Attitudes Towards Risk

14.123 Microeconomic Theory III
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Model

- $C = \mathbb{R}$ = wealth level
- Lottery = cdf F (pdf f)
- Utility function $u : \mathbb{R} \to \mathbb{R}$
- $U(F) \equiv E_F(u) \equiv \int u(x)dF(x)$
- $E_F(x) \equiv \int x dF(x)$

Attitudes Towards Risk

DM is

- risk averse if $E_F(u) \leq u(E_F(x))$ $(\forall F)$
- strictly risk averse if $E_F(u) < u(E_F(x))$ ($\forall$ "risky" F)
- risk neutral if $E_F(u) = u(E_F(x))$ $(\forall F)$
- risk seeking if $E_F(u) \geq u(E_F(x))$ $(\forall F)$

DM is

- risk averse if u is concave
- strictly risk averse if u is strictly concave
- risk neutral if u is linear
- risk seeking if u is convex

Certainty Equivalence

- $CE(F) = u^{-1}(U(F))=u^{-1}(E_F(u))$
- DM is
  - risk averse if $CE(F) \leq E_F(x)$ for all F;
  - risk neutral if $CE(F) = E_F(x)$ for all F;
  - risk seeking if $CE(F) \geq E_F(x)$ for all F.
- Take DM1 and DM2 with u_1 and u_2.
- DM1 is more risk averse than DM2
  - $\Leftrightarrow$ u_1 is more concave than u_2,
  - $\Leftrightarrow$ $u_1 = \varphi \circ u_2$ for some concave function φ,
  - $\Leftrightarrow$ $CE_1(F) \equiv u_1^{-1}(E_F(u_1)) \leq u_2^{-1}(E_F(u_2)) \equiv CE_2(F)$

Absolute Risk Aversion

- absolute risk aversion:

$$r_A(x) = -u''(x)/u'(x)$$

- constant absolute risk aversion (CARA)

$$u(x) = -e^{-\alpha x}$$

- If $x \sim N(\mu,\sigma^2)$, $CE(F) = \mu - \alpha\sigma^2/2$
- Fact: More risk aversion ⇔ higher absolute risk aversion everywhere
- Fact: Decreasing absolute risk aversion (DARA) ⇔ $\forall y>0$, u_2 with $u_2(x) \equiv u(x+y)$ is less risk averse

Relative risk aversion:

- relative risk aversion:

$$r_R(x) = -xu''(x)/u'(x)$$

- constant relative risk aversion (CRRA)

$$u(x) = -x^{1-\rho}/(1-\rho),$$

- When $\rho = 1$, $u(x) = \log(x)$.
- Fact: Decreasing relative risk aversion (DRRA) ⇔ $\forall t>1$, u_2 with $u_2(x) \equiv u(tx)$ is less risk averse

Application: Insurance

- wealth w and a loss of \$1 with probability p.
- Insurance: pays \$1 in case of loss costs q;
- DM buys λ units of insurance.
- Fact: If $p = q$ (fair premium), then $\lambda = 1$ (full insurance).
  - Expected wealth $w - p$ for all λ.
- Fact: If DM1 buys full insurance, a more risk averse DM2 also buys full insurance.
  - $CE_2(\lambda) \leq CE_1(\lambda) \leq CE_1(1) = CE_2(1)$.

Application: Optimal Portfolio Choice

- With initial wealth w, invest $\alpha \in [0,w]$ in a risky asset that pays a return z per each \$ invested; z has cdf F on $[0,\infty)$.
- $U(\alpha) = \int_0^\infty u(w+\alpha z-\alpha)\, dF(z)$; concave
- It is optimal to invest $\alpha > 0$ iff $E[z] > 1$.
  - $U'(0) = \int_0^\infty u'(w)(z-1)\, dF(z) = u'(w)(E[z]-1)$.
- If agent with utility u_1 optimally invests α_1, then an agent with more risk averse u_2 (same w) optimally invests $\alpha_2 \leq \alpha_1$.
- DARA $\Rightarrow$ optimal α increases in w.
- CARA $\Rightarrow$ optimal α is constant in w.
- CRRA (DRRA) $\Rightarrow$ optimal α/w is constant (increasing)

Optimal Portfolio Choice – Proof

- $u_2=g(u_1)$; g is concave; $g'(u_1(w)) = 1$.
- $U_i(\alpha) \equiv \int u_i(w+\alpha(z\text{-}1))(z\text{-}1)\, dF(z)$
- $U_2'(\alpha)\text{-}\ U_1'(\alpha) = \int[u_2(w+\alpha(z\text{-}1))\text{-}\ u_1(w+\alpha(z\text{-}1))](z\text{-}1)dF(z)$
  $\leq 0.$
  - $g'(u_1(w+\alpha_1 z\text{-}\alpha_1)) < g'(u_1(w)) = 1 \Leftrightarrow z > 1.$
  - $u_2(w+\alpha(z\text{-}1)) < u_1(w+\alpha(z\text{-}1)) \Leftrightarrow z > 1.$
- $\alpha_2 \leq \alpha_1$

Stochastic Dominance

- Goal: <u>Compare lotteries</u> with minimal assumptions on preferences
- Assume that the <u>support of all payoff distributions is bounded</u>. Support = $[a,b]$.
- Two main concepts:
  - First-order Stochastic Dominance: A payoff distribution is preferred by all monotonic Expected Utility preferences.
  - Second-order Stochastic Dominance: A payoff distribution is preferred by all risk averse EU preferences.

FSD

- DEF: F first-order stochastically dominates $G \Leftrightarrow$
  $F(x) \leq G(x)$ for all x.
- THM: F first-order stochastically dominates $G \Leftrightarrow$ for every weakly increasing $u: \mathbb{R} \to \mathbb{R}$, $\int u(x)dF(x) \geq \int u(x)dG(x)$.

Proof:

- "If:" for $F(x^*) > G(x^*)$, define $u = \mathbf{1}_{\{x>x^*\}}$.
- "Only if": Assume F and G are strictly increasing and continuous on $[a,b]$.
- Define $y(x) = F^{-1}(G(x))$; $y(x) \geq x$ for all x
- $\int u(y)dF(y) = \int u(y(x))dF(y(x)) = \int u(y(x))dG(x) \geq \int u(x)dG(x)$

MPR and MLR Stochastic Orders

- DEF: F dominates G in the Monotone Probability Ratio (MPR) sense if $k(x) \equiv G(x)/F(x)$ is weakly decreasing in x.
- THM: MPR dominance implies FSD.
- DEF: F dominates G in the Monotone Likelihood Ratio (MLR) sense if $\ell(x) \equiv G'(x)/F'(x)$ is weakly decreasing.
- THM: MLR dominance implies MPR dominance.

SSD

- Assume: F and G has the same mean
- DEF: F second-order stochastically dominates G ⇔ for every non-decreasing concave u, $\int u(x)dF(x) \geq \int u(x)dG(x)$.
- DEF: G is a mean-preserving spread of F ⇔ $y = x + \varepsilon$ for some $x \sim F$, $y \sim G$, and ε with $E[\varepsilon|x] = 0$.
- THM: The following are equivalent:
  - F second-order stochastically dominates G.
  - G is a mean-preserving spread of F .
  - $\forall t \geq 0$, $\int_0^t G(x)dx \geq \int_0^t F(x)dx$.

SSD

- Example: G (dotted) is a mean-preserving spread of F (solid).