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Foundations of Algorithms and Computational Techniques in Systems Biology
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Notes on Kronecker Products

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This note is a brief description of the matrix Kronecker product and matrix stack algebraic operators. For a detailed treatment the reader is referred to [1].

1 The Stack Operator

The stack operator maps an $n \times m$ matrix into an $nm \times 1$ vector. The stack of the $n \times m$ matrix A, denoted A^S, is the vector formed by stacking the columns of A into an $nm \times 1$ vector.

For example if

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}_{2\times 2} \tag{1}$$

then its stack form is

$$A^S = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{4\times 1} . \tag{2}$$

If C is an $n \times m$ matrix comprising m column vectors $\{c_1, c_2, \cdots c_m\}$, where each c_i is an $n \times 1$ vector

$$C = [c_1, c_2, \cdots, c_m]_{n\times m} \tag{3}$$

then

$$C^S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}_{nm\times 1} . \tag{4}$$

1.1 Properties of the Stack Operator

1. If $v \in \mathbb{R}^{n\times 1}$, a vector, then $v^S = v$.
2. If $A \in \mathbb{R}^{m\times n}$, a matrix, and $v \in \mathbb{R}^{n\times 1}$, a vector, then the matrix product $(Av)^S = Av$.

3. $trace(AB) = ((A^T)^S)^T B^S$.

2 The Kronecker Product

The Kronecker product is a binary matrix operator that maps two arbitrarily dimensioned matrices into a larger matrix with special block structure. Given the $n \times m$ matrix $A_{n\times m}$ and the $p \times q$ matrix $B_{p\times q}$

$$A = \begin{bmatrix} a_{1,1} & \dots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,m} \end{bmatrix}_{n\times m} \qquad B = \begin{bmatrix} b_{1,1} & \dots & b_{1,q} \\ \vdots & \ddots & \vdots \\ b_{p,1} & \dots & b_{p,q} \end{bmatrix}_{p\times q} \tag{5}$$

their Kronecker product, denoted $A \otimes B$, is the $np \times mq$ matrix with the block structure

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \dots & a_{1,m}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \dots & a_{n,m}B \end{bmatrix}_{np\times mq} . \tag{6}$$

For example, given

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{2\times 2} \qquad B_{2\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \tag{7}$$

the Kronecker product $A \otimes B$ is

$$A \otimes B = \begin{bmatrix} 1 & 2 & 3 & 2 & 4 & 6 \\ 4 & 5 & 6 & 8 & 10 & 12 \\ 0 & 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & -4 & -5 & -6 \end{bmatrix}_{4\times 6} . \tag{8}$$

2.1 Properties of the Kronecker Product Operator

In the following it is assumed that A, B, C, and D are real valued matrices. Some identities only hold for appropriately dimensioned matrices.

1. The Kronecker product is a bi-linear operator. Given $\alpha \in \mathbb{R}$,
$$\begin{aligned} A \otimes (\alpha B) &= \alpha(A \otimes B) \\ (\alpha A) \otimes B &= \alpha(A \otimes B). \end{aligned} \tag{9}$$
2. Kronecker product distributes over addition:
$$\begin{aligned} (A + B) \otimes C &= (A \otimes C) + (B \otimes C) \\ A \otimes (B + C) &= (A \otimes B) + (A \otimes C). \end{aligned} \tag{10}$$
3. The Kronecker product is associative:
$$(A \otimes B) \otimes C = A \otimes (B \otimes C). \tag{11}$$
4. The Kronecker product is *not* in general commutative, i.e. usually
$$(A \otimes B) \neq (B \otimes A). \tag{12}$$
5. Transpose distributes over the Kronecker product (does *not* invert order)
$$(A \otimes B)^T = A^T \otimes B^T. \tag{13}$$
6. Matrix multiplication, when dimensions are appropriate,
$$(A \otimes B)(C \otimes D) = (AC \otimes BD). \tag{14}$$
7. When A and B are square and full rank
$$(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1}). \tag{15}$$
8. The determinant of a Kronecker product is (note right hand side exponents)
$$det(A_{n \times n} \otimes B_{m \times m}) = det(A)^m \cdot det(B)^n. \tag{16}$$
9. The trace of a Kronecker product is
$$trace(A \otimes B) = trace(A) \cdot trace(B). \tag{17}$$
10. Stack of a matrix multiplication, when dimensions are appropriate for the product ABC to be well defined, is
$$(ABC)^S = (C^T \otimes A)B^S. \tag{18}$$

References

[1] Alexander Graham. *Kronecker Products and Matrix Calculus With Applications*. Halsted Press,John Wiley and Sons, NY, 1981.