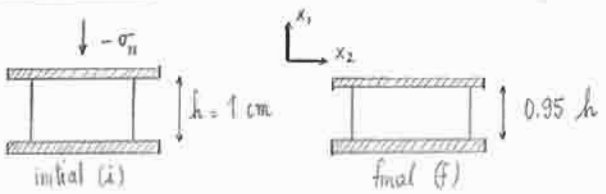
BEH. 410 . March 2003

Problem Set # 3 . SOLUTION

Problem 1



a) Hooke's law states $\varepsilon_{ij} = \dfrac{1+\nu}{E}\sigma_{ij} - \dfrac{\nu}{E}\sigma_{kk}\delta_{ij}$

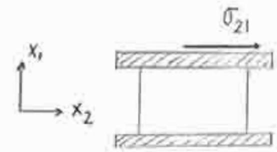
Here $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{11} + 0 + 0 = \sigma_{11}$

$$\varepsilon_{11} = 0.05 = \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{11} = \frac{\sigma_{11}}{E} \qquad (1)$$

(1) can be written as $\sigma_{11} = E\,\varepsilon_{11} = 10^6 \cdot 0.05$
$= 5 .\, 10^4$ Pa

One need to <u>apply a force of $5 . 10^4$ newtons per square meter</u>

b) <u>Shear modulus G</u>



- Hooke's law $\sigma_{ij} = \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2G\,\varepsilon_{ij}$

$$\sigma_{11} = \lambda\,\varepsilon_{kk} + 2G\,\varepsilon_{11} \qquad (1)$$

- Express λ in terms of G and ν (solution given in lecture 11) in the unconfined compression configuration $\sigma_{22} = \sigma_{33} = 0$

$$\left.\begin{aligned}
\varepsilon_{11} &= \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{11} = \frac{\sigma_{11}}{E}\\
\varepsilon_{22} &= 0 - \frac{\nu}{E}(\sigma_{11} + 0 + 0) = -\nu\frac{\sigma_{11}}{E}\\
\varepsilon_{33} &= 0 - \frac{\nu}{E}(\sigma_{11} + 0 + 0) = -\nu\frac{\sigma_{11}}{E}
\end{aligned}\right\} \Rightarrow \boxed{\varepsilon_{22} = \varepsilon_{33} = -\nu\,\varepsilon_{11}}$$

now writing $\sigma_{22} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2G\,\varepsilon_{22}$

or $0 = \lambda(\varepsilon_{11} - \nu\varepsilon_{11} - \nu\varepsilon_{11}) + 2G(-\nu\,\varepsilon_{11})$, one gets $\boxed{\lambda = \dfrac{2\nu}{1-2\nu}G}$

- in equation (1) $\sigma_{11} = \dfrac{2\nu G}{1-2\nu}(1-2\nu)\varepsilon_{11} + 2G\,\varepsilon_{11} = 2G(\nu+1)\,\varepsilon_{11}$

$G = \dfrac{1}{2(1+\nu)}\cdot\dfrac{\sigma_{11}}{\varepsilon_{11}} = \dfrac{E}{2(1+\nu)}$ YOU COULD USE THIS RESULT FROM L# 11 WITHOUT DERIVING IT

for an incompressible material $\nu = 0.5$; hence <u>$G = 3.3 \cdot 10^5$ Pa</u>

- From Hooke's law $\sigma_{21} = 2G\,\varepsilon_{21}$ with $\left\{\begin{array}{l}\varepsilon_{21} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right)\\ u_i : \text{displacement on the } i \text{ axis}\end{array}\right.$

For small displacement $\varepsilon_{21} = \dfrac{1}{2}\dfrac{\Delta L}{h}$ where ΔL : displacement of the top surface

Hence $\sigma_{21} = \dfrac{\Delta L}{h}G = 0.1 * 3.3 * 10^5 =$ <u>3.3×10^4 Pa : shear force to be applied</u>

c) $V_f = V_i\,(1+\varepsilon_{11})(1+\varepsilon_{22})(1+\varepsilon_{33}) \qquad (2)$

Here, we are told that $V_i = h^3$, and that $\varepsilon_{11} = -0.05$ whereas $\varepsilon_{22} = \varepsilon_{33} = +0.02$

Thus <u>$V_f = 0.988\,h^3$</u> $\neq V_i \Rightarrow$ the specimen is <u>not incompressible</u>.

- Rather, using the relationship established in question b), $\varepsilon_{22} = \varepsilon_{33} = -\nu\,\varepsilon_{11}$, we get

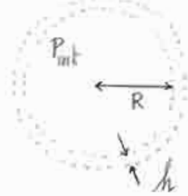
$V_f = V_i\,(1+\varepsilon_{11})(1-\nu\,\varepsilon_{11})^2$

$$\boxed{\nu = \frac{1}{\varepsilon_{11}}\left(1 - \sqrt{\frac{V_f}{V_i\,(1+\varepsilon_{11})}}\right)} = \frac{1}{-0.05}\left(1 - \sqrt{\frac{0.988\,h^3}{h^3\,(1-0.05)}}\right) = \underline{0.4 \text{ : Poisson ratio of specimen}}.$$

d)
$$\left.\begin{aligned}
\sigma_{ij} &= \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2G\,\varepsilon_{ij} \Rightarrow \sigma_{21} = 2G\,\varepsilon_{21}\\
\varepsilon_{ij} &= \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\,\delta_{ij} \Rightarrow \varepsilon_{21} = \frac{1+\nu}{E}\sigma_{21}
\end{aligned}\right\} \Rightarrow \boxed{E = 2G(1+\nu)}$$

Problem 2

$P_{int} = 100$ mm Hg $= 1.32 \times 10^4$ Pa $P_{ext} = 0$
$\nu = 0.45$
$E = 10^5$ Pa
$R = 10^{-2}$ m
$h = 10^{-3}$ m



a) Average circumferential stress on wall

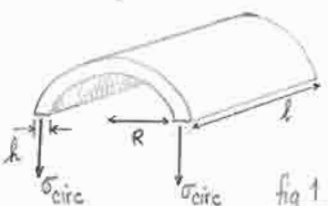


fig 1

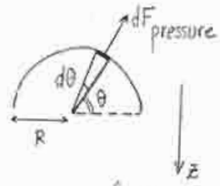


fig 2

Force balance on hemicylinder (fig 1):

$$\cdot \begin{cases} dF_{pressure} = P_{int} \cdot R\,d\theta \cdot l \quad \text{and onto } z\text{-axis} \\ dF_{pressure}|_z = dF_{pressure} \sin\theta = P_{int} R\, l \sin\theta\, d\theta \\ \text{on volume } F_{pressure}|_z = \int_0^{\pi} P_{int} R\, l \sin\theta\, d\theta = 2 P_{int} R\, l \end{cases}$$

$$\cdot\ 2\sigma_{circ} \cdot h \cdot l = 2 P_{int} R\, l$$

Under the thin wall assumption, $\boxed{\sigma_{circ} = \dfrac{P_{int} R}{h}} = \underline{1.3 \times 10^5 \text{ Pa}}$

Radial compressive stress:

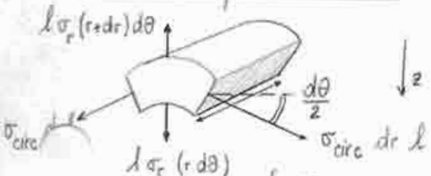


fig 3

Force balance of infinitesimal volume (fig 3) onto z-axis

$$\left\{ (\sigma_r\, r)|_r - (\sigma_r\, r)|_{r+dr} \right\} d\theta \cdot l + 2\, \sigma_{circ}\, dr\, l\, \frac{d\theta}{2} = 0 \qquad \text{with } \sin\frac{d\theta}{2} \simeq \frac{\theta}{2}$$

$$-\frac{d}{dr}(\sigma_r\, r) + \sigma_{circ} = 0 \quad \text{which can be written} \quad \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_{circ}}{r} = 0 \quad (1)$$

solve homogeneous equation associated to (1) $\frac{d\sigma_r}{dr} + \frac{\sigma_r}{r} = 0 \Rightarrow \sigma_r(r) = \frac{A}{r}$, A const.
add particular solution to (1): $\sigma_r = \sigma_{circ}$ and solve $\sigma_r(r) = \sigma_{circ} + \frac{A}{r}$ knowing boundary conditions $\sigma_r(R+h) = 0$

$$\boxed{\sigma_r = \sigma_{circ}\left(1 - \frac{R+h}{r}\right)}$$

b) Hooke's law (with $\varepsilon_{33} = 0$, $\sigma_{11} = \sigma_r$, $\sigma_{22} = \sigma_{circ}$)

$$\varepsilon_{11} = \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \quad (1)$$

$$\varepsilon_{22} = \frac{1+\nu}{E}\sigma_{22} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \quad (2)$$

$$0 = \frac{1+\nu}{E}\sigma_{33} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \quad (3)$$

(3) gives us and into (1)

$$\boxed{\begin{aligned} \sigma_{33} &= \nu(\sigma_{11} + \sigma_{22}) \\ \varepsilon_{11} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{11} - \nu\,\sigma_{22}\right] \\ \varepsilon_{22} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{22} - \nu\,\sigma_{11}\right] \end{aligned}}$$

and using $\sigma_{11}(R + \frac{h}{2}) = -6.3 \times 10^3$ Pa, we get
$\underline{\sigma_{33} = -5.6 \times 10^4 \text{ Pa}, \quad \varepsilon_{11} = -0.9, \quad \varepsilon_{22} = 1.1}$

Note that this is NOT a small strain (more realistic values for E and h would be 10^6 Pa and 2 mm, making strains much lower)

c) Change in volume ($V_f - V_i$)

By definition of the strain $V_f = V_i(1 + \varepsilon_{11})(1 + \varepsilon_{22})(1 + \varepsilon_{33}) = 0.21\ V_i$
($V_f = V_i(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} + 1)$ for small strain approximation, not valid here)

$\underline{V_f - V_i = 0.79\ V_i \simeq -8 \times 10^{-10}\ m^3}$

Bulk compressive modulus $K = \dfrac{E}{3(1-2\nu)} = \underline{3.3 \times 10^5 \text{ Pa}}$

This positive change in volume under applied hydrostatic pressure may be due to water fluxes (in particular entrance) through the tissue.

Problem 2 (cont'd)

d) Small strain approximation $\Delta V = 0 \iff \varepsilon_{kk} = 0$ incompressibility condition

Writing Hooke's law $\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2G \varepsilon_{ij}$,

$$\left.\begin{aligned} \sigma_{11} &= \lambda \varepsilon_{kk} + 2G \varepsilon_{11} \\ \sigma_{22} &= \lambda \varepsilon_{kk} + 2G \varepsilon_{22} \\ \sigma_{33} &= \lambda \varepsilon_{kk} + 2G \varepsilon_{33} \end{aligned}\right\} \Rightarrow \sigma_{kk} = (3\lambda + 2G)\,\varepsilon_{kk} \quad \text{or} \quad \frac{\sigma_{kk}}{3} = -p = K\,\varepsilon_{kk}$$

$$\text{or} \quad \boxed{K = \frac{\sigma_{kk}}{3\,\varepsilon_{kk}} \xrightarrow[\varepsilon_{kk}\to 0]{} +\infty} \quad \left(\text{you could also say } K = \frac{E}{3(1-2\nu)} \xrightarrow[\nu\to 0.5]{} +\infty \right)$$

If $\varepsilon_{11} = 0.1$, $\varepsilon_{22} = -0.2$, $\varepsilon_{33} = -0.2$, then $\varepsilon_{kk} = -0.3$.
From a) and b) we know that $\sigma_{kk} = 1.82 \times 10^5$ Pa

$$\Big\}\; K = \frac{\sigma_{kk}}{3\varepsilon_{kk}} = -2 \times 10^5 \text{ Pa}$$

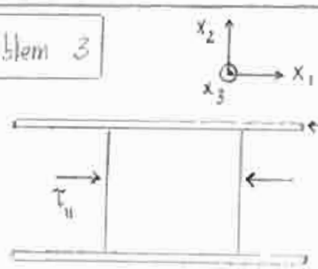
And with the assumption that E remains 10^5 Pa ▲, $\boxed{\nu = \frac{1}{2}\left[1 - \frac{E}{3K}\right] \simeq 0.6}$

This can't be biologically meaningful, since our sample wouldn't dilate when compressed

▲ note : if we solve Hooke's law $\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$ with the new values for ε_{ii}, we find $E < 0$!!
This imaginary result cannot represent our previous sample

Problem 3

a)



From the experiment description we know that $\varepsilon_{22} = 0$ and $\tau_{33} = 0$.
Writing Hooke's law for the normal strain on axes 1 and 2

$$(1) \quad \varepsilon_{11} = \frac{1+\nu}{E}\tau_{11} - \frac{\nu}{E}\left(\tau_{11} + \tau_{22} + 0\right)$$

$$(2) \quad 0 = \frac{1+\nu}{E}\tau_{22} - \frac{\nu}{E}\left(\tau_{11} + \tau_{22} + 0\right)$$

(2) yields $\tau_{22} = \nu\,\tau_{11}$, which, injected into (1) gives $\varepsilon_{11} = \tau_{11} \cdot \frac{1+\nu}{E}(1-\nu)$

$$\boxed{E_a = \frac{\tau_{11}}{\varepsilon_{11}} = \frac{E}{1-\nu^2}}$$

The apparent equilibrium modulus E_a in the x_1 direction is larger than the overall tissue Young's modulus E.

b) Charge repulsion within the tissue is responsible for about 50 % of its stiffness (the other 50% being due to the intrinsic stiffness of its constituents). Hence, the higher the net charge of the specimen, the stiffer the specimen.