units. A derived quantity does not necessarily represent something tangible in the same sense as a base quantity, although it may. The square root of time, for example, is a derived quantity because it has the required power-law form, but we cannot point to any physical thing that “is” the square root of time.

To avoid talking of "units" for quantities that may have no physical representation, but whose numerical values nevertheless depend on the choice of base units, we introduce the concept of *dimension*. Each type of base quantity has by definition its own dimension. If A is the numerical value of a length, we say it “has the dimension of length”, and write this as $[A]=L$ where the square brackets imply “the dimension of” and L symbolizes the concept of length. By this we mean simply that if the length unit size is increased by a factor n, the numerical value A will increase by a factor n^{-1}.

The dimension of a derived quantity conveys the same information in generalized form, for derived as well as base quantities. Consider a quantity defined by the formula

$$Q = \alpha L_1^{l_1} L_2^{l_2} \ldots M_1^{m_1} M_2^{m_2} \ldots t_1^{\tau_1} t_2^{\tau_2} \ldots \tag{2.6}$$

where the L_i's are numerical values of certain lengths, M_i's of certain masses, and t_i's of certain times, and α and all exponents are real numbers. If the length unit is changed by a factor n_L, the mass unit by a factor n_m, and the time unit by a factor n_t, it follows from equations (2.4) and (2.6) that the value of Q changes to

$$Q' = n^{-1} Q \tag{2.7a}$$

where

$$n = (n_L)^{\Sigma l_i} (n_m)^{\Sigma m_i} (n_t)^{\Sigma t_i} . \qquad (2.7b)$$

This implies that Q transforms like the numerical value of a base quantity with a unit whose size is proportional to $L^{\Sigma l_i} M^{\Sigma m_i} t^{\Sigma \tau_i}$, where L, M, and t represent the sizes of the length, mass, and time units, respectively. By analogy with the meaning of dimension for base quantities [see equation (2.4)], we therefore say that the derived quantity Q has the dimension

$$[Q] = (L)^{\Sigma l_i} (M)^{\Sigma m_i} (t)^{\Sigma t_i} . \qquad (2.8)$$

Whether applied to a base or derived quantity, the dimension is simply a formulaic indication of how the quantity's numerical value transforms when the sizes of the base units are changed. A derived quantity's dimension follows from its defining equation. To obtain the result (2.8) for the quantity defined in equation (2.6), we simply substitute for each base quantity in equation (2.6) the symbol for its dimension, omit the numerical coefficient α, and obtain equation (2.8) by algebra. The dimension of a kinetic energy $mV^2/2$, for example, is $M(L/t)^2 = ML^2t^{-2}$.

Statements to the effect that a quantity's dimension is an "expression of its essential physical nature" (Tolman, 1917) are either a meaningless tautology (in the case of base quantities) or nonsense (in the case of derived quantities). We shall see that a quantity's dimension depends on the choice of the system of units, and is therefore under the control of the observer rather than an inherent attribute of that quantity.

By convention, a particular derived quantity is specified by its numerical value followed by the base units upon which that value is based, the latter arranged in a form which reflects the quantity's dimension. The statement $Q = 0.021$kg s$^{1/2}$ implies that Q is a quantity with dimension $Mt^{1/2}$ and has a magnitude 0.021 if mass is measured in kilograms and time in seconds. Speaking loosely, it may be said that the quantity Q is "measured in units of kg s$^{1/2}$", which implies both its dimension and the base units on which the indicated value is based. In common parlance the terms unit and dimension are often used synonymously, but such usage is undesirable in a treatise where fundamental understanding is paramount.

Some important points about derived quantities:

1. The dimension of any derived physical quantity is a product of powers of the base quantity dimensions.

2. Sums of derived quantities with the same dimension are derived quantities of the same dimension. Products and ratios of derived quantities are also derived quantities, with dimensions which are usually different from the original quantities.

3. All derived quantities with the same dimension change their values by the same factor when the sizes of the base units are changed.

4. A derived quantity is *dimensionless* if its numerical value remains invariant when the base units are changed. An example is Vt/L, where $V = dx/dt$ is a velocity, t is a time and L is a length. The *dimension of a dimensionless quantity is unity*, the factor by which the quantity's numerical value changes when base units sizes are changed.

5. Special functions (logarithmic, exponential, trigonometric, etc.) of *dimensional* derived quantities are in general *not* derived quantities because their values do not in general transform like derived quantities when base unit size changes. Only when the arguments of these functions are *dimensionless* will the arguments and therefore the values of the functions remain invariant when units are changed. Special functions with dimensionless arguments are therefore derived quantities with dimension unity.

2.5 Physical equations, dimensional homogeneity, and physical constants

In quantitative analysis of physical events one seeks mathematical relationships between the numerical values of the physical quantities that describe the event. We are not, however, interested in just *any* relationships that may apply between the values of physical quantities. A primitive soul may find it remarkable, or even miraculous, that his own mass in kilograms is exactly equal to his height in inches. We dismiss this kind of "relationship" as an accidental result of the choice of units. Science is concerned only with expressing a *physical* relationship between one quantity and a set of others, that is, with “physical equations.” Nature is indifferent to the arbitrary choices we make when we pick base units. We are interested, therefore, only in numerical relationships that remain true independent of base unit size.

This puts certain constraints on the allowable form of physical equations. Suppose that, in a specified physical event, the numerical value Q_o of a physical quantity is determined by the numerical values of a set $Q_1 ... Q_n$ other physical quantities, that is,

$$Q_o = f\,(Q_1, Q_2, \ldots, Q_n), \tag{2.9}$$

The principle of absolute significance of relative values tells us that the relationship implied by equation (2.9) can be physically relevant only if Q_o and f change by the same factor when the magnitudes of any base units are changed. In other words, *a physical equation must be dimensionally homogeneous.* Some reflection based on the points summarized at the end of Section 2.4 will show that dimensional homogeneity imposes the following constraints on any mathematical representation of a relationship like equation (2.9):

(1) both sides of the equation must have the same dimension;
(2) wherever a sum of quantities appears in f, all the terms in the sum must have the same dimension;
(3) all arguments of any exponential, logarithmic, trigonometric or other special functions that appear in f must be dimensionless.

For example, if a physical equation is represented by

$$A = Be^{-C} - \frac{(D_1 + D_2)}{E} + F \ , \tag{2.10}$$

C must be dimensionless, D_1 and D_2 must have the same dimension, and A, B, D/E and F must have the same dimension.

An important consequence of dimensional homogeneity is that the *form* of a physical equation is independent of the size of the base units.

The following example may help to illustrate the reason for dimensional homogeneity in physical equations and show how conceptual errors that may arise if homogeneity isn't recognized. Suppose we release an object from rest in a uniform gravitational field, in vacuum, and ask what distance x it will fall in a time t. We know of course that elementary Newtonian mechanics gives the answer as

$$x = \frac{1}{2} g t^2 \ , \tag{2.11}$$

where g is the local acceleration of gravity and has the dimension Lt^{-2}. This equation expresses the result of a general physical law, and is clearly dimensionally homogeneous.

The physical basis of dimensional homogeneity becomes apparent when we consider the same phenomenon from a different perspective. Suppose we are ignorant of mechanics and conduct a large variety of experiments in Cambridge, Massachusetts, on the time t it takes a body with mass m to fall a distance x from rest in an evacuated chamber. After performing experiments with numerous masses and distances, we find that if t is measured in seconds and x in meters, all our data can be accurately represented, regardless of mass, with the single equation

$$x = 4.91 t^2 . \tag{2.12}$$

This is a perfectly *correct* equation. It describes and predicts all experiments (in Cambridge, Massachusetts) to a very good accuracy. However, it appears at first glance to be dimensionally non-homogeneous, the two sides seemingly having different dimensions, and thus appears not to be a true physical equation. This impression is, however, based on the false presumption that the coefficient 4.91 remains invariant when units are changed. In fact, the coefficient 4.91 represents not a dimensionless number, but a particular numerical value of a *dimensional* physical quantity which characterizes the relationship between x and t in the Cambridge area. That this must be so becomes clear when we consider how equation (2.12) must transform when units are changed. We know that when units are changed, the actual *physical* distance x remains invariant, and we therefore argue that to obtain the falling distance in feet, for example, the right hand side of equation (2.12), which gives it in meters, must be multiplied by 3.28, the number of feet in one meter. Thus, if x is measured in feet and t in seconds, the correct version of equation (2.12) is

$$x = 16.1 t^2 \ . \tag{2.13}$$

This same transformation could also have been obtained by arguing that equation (2.12), being an expression of a general physical law, *must*, according to Bridgman's principle of absolute significance of relative magnitude, be dimensionally homogeneous, and therefore should properly have been written

$$x = ct^2 \qquad (c = 4.91 ms^{-2}) \qquad (2.14)$$

This form makes clear that the coefficient *c* is a physical quantity rather than a numerical coefficient. The units of c indicate its dimension and show that a change of the length unit from meters to feet, with the time unit remaining invariant, changes *c* by the factor 3.28, the inverse of the factor by which the length unit is changed. This gives c=16.1 ft s^{-2}, as in equation (2.13).

Equation (2.14) is the correct way of representing the data of equation (2.11). It is dimensionally homogeneous, and makes the transformation to different base units straightforward.

Every correct physical equation—that is, every equation that expresses a physically significant relationship between numerical values of physical quantities—must be dimensionally homogeneous. A fitting formula derived from correct empirical data may at first sight appear dimensionally non-homogeneous because it is intended for particular base units. Such formulas can always be rewritten in general, homogeneous form by the following procedure (Bridgman, 1931):

(1) Replace all the numerical coefficients in the equation with unknown dimensional constants.
(2) Determine the dimensions of these constants by requiring that the new equation be dimensionally homogeneous.
(3) Determine the numerical values of the constants by matching them with those in the original equation when the units are the same.

This is of course how equation (2.14) was derived from equation (2.12).

Another example serves to reinforce this point. Suppose it is found that the pressure distribution in the earth's atmosphere over much of the United States can be represented (approximately) by the formula

$$p = 1.01x10^5 e^{-0.00012z} \qquad (2.15)$$

where p is the pressure in Nm^{-2} and z is the altitude in meters. This expression applies only with the cited units. The correct, dimensionally homogeneous form of this equation is

$$p = ae^{-bz} \quad (a = 1.01 \times 10^5 Nm^{-2},\ b = 0.00012m^{-1}) \qquad (2.16)$$

where a and b are physical quantities. In this form the equation is valid independent of the chosen base units. The dimensions of a and b indicate how these quantities change when units are changed.

The two quantities a and b in equation (2.16) are *physical constants* in the sense that their values are fixed once the system of units is chosen. In this case the constants characterize a particular environment□the pressure distribution in the earth's atmosphere over the US. Similarly, the acceleration of gravity g in equation (2.11) is a physical constant that characterizes the (local!) gravitational force field at the earth's surface.

The basic laws of physics also contain a number of *universal physical constants* whose magnitudes are the same in all problems once the system of units is chosen: the speed of light in vacuum c, the universal gravitational constant G, Planck's constant h, Boltzmann's constant k_B, and many others.

2.6 Derived quantities of the second kind

The classification of quantities as base or derived is not unique. There exist general laws that bind different kinds of quantities together in certain relationships, and these laws can be used to transform base quantities into derived ones. Such transformations are useful because they reduce the number of units that must be chosen arbitrarily, and simplify the forms of physical laws.

Area, for example, may be taken as a base quantity with its own comparison and addition operations, and measured in terms of an arbitrarily chosen (base) unit: a certain postage stamp, say, to use an absurd example. The floor area of a room may be measured by covering the floor with copies of this postage stamp and parts thereof, and counting the number of whole stamps required. If we adopt this practice we will eventually find that, regardless of the unit we have chosen for measuring an area ***A***, its numerical value *A* will depend on its linear dimensions according to

$$A = c\int dxdy \tag{2.17}$$

where the integral is taken over the area and c is a dimensional constant the magnitude of which depends on the choice of base units of area and length. Dimensional homogeneity requires that c be a derived quantity,

$$c = \frac{A}{\int dxdy} \tag{2.18}$$

with dimension AL^{-2}.

Equation (2.17) can be thought of as a "physical law." A little reflection reveals, however, that it holds simply because the quantity we have symbolized as $\int dxdy$ is defined by comparison and addition rules that happen to be similar to those we have chosen for area. Equation (2.17) is thus not really a law of nature, but one have crafted ourselves.

Dimensional constants like *c* in Equation (2.17) whose values depend on the choice of units but are entirely independent of the problem being considered are called *universal constant*.

The law (2.17) suggests a simplification. If we choose to measure area in terms of a unit that is defined by the area of a square with sides equal to the length unit, the physical coefficient *c* becomes unity, and equation (2.17) becomes

$$A = \int dxdy \tag{2.19}$$

Area, in effect, has become a *derived quantity* that is defined in terms of operations involving length. Equation (2.19) does *not* imply that area now "is", in any physical sense, length squared. The physical property that is area remains what it was before, quite distinct from length, and for that matter quite distinct from "length squared", of which we have no physical concept whatsoever. We have simply noted that, because our concepts of "integral" and "area" are in fact similar, we may choose to measure area via operations involving length, and have made a decision to do so. The fact that area now has the dimension L^2 simply indicates that the numerical value of any area so defined will change by a factor n^{-2} when the length unit is changed by a factor *n*.

By transforming area in this way from a base to a derived quantity, we have accomplished two simplifications: the length unit automatically determines the area unit, and the dimensional physical constant *c* in equation (2.17) is replaced with unity.

There is nothing sacred about choosing c=1 in this kind of transformation. We could have adopted as the unit for area circle with diameter equal to one length unit, in which case area would have been defined by $A = (4/\pi)\int dxdy$ instead of by equation (2.19). This would have been perfectly acceptable, but less convenient since it would force us to waste a lot of paper writing unnecessary $4/\pi$'s. In other instances, dimensionless coefficients other than unity are introduced deliberately to make things more convenient. For example, by defining the kinetic energy as $K = mV^2/2$ instead of $K = mV^2$ we make the energy equation for a point mass in a force field read $K+U$=constant instead of $K/2+U$=constant, where U is the force potential. Similarly, the 4π terms that appear in the SI system of units in the integral forms of Gauss's and Ampere's laws are placed there in order to eliminate numerical coefficients in the differential forms (Maxwell’s field equations).

Speed offers an example similar to area. We may define speed as a base quantity with its own comparison and addition operations, and choose for it a base unit—the speed of a certain very reliable wind-up toy on a horizontal surface, say, to use again an absurd example. If we take this route we would eventually discover that the quantity we have identified as speed is in fact proportional to the distance covered in unit time. We can then choose to *define* speed as the derived quantity $V=dx/dt$, which is equivalent to choosing a speed unit such that unit distance is covered in unit time.

A more interesting example arises with force. Force may be taken as a base quantity with an arbitrarily specified unit—the (equilibrium) force required to extend a standard spring a given distance, say. Newton's original law of motion states that if a body (a "point mass", to be precise) of mass m is subjected to a net force F (in the x-direction, say) under non-relativistic conditions, it accelerates in the direction of the force and the acceleration $a = d^2x/dt^2$ is related to the force F and mass m by

$$F = cma \qquad (2.20)$$

where the coefficient c is a universal constant with dimension $Ft^2M^{-1}L^{-1}$ if force, length, mass and time are all selected as base quantities. Equation (2.20) is a general physical law which expresses a relationship between the numerical values of three different physical quantities that are involved in the dynamics of a point mass—force, (gravitational) mass, and acceleration.

We are at liberty, however, to choose a force unit such that one unit of force will give unit mass unit acceleration under non-relativistic conditions. If we do this, we make c=1 in equation (2.20) and throw Newton's law into the coefficient-less form

$$F = ma \tag{2.21}$$

where force has a dimension MLt^{-2}. Equation (2.21) does *not* imply that force "is" in any physical sense a mass times an acceleration. The product of a mass times an acceleration is in fact *not defined* in physical terms, and in any case, a force can exist without there being any mass or acceleration involved, as for example when I push on an immobile brick wall (and the wall exerts an equal and opposite counter-force on me). In contrast with the examples of area and speed, equation (2.21) does *not* in all cases provide a recipe for directly evaluating a force by making in situ measurements of base quantities. Instead, we have imparted to force the character of a derived quantity by making the force *unit* depend in a particular way on the units of mass and length.

Quantities that are transferred into the derived category by choosing a unit motivated by a general physical law are called *derived quantities of the second kind*. Force is one example; heat and electric charge are also treated in this way in the SI system.

2.7 Systems of units

A system of units is defined by

(1) a complete set of base quantities with their defining comparison and addition operations,
(2) the base units, and

(3) all relevant derived quantities, expressed in terms of their defining equations (e.g $V=dx/dt$), if they are of the first kind, or the forms of the physical laws that define their units (e.g. $F=ma$ defines the force unit in the SI system), if they are of the second kind.

The set of derived quantities is open-ended; new ones may be introduced at will in any analysis.

Systems of units are said to be of the same type if they differ only in the *magnitudes* of the base units.

In the SI system (Système International) there are six base quantities (table 2.1): length, time, mass, temperature, current, number of elementary particles, and luminous intensity. The units of length, time and mass are the meter (m), the second (s) and the kilogram (kg), respectively. Force is made a derived quantity by writing Newton's law as $F=ma$.

The temperature in any system of units must be a *thermodynamic* temperature. Numbers read from an arbitrarily constructed thermometer scale can be used for establishing whether two temperatures are equal, but they are not numerical values of a physical quantity. (Is 2°C "twice as hot" as 1°C ? Is 0°C "zero" temperature in the sense of there being an absence of temperature? For that matter, if temperature is defined in terms of a thermometer with an arbitrarily marked temperature *scale*, is there any reason why heat should flow from a "higher" to a "lower" temperature?) The *thermodynamic* (or absolute) temperature is, however, defined in terms of physical comparison and addition operations appropriate to a base quantity[7]. Measurements via a Carnot engine or equivalent device yield the ratio of two absolute temperatures; the temperature *unit* (i.e. the size of the "degree") must be chosen arbitrarily. The SI temperature unit is the kelvin (K), which is defined as the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.

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[7] The ratio of the numerical values of two physical thermodynamic temperatures $\boldsymbol{T_1}$ and $\boldsymbol{T_2}$ can for example be defined in terms of the (physically measurable) efficiency η_{12} of a Carnot engine which operates between heat reservoirs at the two temperatures. If $\boldsymbol{T_1}$ is the temperature of the reservoir into which heat flows, $T_1/T_2 \equiv 1-\eta_{12}$, where η_{12} is a pure number. This provides a physical operation for determining the ratio of the numerical values of T_1 and T_2, and implies a physical addition for the two quantities: if $\boldsymbol{T_2}=n\boldsymbol{T_1}$,where $n=T_2/T_1$, then $\boldsymbol{T_1}+\boldsymbol{T_2}\equiv\boldsymbol{T_3}=(1+n)\boldsymbol{T_1}$. The temperature $\boldsymbol{T_3}=(1+n)\boldsymbol{T_1}$ can be physically identified by noting that if a Carnot engine is run between this temperature and $\boldsymbol{T_1}$, it will have an efficiency n/(1+n).

Table 2.1: The SI system of units

Base quantities (complete set)

Quantity	SI name	SI Symbol
length, L	meter	m
time, t	second	s
mass, M	kilogram	kg
temperature, T	kelvin	K
current, I	ampere	A
number of elementary particles	mole	mol
luminous intensity	candela	cd

Derived quantities (incomplete set)

Quantity	Defining equation/law	Dimension	Dimensional Symbol	Name
area	A= ∫dxdy	L^2	m^2	- - -
volume	V= ∫dxdydz	L^3	m^3	- - -
frequency	f=l/τ	t^{-1}	$s^{-}1$	hertz (Hz)
velocity	v=dx/dt	Lt^{-1}	ms^{-1}	- - -
acceleration	$a=d^2x/dt^2$	Lt^{-2}	ms^{-2}	- - -
density	ρ=M/V	ML^{-3}	kg m^{-3}	- - -
force	F=Ma	MLt^{-2}	kg m s^{-2}	newton (N)
stress/pressure	p=F/A	$ML^{-1}\ t^{-2}$	N m^{-2}=kg $m^{-1}s^{-2}$	pascal (Pa)
work/energy	W=∫Fdx	ML^2t^{-2}	N m = kg m^2s^{-2}	joule (J)
torque	T=Fl	ML^2t^{-2}	N m = kg m^2s^{-2}	- - -
power	dW/dt	ML^2t^{-3}	J s^{-1} = kg m^2s^{-3}	watt (W)
charge	Q=∫ ldt	It	A s	coulomb (C)

In SI the electric current is adopted as a base quantity and charge is thrown into the derived category (via Q=∫Idt), as are all other electric and magnetic quantities. The unit for current is the ampere (A), which is defined as the current which, when passed through each of two infinite, parallel conductors in vacuum one meter apart, will produce a force of $2x10^{-7}$ N per unit length on each conductor. The conventional factor $2x10^{-7}$ is introduced for convenience and historical reasons, and appears in the SI version of Ampere's law.

The mole unit is retained in SI as an alternative way of specifying number of things: instead of counting things one by one, one counts them in lots of $6.02x10^{23}$ (Avagadro’s number). One must, however, specify what things one is counting, e.g. "one mole of atoms, of molecules, of tennis balls", or whatever. Note that the SI mole is in fact the old "gram mole".

Luminous intensity does not refer to radiant energy flow per unit solid angle as such, which could be measured in watts per steradian, but only to that portion of it to which "the human eye" (as defined by a standard response curve) is sensitive. At a wavelength of 555 nanometers, where the human eye is most sensitive to light, one candela is equivalent to $1.46x10^{-3}$ watt per steradian.

Also included sometimes among the base quantities are two dimensionless quantities, plane angle and solid angle, which are measured in radians and steradians, respectively. We consider them derived quantities because, though dimensionless, they are defined in terms of operations involving length, much like area is defined in terms of length operations.

We note again that the dimension of a derived quantity depends on the choice of system of units, which is under the control of the observer and has nothing to do with the quantity's intrinsic nature. Indeed, quantities with quite different physical meaning, like work and torque, can have the same dimension.

The effect of the system of units on the dimensions of quantities is illustrated in table 2.2, which shows the dimensions of some mechanical quantities in three different types of systems of units. Systems of units are of the same type if they have base units of the same kind but different magnitude, and the derived quantities and the basic physical laws have the same mathematical forms, differing only in the values of the physical constants that appear in them.

In the first type of system illustrated in the table, length, time and mass are base quantities and the force unit is measured in terms of mass and acceleration via Newton's law in the form $F=ma$. The SI and the cgs (centimeter, gram, second) systems are in this category. In the second type, length, time and force are base, and mass is measured via *F=ma*. An example is the British Gravitational System, in which the base units are the foot, the second and the pound-force (lbf). The pound-force is the force exerted by *standard* gravity (32.2 ft/s^2) on a standard mass sample, the pound-mass (hence the term "gravitational system"). Mass in this

Table 2.2: Dimensions of some mechanical quantities in different types of systems of units

Base quantities and their dimensions in three types of system of units

	Type 1	Type 2	Type 3
Base quantities & dimensions	L, M, t	L, F, t	L, M, F, t

Dimensions of some derived quantities

velocity = dx/dt	Lt^{-1}	Lt^{-1}	Lt^{-1}
acceleration = d^2x/dt^2	Lt^{-2}	Lt^{-2}	Lt^{-2}
mass	M	Ft^2L^{-1}	M
area=∫dxdy	L^2	L^2	L^2
force	MLt^{-2}	F	F
c=F/ma in Newton's law	c=l	c=l	$FM^{-1}L^{-1}t^2$
work = ∫Fdx	ML^2t^{-2}	FL	FL
stress = F/A	$ML^{-1}t^{-2}$	FL^{-2}	FL^{-2}
viscosity=τ/(∂u/∂y)	$ML^{-1}t^{-1}$	$FL^{-2}t$	$FL^{-2}t$

system is a derived quantity with dimension Ft^2L^{-1} and a unit (the "slug") which turns out to be *g*=32.2 times as large as the pound-mass. In the third type, length, time, mass, *and* force are taken as base quantities, and Newton's law reads *F=cma*, where *c* is a physical constant with dimension $Ft^2M^{-1}L^{-1}$. The British Engineering System is an example. In this system the base units taken are the foot, the second, the pound-mass (lbm), and the pound-force (lbf), and the constant in Newton's law has the value c=1/32.2 lbf s^2 lbm^{-1} ft^{-1}.

Table 2.2 illustrates the fact that, while an actual *physical* quantity like force is the same regardless of the (arbitrary) choice of the system of units, its dimension depends on that choice. What is more, depending on how derived quantities are defined, a given physical law may contain a dimensional physical constant the value of which must be specified (as in F=cma), or it may contain no physical constant (as in $F=ma$).

An interesting point to note is that only a few of the available universal laws are usually "used up" to make base quantities into derived ones of the second kind. There are many laws left with universal dimensional physical constants that could in principle be set equal to unity: the gravitational constant G, Planck's constant h, Boltzmann's constant k_B, the speed of light in vacuum c, etc. This leaves us with some interesting possibilities. For example, it is possible to define systems of units that have no base quantities at all (see Bridgman, 1931). In such systems all units of measurement are related to some of the universal constants that describe our universe. In effect, there exist in the universe "natural" units for all base quantities, based on universal constants such as the speed of light, the quantum of energy, etc. Unfortunately, the choice of such “natural” systems of units turns out to be far from unique, which renders futile any attempt to endow any one of them with unique significance.

2.8 Recapitulation

1. A *base quantity* is a property that is defined in physical terms by two operations: a comparison operation, and an addition operation. The comparison operation is a physical procedure for establishing whether two samples of the quantity are equal or unequal; the addition operation defines what is meant by the sum of two samples of that property.

2. Base quantities are properties for which the following concepts are defined in terms of physical operations: equality, addition, subtraction, multiplication by a pure number, and division by a pure number. Not defined in terms of physical operations are: product, ratio, power, and logarithmic, exponential, trigonometric and other special functions of physical quantities.

3. A base quantity can be measured in terms of an arbitrarily chosen *unit* of its own kind and a *numerical value*.

4. A *derived quantity* of the first kind is a product of various powers of numerical values of base quantities. A derived quantity is defined in terms of numerical value (which depends on base unit size) and does not necessarily have a tangible physical representation.

5. The *dimension* of any physical quantity, whether base or derived, is a formula that defines how the numerical value of the quantity changes when the base unit sizes are changed. The dimension of a quantity does not by itself provide any information on the quantity's intrinsic nature. The same quantity (e.g. force) may have different dimensions in different systems of units, and quantities that are clearly physically different (e.g. work and torque) may have the same dimension.

6. Relationships between physical quantities may be represented by mathematical relationships between their numerical values. A mathematical equation that correctly describes a physical relationship between quantities is *dimensionally homogeneous* (see section 2.5). Such equations remain valid when base unit sizes are changed arbitrarily.

7. The categorization of physical quantities as either base or derived is to some extent arbitrary. If a particular base quantity turns out to be uniquely related to some other base quantities via some universal law, then we can, if we so desire, use the law to redefine that quantity as a derived quantity of the second kind whose magnitude depends on the units chosen for the others. All base quantities that are transformed into derived quantities in this way retain their original physical identities (i.e. their comparison and addition operations), but their numerical values are measured in terms of the remaining base quantities, either directly via a defining equation or indirectly by using a unit that is derivable from the remaining base units.

8. A system of units is defined by (a) the base quantities, (b) their units, and (c) the derived quantities, each with either its defining equation or the form of the physical law that has been used to cast the quantity into the derived category. Both the type and the number of base quantities are open to choice.

3. Dimensional Analysis

This chapter introduces the procedure of dimensional analysis and describes Buckingham's π-theorem, which follows from it. Section 3.1 lays down the procedure in general terms and defines the vocabulary. Section 3.2 gives an example, which the reader may wish to read in parallel with section 3.1 step by step. Section 3.3 discusses the utility of dimensional analysis and some of the pitfalls and questions that arise in its application to real problems.

3.1 The steps of dimensional analysis and Buckingham's π-theorem

The premise of dimensional analysis is that the form of any physically significant equation must be such that the relationship between the actual physical quantities remains valid independent the magnitudes of the base units. Dimensional analysis derives the logical consequences of this premise.

Suppose we are interested in some particular physical quantity Q_0 that is a "dependent variable" in a well defined physical process or event. By this we mean that, once all the quantities that define the particular process or event are specified, the value of Q_0 follows uniquely.

Step 1: The independent variables

The first and most important step in dimensional analysis is to identify a *complete* set of *independent* quantities $Q_1 \ldots Q_n$ that determine the value of Q_0,

$$Q_0 = f(Q_1, Q_2, \ldots , Q_n). \tag{3.1}$$

A set $Q_1 \ldots Q_n$ is *complete* if, once the values of the members are specified, no other quantity can affect the value of Q_0, and *independent* if the value of each member can be adjusted arbitrarily without affecting the value of any other member.

Starting with a correct set $Q_1 \ldots Q_n$ is as important in dimensional analysis as it is in mathematical physics to start with the correct fundamental equations and boundary conditions. If the starting point is wrong, so is the answer. We defer to section 3.2 the question of how a correct set is to be identified.

The relationship expressed symbolically in equation (3.1) is the result of the physical laws that govern the phenomenon of interest. It is our premise that its form must be such that, once the values $Q_1...Q_n$ are specified, the equality holds regardless of the *sizes* of the base units in terms of which the quantities are measured. The steps that follow derive the consequences of this premise.

Step 2: Dimensional considerations

Next we list the dimensions of the dependent variable Q_0 and the independent variables $Q_1...Q_n$. As we have discussed, the dimension of a quantity depends on the *type* of system of units (see table 2.2), and we must specify at least the type the system of units before we do this. For example, if we use a system of type 1 in table 2.2 and are dealing with a purely mechanical problem, all quantities have dimensions of the form

$$\left[Q_i\right] = L^{l_i} M^{m_i} t^{\tau_i} \tag{3.2}$$

where the exponents l_i, m_i and τ_i are dimensionless numbers that follow from each quantity's definition.

We now pick from the complete set of physically independent variables $Q_1...Q_n$ a *complete, dimensionally independent subset* $Q_1...Q_k$ ($k \leq n$), and express the dimension of each of the remaining independent variables $Q_{k+1}...Q_n$ and the dependent variable Q_0 as a product of powers of $Q_1...Q_k$. All physical quantities have dimensions which can be expressed as products of powers of the set of base dimensions. Alternatively, it is possible to express the dimension of one quantity as a product of powers of the dimensions of other quantities which are not necessarily base quantities. A subset $Q_1...Q_k$ of the set $Q_1...Q_n$ is *dimensionally independent* if none of its members has a dimension that can be expressed in terms of the dimensions of the remaining members. And *complete* if the dimensions of all the remaining quantities $Q_{k+1}...Q_n$ of the full set *can* be expressed in terms of the dimensions of the subset $Q_1...Q_k$.

Since equation (3.1) is dimensionally homogeneous, the dimension of the dependent variable Q_0 is also expressible in terms of the dimensions of $Q_1...Q_k$.

The dimensionally independent subset $Q_1...Q_k$ is picked by trial and error. Its members may be picked in different ways (see section 3.2), but the *number* k of dimensionally independent quantities in the full set $Q_1...Q_n$ is unique to the set, and cannot exceed the number of base dimensions which appear in the dimensions the quantities in that set. For example, if the dimensions of $Q_1...Q_n$ involve only length, mass, and time, then k≤3.

Having chosen a complete, dimensionally independent subset $Q_1...Q_k$, we express the dimensions of Q_0 and the remaining quantities $Q_{k+1}...Q_n$ in terms of the dimensions of $Q_1...Q_k$. These will have the form

$$[Q_i] = [Q_1^{N_{i1}} Q_2^{N_{i2}} ... Q_k^{N_{ik}}] \tag{3.3}$$

if $i>k$ or $i=0$. The exponents N_{ij} are dimensionless real numbers and can in most cases be found quickly by inspection (see section 3.2), although a formal algebraic method can be used.

The formal procedure can be illustrated with an example where length, mass and time are the only base quantities, in which case all dimensions have the form of equation (3.2). Let us take Q_1, Q_2, and Q_3 as the complete dimensionally independent subset. Equating the dimension given by equation (3.2) with that of equation (3.3), we obtain three equations

$$l_i = \sum_{j=1}^{3} N_{ij} l_j$$

$$m_i = \sum_{j=1}^{3} N_{ij} m_j \tag{3.4}$$

$$t_i = \sum_{j=1}^{3} N_{ij} t_j$$

which can be solved for the three unknowns N_{i1}, N_{i2}, and N_{i3}.

Step 3: Dimensionless variables

We now define *dimensionless forms* of the n-k remaining independent variables by dividing each one with the product of powers of $Q_1 ... Q_k$ which has the same dimension,

$$\Pi_i = \frac{Q_{k+i}}{Q_1^{N_{(k+i)1}} Q_2^{N_{(k+i)2}} \ldots Q_k^{N_{(k+i)k}}} \quad , \tag{3.5}$$

where i=1, 2,..., n-k, and a dimensionless form of the dependent variable Q_0,

$$\Pi_0 = \frac{Q_0}{Q_1^{N_{01}} Q_2^{N_{02}} \ldots Q_k^{N_{0k}}} \quad . \tag{3.6}$$

Step 4: The end game and Buckingham's π-theorem

An alternative form of equation (3.1) is

$$\Pi_0 = f(Q_1, Q_2, ..., Q_k; \Pi_1, \Pi_2, ..., \Pi_{n-k}) \tag{3.7}$$

in which all quantities are dimensionless except $Q_1 ... Q_k$. The values of the dimensionless quantities are independent of the sizes of the base units. The values of $Q_1 ... Q_k$, on the other hand, do depend on base unit size. They *cannot* be put into dimensionless form since they are (by definition) dimensionally independent of each other. From the principle that any physically meaningful equation must be dimensionally homogeneous, that is, valid independent of the sizes of the base units, it follows that $Q_1 ... Q_k$ must in fact be *absent* from equation (3.7), that is,

$$\Pi_0 = f(\Pi_1, \Pi_2, ..., \Pi_{n-k}) \quad . \tag{3.8}$$

This equation is the final result of the dimensional analysis, and contains

Buckingham's π-theorem:

> When a complete relationship between dimensional physical quantities is expressed in dimensionless form, the number of independent quantities that appear in it is reduced from the original n to n-k, where k is the maximum number of the original n that are dimensionally independent.

The theorem derives its name from Buckingham's use of the symbol Π for the dimensionless variables in his original 1914 paper. The π-theorem tells us that, because all complete physical equations must be dimensionally homogeneous, a restatement of any such equation in an appropriate dimensionless form will reduce the number of independent quantities in the problem by k. This can simplify the problem enormously, as will be evident from the example that follows.

The π-theorem itself merely tells us the *number* of dimensionless quantities that affect the value of a particular dimensionless dependent variable. It does not tell us the forms of the dimensionless variables. That has to be discovered in the third and fourth steps described above. Nor does the π-theorem, or for that matter dimensional analysis as such, say anything about the form of the functional relationship expressed by equation (3.7). That form has to be discovered by experimentation or by solving the problem theoretically.

3.2 An example: Deformation of an elastic ball striking a wall

Suppose we wish to investigate the deformation that occurs in elastic balls when they impact on a wall. We might be interested, for example, in finding out what determines the diameter d of the circular imprint left on the wall after a freshly dyed ball has rebounded from it (figure 3.1).

Step 1: The independent variables

The first step is to identify a complete set of independent quantities that determine the imprint radius d. We begin by specifying the problem more clearly. We agree to restrict our attention to (initially) spherical, homogeneous balls made of perfectly elastic material, to impacts at perpendicular to the wall, and to walls that are perfectly smooth and flat and so stiff and heavy that they do not deform or move during the impact process. We also agree to adopt a system of units of type 1 in table 2.2, like the SI system.

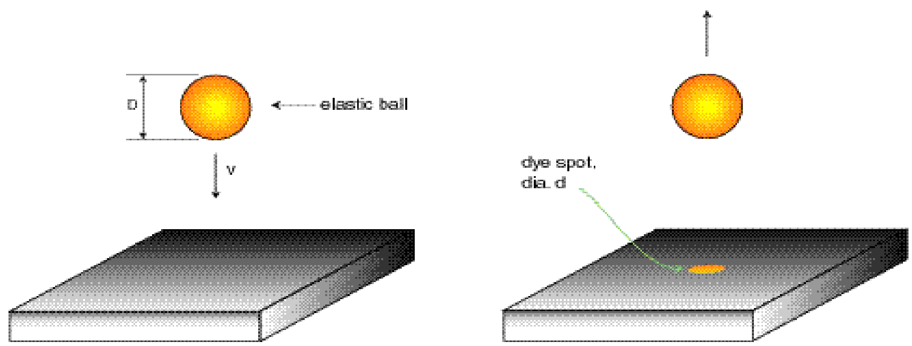

Fig. 3.1: A freshly dyed elastic ball leaving imprint after impact with rigid wall.

The numerical value of a dependent variable like d will be depend on the values of all quantities that distinguish one impact event from another. Experience suggests that these should include at least the following: the ball's diameter D and velocity V just prior to contact (the initial conditions) and its mass m. The ball's intrinsic material properties will also play a role. Our theoretical understanding of solid mechanics tells us that the quasi-static response of a *perfectly elastic* material is characterized by two material properties, the modulus of elasticity E and Poisson's ratio γ, and that the inertial effects which inevitably come into play during collision and rebound will also depend on the material's density ρ. The properties of the wall are irrelevant if it is indeed perfectly rigid, as we assumed. We know, however, by thinking of how the problem would have to be set up as a theoretical one, that the answer for the numerical value d will also depend on the values of all universal constants that appear in the physical laws that control the ball's impact dynamics. In this case the process is governed by Newton's law of motion and the law of mass conservation. Having chosen a system of units of type 1 in table 2.2, we know that Newton's law has the form $F=ma$ and contains no universal constants[8]. Nor are there any physical constants in the law of mass conservation.

We seem to arrive at the conclusion that d depends on six quantities: D, V, m, E, γ, ρ. This is a *complete* set, as required, but *not an independent set:* once the ball's mass and diameter are specified, its density follows. We must therefore exclude either the density or the mass. (Other quantities like V^2, $DE^{1/2}$, etcetera, all involving quantities that affect the value of d, are excluded for the same reason: they are not independent of the quantities already included.) We conclude that the following relationship expresses the impact diameter in terms of a complete set of independent variables:

$$d = d(V, \rho, D, E, \gamma). \tag{3.9}$$

Note that the choice of a complete, independent set for a specified problem is *not unique* except for the number n of its members (n=5 in this case). One could just as well have chosen $V^2, \rho, D, E,$ and γ, say, or V, m, D, E, γ—see section 3.3. It should also be noted that further assumptions have been taken for granted in equation (3.9). We have presumed, for example, that the ball's motion is unaffected by the properties of the fluid through which it approaches the wall (which is certainly OK if the ball moves through vacuum and a good approximation in air, but may not apply to small balls in viscous liquids), and that gravitational effects play a negligible role. See the discussion in section 3.3.

Step 2: Dimensional considerations

In the type of system of units we have adopted in step 1, the dimensions of the quantities in equation (3.9) are:

independent:

$$\begin{aligned} &[V]=\mathrm{Lt}^{-1} \\ &[\rho]=\mathrm{ML}^{-3} \\ &[D]=\mathrm{L} \\ &[E]=\mathrm{ML}^{-1}\mathrm{t}^{-2} \\ &[\gamma]=1 \end{aligned} \tag{3.10}$$

---

[8] If, on the other hand, we had decided to use a system of type 3 in Table 2.1, the value of the dimensional constant c=F/ma for the chosen system of units would have to be specified, and would affect the value of d.

dependent: $[d]=L$

Inspection of the above shows that the three quantities V, ρ, and D, for example, comprise a complete, dimensionally independent subset of the five independent variables. The dimension of any one of these three cannot be made up of the dimensions of the other two. The dimensions of the remaining independent variables E and γ and the dependent variable d can, however, be made up of those of V, ρ and D as follows:

independent: $[E] = ML^{-1}t^{-2} = (ML^{-3})(Lt)^2 = [\rho V^2]$

$[\gamma] = 1$

$$\tag{3.11}$$

dependent: $[d] = L = [D]$

We have written down these results very simply by inspection. Accomplished practitioners seldom use the formal algebraic method of section 3.1. Note again that the dimension of a dimensionless quantity like γ is unity, the factor by which dimensionless numbers change when the sizes of the base units are changed.

Step 3: Dimensionless similarity parameters

We non-dimensionalize the remaining independent variables E and γ and the dependent variable d by dividing them by ρV^2, D, and unity, respectively, as suggested by equation (3.11):

independent: $\Pi_1 = \dfrac{ED^3}{mV^2}$

$$\Pi_2 = \gamma \tag{3.12}$$

dependent: $\Pi_0 = \dfrac{d}{D}$

Step 4: The end game

Using the logic that led to Buckingham's π-theorem, we now conclude that

$$\Pi_o = f(\Pi_1, \Pi_2),$$

or

$$\frac{d}{D} = f\left(\frac{E}{\rho V^2}, \gamma\right). \tag{3.13}$$

The number of independent variables has been reduced from the original n=5 that define the problem to n-k =2.

3.3 On the utility of dimensional analysis, and some difficulties and questions that arise in its application

Similarity

Dimensional analysis provides a *similarity law* for the phenomenon under consideration. Similarity in this context implies a certain equivalence between two physical phenomena that are actually different. The collisions of two different elastic spheres 1 and 2 with a rigid wall, each with its own values of V, ρ, D, E, and γ, may appear to be quite different. However, under particular conditions where the parameters of the two events are such that Π_1 and Π_2 have the same values, that is, where

$$\begin{gathered} \frac{E_2}{\rho_2 V_2^2} = \frac{E_1}{\rho_1 V_1^2} \\ \gamma_2 = \gamma_1, \end{gathered} \tag{3.14}$$

equation (3.13) informs us that Π_0 has the same value in both cases, that is,

$$\frac{d_2}{D_2} = \frac{d_1}{D_1}. \tag{3.15}$$

When the relationships in equation (3.14) apply, the two dynamic events are *similar* in the sense of equation (3.15).

Out-of-scale modeling

Scale modeling deals with the following question: If we want to learn something about the performance of a full-scale system 1 by testing a *geometrically similar* small-scale system model 2 (or vice-versa, if the system of interest inaccessibly small), at what conditions should we test the model, and how should we obtain the full-scale performance from measurements at the small scale? Dimensional analysis provides the answer.

Suppose we need to know the deformation diameter of a huge, soft rubber ball with a diameter D_1 of 5 meters and properties E_1, ρ_1, and γ_1, as it hits the pavement with a speed V_2 of 10 m/s, but are unable to compute it from basic principles. In that case we need only perform *one* small-scale test with a model 2 of diameter D_2, selecting its properties and test conditions such that equations (3.14) are satisfied, and measure its imprint diameter d_2. The full-scale value d_1 of the big ball's imprint diameter at its "design conditions" can then be obtained from equation (3.15).

Dimensional analysis reduces the number of variables and minimizes work

Dimensional analysis reduces the number of variables that must be specified to describe an event. This often leads to an enormous simplification. In our example of the impacting ball the answer depends on five independent variables (equation 3.9), that is, a particular event may be represented as a distribution of *d* defined in a five-dimensional space of independent variables. Suppose we set out to obtain the answer in a certain region (a certain volume) of this variable-space, by either computation or experimentation, and decide that 10 data points will be required in each variable, with the other four being held constant. This would require obtaining 10^5 data points. Dimensional analysis, however, shows us that in dimensionless form the answer depends only on two similarity parameters. This two-dimensional space can be explored with similar resolution with only 10^2 data points, that is, with 0.1% of the effort.

Table 3.1 shows some “experimental data” for impacts with balls of three materials and various values of impact velocity. These results were actually computed by Mark Bathe (2001) using the finite-element code

Table 3.1: Computed “Experimental data”

Material	E (MPa)	ρ (kg m^{-3})	V (m s^{-1})	γ	$\frac{E}{\rho V^2}$	$\frac{d}{D}$	Symbol (Fig. 3.2)
Alumina	3.66E+05	3960	43	0.22	50133	0.150	☆
	3.66E+05	3960	59	0.22	26937	0.170	◁
	3.66E+05	3960	77	0.22	15511	0.190	▷
Aluminum	6.90E+04	2705	80	0.33	3973	0.250	△
	6.90E+04	2705	126	0.33	1608	0.300	○
	6.90E+04	2705	345	0.33	215	0.450	◇
Rubber	3.93E+00	1060	5	0.47	127	0.500	□
	3.93E+00	1060	7	0.47	79	0.550	·
	3.93E+00	1060	12	0.47	24	0.700	✳

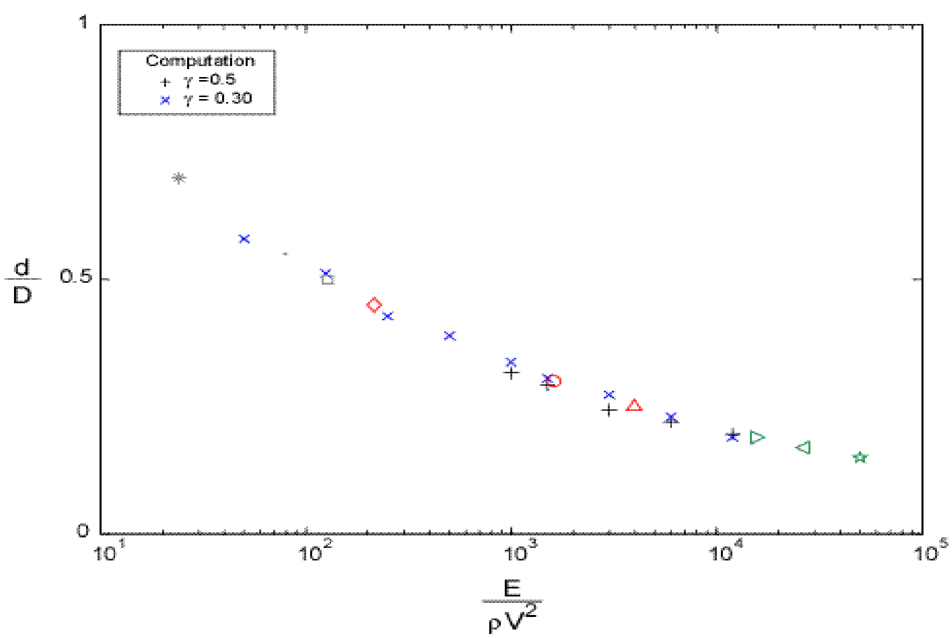

Figure 3.2: Plot of “experimental data” in dimensionless form.

ADINA. Fig. 3.2 shows these data as a plot of d/D vs. $E/\rho V^2$ with γ as a parameter. The simulated "experimental scatter" in Fig. 3.2 actually results from the coarseness of the computational grid. Disregarding the scatter, all points fall essentially on a common curve, as predicted by dimensional analysis. The influence of the Poisson's ratio turns out to be virtually negligible, given the scatter, and all the data may be curve-fitted with an equation of the form $d/D = f(E/\rho V^2)$.

An incomplete set of independent quantities may destroy the analysis

Assuming competence on the part of the analyst, the correctness of the dimensional analysis will depend entirely on whether a complete set of independent quantities $Q_1 ... Q_n$ is in fact properly identified in step 1. Any complete set will yield correct results. If, however, the analysis is based on a set which omits even one independent quantity that affects the value of Q_o, dimensional analysis will give erroneous results.

Suppose that in our example we had omitted the sphere's modulus of elasticity E in equation (3.9). Instead of equation (3.13), we would then have obtained the absurd result

$$\frac{d}{D} = f(\gamma), \tag{3.16}$$

which implies that the maximum deformation depends on the ball's Poisson ratio, but is independent of its elasticity, mass and approach velocity! This single error of omission is clearly fatal to the analysis.

Superfluous independent quantities complicate the result unnecessarily

Errors on the side of excess have a less traumatic effect. Over-specification of independent variables does not destroy the analysis, but robs it of its power. For every superfluous independent quantity included in the set, there will be in the final dimensionless relationship a superfluous dimensionless similarity parameter.

Suppose we argue (quite reasonably!) that the ball's deformation upon impact will in general also depend on the local gravitational acceleration g (which we assume to be in the direction into the wall on which the impact occurs). This would change equation (3.13) to

$$\frac{d}{D} = f\left(\frac{E}{\rho V^2}, \gamma, \frac{gD}{V^2}\right) \tag{3.17}$$

where gD/V^2 is a dimensionless gravity. Under conditions where the deformation is in fact insensitive to gravity, as we implicitly assumed earlier, equation (3.17) is “wrong” only in the sense that it suggests a dependence on g that is not noticeably there, and thus unnecessarily complicates our thinking. If by experimentation or computation we eventually discover that there exists a broad range of conditions where the similarity parameter involving g has in fact no measurable effect on d/D, and that the conditions of interest fall into this range[6], we omit the parameter involving g and arrive at the same simpler conclusion as before, but only after due payment in effort for our lack of insight.

On the importance of simplifying assumptions

The previous example illustrates an important point about most problems in dimensional analysis: Completeness in the set of independent variables is not an absolute matter, but depends on how we choose to circumscribe the problem. It is quite conceivable, for example, that there are “balls” whose deformation upon impact with a wall is gravity-dependent. This would be the case for balls with such low coefficient of elasticity, or large diameter, or low impact velocity, that their deformation at rest on a wall in the gravitational field would be significant compared with their deformation upon impact. When we say “the deformation does not depend on gravity”, we imply that we know with some confidence that there exist conditions where this is true, and choose to confine our attention to those conditions.

If dimensional analysis depended on a truly complete identification of the independent variables that specify a given physical event, we would in most cases be reduced to impotence. Those familiar with the theory of chaos and the mechanics of many-body systems may sympathize with the view that the wind at a certain street corner in New York, say, may in

[6] To prove this with earthbound experimentation one would have to demonstrate that d/D remains constant when m, D and/or V are changed so as to vary gD/V^2 *while keeping* ED^3/mV_2 *constant.*

some measure have been affected by the wing-beat of a butterfly in Brazil a month earlier. This kind of fastidiousness elicits a shrug from the pragmatist, who proceeds with a problem by saying: “Based on my experience as a scientist and engineer, I argue that this event should be controlled by the following complete set of independent variables. Assuming tentatively that I am correct, what does dimensional analysis tell me?”

On choosing a complete set of independent variables

Given what has been said above, how does one go about choosing a complete set of independent variables that define a particular problem?

If we know the mathematical forms of all the equations and boundary (and initial) conditions that completely specify a particular type of process or event, one can deduce from them a complete set of independent parameters that define the event. This we do by simply examining all the equations and listing all the quantities whose values would have to be specified to define a particular event through all time. The set may include position and time, if the variable of interest depends on them, universal physical constants (e.g. gravitational constant, universal gas constant R, etc.), material properties (e.g. density, E, etc.) and all other quantities that appear not only in the equations but also in the boundary and initial conditions that determine the answer to the particular problem at hand. The last point is not as straightforward as it sounds: the question of what constitutes a well-posed mathematical problem is often difficult to answer in non-linear systems.

If the equations and boundary conditions are *not* well known, as for example when one is trying to use dimensional analysis to help correlate experimental data for a complex phenomenon that is not well understood, one has to proceed by trial and error based on an (educated) guess about the physics of the problem at hand. Support for the analysis (and thus for the guess about the physics) may be obtained *à posteriori* by showing that all experimental data can indeed be accurately correlated in the dimensionless form suggested by the analysis. There should be sufficient data to be able to show that one has neither missed an important quantity nor included one that is irrelevant.

The result is independent of how one chooses a dimensionally independent subset

Suppose we had chosen the dimensionally independent subset *V*, *E*, and ρ instead of *V*, *D* and ρ. Non-dimensionalizing *d* and *D* with combinations of *V*, *E* and ρ, we might have obtained the result

$$\frac{d}{(\rho V^2/E)^{1/3}} = F\left(\frac{D}{(mV^2/E)^{1/3}}, \gamma\right) \tag{3.18}$$

This can, however, be rewritten as

$$\frac{d}{D} = \left(\frac{mV^2}{ED^3}\right)^{1/3} F\left(\left(\frac{ED^3}{mV^2}\right)^{1/3}, \gamma\right) = f\left(\frac{ED^3}{mV^2}, \gamma\right) \tag{3.19}$$

where *F* and *f* are different functions of their arguments. Equation (3.19) is of course identical to equation (3.13).

The result is independent of the type of system of units

The choice of system of units may affect the dimensions of physical quantities as well as the values of the physical constants that appear in the underlying physical laws. What effect, if any, does this have on dimensional analysis? Reason dictates there should be no effect on the "bottom line", since the observer (the analyst) is free to choose or make up whatever system of units he wants, and his arbitrary choice should not affect the laws of physics.

Consider our example of the dyed ball, but viewed in terms of a system of units like the British Engineering System (type3 in table 2.2), where mass, length, time *and* force are taken as base units. In such a system Newton's law reads $F = cma$, where *c* is a physical constant with dimension $Ft^2m^{-1}L^{-1}$. This affects the very first step of the analysis. Since the impact process is controlled by Newton's law, which now contains the constant *c*, the value of which must be specified, we now have

$$d = d(V, D, E, m, \gamma, c). \tag{3.20}$$

The number of independent variable (n=6) has increased by one. The dimensions of the variable are now:

independent: $[V]=\mathrm{Lt}^{-1}$
$[D]=\mathrm{L}$
$[E]=\mathrm{FL}^{-2}$ (3.21)
$[m]=\mathrm{M}$
$[\gamma]=0$
$[c]=\mathrm{Ft}^{2}\mathrm{M}^{-1}\mathrm{L}^{-1}$

dependent: $[d]=\mathrm{L}$

The quantities V, D, m and c comprise a convenient dimensionally independent subset. The number of this subset has also increased by one (k=4). The dimensions of the remaining quantities can be expressed in terms of these four as

$$[d]=[D] \qquad [E]=[cV^2D^{-3}] \qquad [\gamma]=0 \tag{3.22}$$

and the final result of the analysis is

$$\frac{d}{D} = f\left(\frac{ED^3}{cmV^2}, \gamma\right) \tag{3.23}$$

This differs from our previous result, equation (3.13), only in that the physical constant c appears in the non-dimensionalization of E. Equations (3.13) and (3.23) are, however, functionally identical. Since n-k=2 in both cases, both analyses imply that d/D depends on two dimensionless parameters, γ and a dimensionless E. It is just that in the second system of units, E must be non-dimensionalized with cmV^2/D^3 instead of mV^2/D^3, since the latter no longer has the same dimension as E. In short, the forms of some of the dimensional parameters may change with the system of units, but the physical content of analysis remains invariant. This is, of course, as we expected; the choice of system of units is arbitrary, and should not affect the physical "bottom line."

4. Dimensional Analysis in Problems where Some Independent Quantities Have Fixed Values

Engineering practice often involves problems where some of the quantities that define the problem have the same fixed values in all the applications being considered. As a simple example, suppose we are interested in determining the hydrodynamic drag force *D* on a fully submerged, very long, neutrally buoyant cable being dragged behind a ship, and propose to do this by making small scale experiments in a water tunnel. Basic fluid mechanics tells us that, barring surface roughness effects, the drag force should be completely determined by the cable's length *L* and diameter *d*, the ship's (or water's) velocity *V*, and the water's density ρ and viscosity μ. Three of these five quantities are dimensionally independent, and dimensional analysis (or Buckingham's pi-theorem) tells us that an appropriately defined dimensionless drag is a function of n-k=2 dimensionless similarity parameters. One way of writing this relationship is

$$\frac{D}{\rho V^2 L^2} = f\left(\mathrm{Re}, \frac{d}{L}\right) \tag{4.1}$$

where

$$\mathrm{Re} = \frac{\rho V L}{\mu} \qquad 4.2)$$

is a Reynolds number based on cable length and *d/L* is the cable's "fineness coefficient," which defines its geometry. Equation (4.1) is a general relationship for the cable-towing problem as stated.

We observe, however, that the density and viscosity have essentially the same values—those of sea water at its typical temperature—in all the applications that are of interest. The number of quantities that actually vary from case to case are thus actually three, not five.

The question arises, does this lead to a simplification of the similarity law expressed by equation (4.1), that is, to a reduction in the number of independent similarity parameters? Simply omitting the quantities that have fixed values and performing dimensional analysis on the rest cannot answer this question[9]. Dimensional analysis must be based on a *complete* set of independent quantities that determine the quantity of interest. *All* the quantities whose values determine the quantity of interest must be included, regardless of whether some of them happen to be the same values in the problems that are of interest. As we have seen in section 3, omitting even one independent variable can fatally damage the analysis.

The general question is the following: What reduction, if any, can be obtained in the number of similarity parameters if a certain number of the independent quantities that define the problem always have the same fixed values? This can be answered by the following analysis.

Suppose we are interested in a quantity Q that is *completely determined* by the values of n independent quantities Q_i, of which n_F are held at fixed values in all the cases that concern us. Let the quantities that may vary be the first $(n\text{-}n_F)$ of Q_i, and designate the n_F quantities with fixed values by F_i:

$$Q = f(Q_1, Q_2, .., Q_{n-n_F}; F_1, F_2, .., F_{k_F}, F_{k_F+1}, F_{k_F+2}, .., F_{n_F}) \qquad (4.3)$$

Choose a complete, *dimensionally* independent subset of the set F_i. Let these be the first k_F of the fixed set, as indicated in equation (4.3). Using this subset, non-dimensionalize the remaining $(n_F\text{-}k_F)$ fixed quantities and write the relationship (4.3) in the alternative form

$$Q = f(Q_1, Q_2, ..., Q_{n-n_F}; F_1, F_2, ..., F_{k_F}, F^{*}_{k+1}, F^{*}_{k+1}, ..., F^{*}_{n_F}) \qquad (4.4)$$

where the asterisked indicate dimensionless quantities involving only quantities with fixed values. These dimensionless quantities thus have the same *fixed values* in the cases that concern us. For these cases, therefore, we can write (4.4) as

$$Q = f(Q_1, Q_2, ..., Q_{n-n_F}; F_1, F_2, ..., F_{k_F}) \qquad (4.5)$$

The value of Q is thus completely determined by a set of $n\text{-}n_F\text{+}k_F$ independent quantities consisting of those dependent quantities that are

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9 Were we to simply omit the density and viscosity in the present problem, for example, we would imply a relationship *D=f(L,h,V)* which is not dimensionally homogeneous and therefore unacceptable. (There is no way of writing the dimension of force in terms of just length and velocity).

not fixed plus the dimensionally independent subset of the fixed quantities.

Now perform dimensional analysis on the relationship in equation (4.5). First, select from the set of n-n_F+k_F independent quantities in equation (4.5) a complete, dimensionally independent subset of k quantities. Let this subset be the first k of the quantities. Since equation (4.5) contains all the variable independent quantities plus the dimensionally independent subset of the fixed (independent) quantities, the subset we thus obtain is also complete, dimensionally independent subset for the whole original set of n quantities.

According to equation (4.5), Q depends on n-n_F+k_F independent variables, of which k are independent. Dimensional analysis thus yields the result

$$Q^{\square} = f(\Pi_1, \Pi_2, ..., \Pi_N) \tag{4.6}$$

where

$$N = (n - k) - (n_F - k_F)\,. \tag{4.7}$$

We have arrived at the following theorem:

Theorem

> If a quantity Q is completely determined by a set of n independent quantities, of which k are dimensionally independent, and if n_F of these quantities have fixed values in all the cases being considered, a number k_F of these being dimensionally independent, then a suitable dimensionless Q will be completely determined by (n-k)-(n_F-k_F) dimensionless similarity parameters.

In other words, the fact that a number n_F of the quantities always have fixed values reduces the number of independent similarity parameters by (n_F-k_F). This theorem is a generalization of Buckingham's Π-theorem, and reduces to it when n_F=0.

Returning to the cable-towing example, in which n_F=2 and k_F=2, we now see immediately that no gain in similarity (no reduction in the number of similarity parameters) accrues from the fact that the viscosity and density are essentially the same in all the applications that interest us. The similarity law equation (4.1) cannot be simplified.

Simplification occurs only when some of the fixed quantities are dimensionally dependent on the rest.

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