Chapter 8

Rationalizability

The definition of a game $(N, S, u_1, \ldots, u_n)$ implicitly assumes that

1. the set of players is N, the set of available strategies to a player i is S_i, and the player i tries to maximize the expected value of $u_i : S \to R$ according to some belief, and that

2. each player knows 1, and that

3. each player knows 2, and that

   ...

n each player knows $n - 1$

   ...

   ad infinitum.

That is, it is implicitly assumed that it is *common knowledge* among the players that the game is $(N, S, u_1, \ldots, u_n)$ and that players are *rational* (i.e. they are expected utility maximizers). As a solution concept, Rationalizability yields the strategies that are consistent with these assumptions, capturing what is implied by the model (i.e. the game). Other solution concepts impose further assumptions, usually on players' beliefs, to obtain sharper predictions. In this lecture, I will formally introduce rationalizability and present some of its applications. The outline is as follows. I will first illustrate the idea on a simple example. I will then present the formal theory. I will finally apply rationalizability to Cournot and Bertrand competitions.

8.1   Example

Consider the following game.

1\2	L	R
T	2,0	-1,1
M	0,10	0,0
B	-1,-6	2,0

(8.1)

A player is said to be rational if he plays a best response to a belief about the other players' strategies. What does rationality imply for this game?
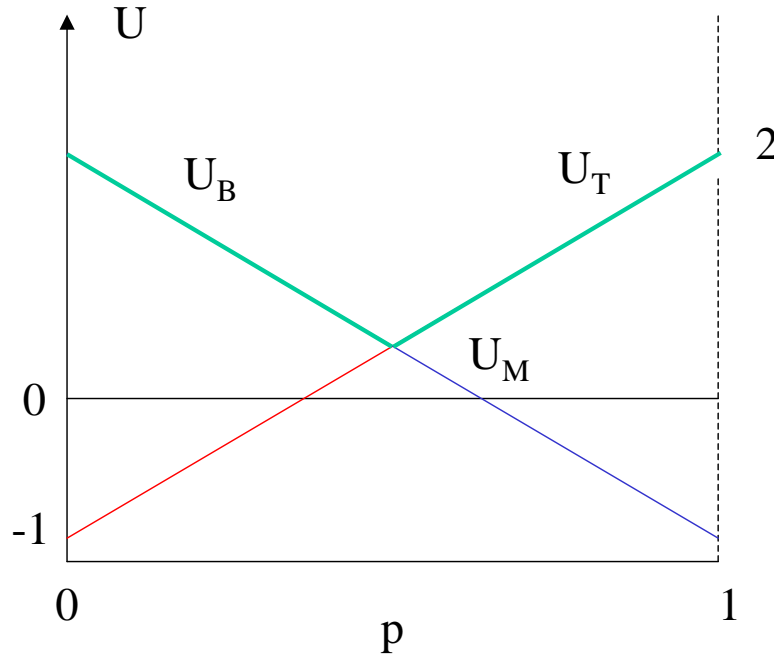
Consider Player 1. He is contemplating about whether to play T, or M, or B. A quick inspection of his payoffs reveals that his best play depends on what he thinks the other player does. Let's then write p for the probability he assigns to L (as Player 2's play), representing his belief about Player 2's strategy. His expected payoffs from playing T, M, and B are

$$
\begin{aligned}
U_T &= 2p - (1 - p) = 3p - 1, \\
U_M &= 0, \\
U_B &= -p + 2(1 - p) = 2 - 3p,
\end{aligned}
$$

respectively. These values as a function of p are plotted in the following graph:

As it is clear from the graph, U_T is the largest when $p > 1/2$, and U_B is the largest when $p < 1/2$. At $p = 1/2$, $U_T = U_B > 0$. Hence, if player 1 is rational, then he will play B when $p < 1/2$, D when $p > 1/2$, and B or D if $p = 1/2$.

Notice that, if Player 1 is rational, then he will never play M—no matter what he believes about the Player 2's play. Therefore, if we assume that Player 1 is rational (and that the game is as it is described above), then we can conclude that Player 1 will not play M. This is because M is a *strictly dominated strategy.* In particular, the mixed strategy that puts probability 1/2 on T and probability 1/2 on B yields a higher expected payoff than strategy M no matter what (pure) strategy Player 2 plays. A consequence of this is that M is never a weak best response to a belief p, a general fact that will be established momentarily.

We now want to understand the implications of the assumption that players know that the other players are also rational. Now, rationality of player 1 requires that he does not play M. For Player 2, her both actions can be a best reply. If she thinks that Player 1 is not likely to play M, then she must play R, and if she thinks that it is very likely that Player 1 will play M, then she must play L. Hence, rationality of player 2 does not put any restriction on her behavior. But, what if she thinks that it is very likely that player 1 is rational (and that his payoff are as in (8.1))? In that case, since a rational player 1 does not play M, she must assign very small probability for player 1 playing M. In fact, if she knows that player 1 is rational, then she must be sure that he will not play M. In that case, being rational, she must play R. In summary, *if player 2 is rational and she knows that player 1 is rational, then she must play R.*

Notice that we first eliminated all of the strategies that are strictly dominated (namely M), then taking the resulting game, we eliminated again all of the strategies that are strictly dominated (namely L). This is called *twice iterated elimination of strictly dominated strategies.* The resulting strategies are the strategies that are consistent with the assumption that players are rational and they know that the other players are rational.

As we impose further assumptions about rationality, we keep iteratively eliminating all strictly dominated strategies (if there remains any). Recall that rationality of player 1 requires him to play T or B, and knowledge of the fact that player 2 is also rational does not put any restriction on his behavior—as rationality itself does not restrict Player

2's behavior. Now, assume that Player 1 also knows (i) that Player 2 is rational and (ii) that Player 2 knows that Player 1 is rational (and that the game is as in (8.1)). Then, as the above analysis snows, Player 1 must know that Player 2 will play R. In that case, being rational he must play B. Therefore, common knowledge of rationality implies that Player 1 plays B and Player 2 plays R.

In the next section, I will apply these ideas more generally.

8.2   Theory

Fix a game $(N, S, u_1, \ldots, u_n)$. To be concrete, define the concepts of belief, best response, and rationality as follows.

Definition 26 *For any player i, a (correlated)* belief *of i about the other players' strategies is a probability distribution μ_{-i} on $S_{-i} = \prod_{j \neq i} S_j$.*

The essential part of this definition is that the belief μ_{-i} of player i allows correlation between the other players' strategies. For example, in a game of three players in which each player is to choose between Left and Right, Player 1 may believe that with probability 1/2 both of the other players will play Left and with probability 1/2 both players will play Right. Hence, viewed as mixed strategies, it may appear as though Players 2 and 3 use a common randomization device, contradicting the fact that Players 2 and 3 make their decisions independently. One may then find such a correlated belief unreasonable. This line of reasoning is based on mistakenly identifying a player's belief with other players' conscious randomization. For Player 1 to have such a correlated belief, he does not need to believe that the other players choose their decisions together. Indeed, he does not think that the other players are using randomization device. He thinks that each of the other players play a pure strategy that he does not know. He may assign correlated probabilities on the other players strategies because he may assign positive probability to various theories and each of these theories may lead to a prediction about how the players play. For example, he may think that players play Left (as in the cars in England) or players play Right (as in the cars in France) without knowing which of the theories is correct.

Depending on whether one allows correlated beliefs, there are two versions of Rationalizability. Because of the above reasoning, in this course, I will focus on correlated

version of Rationalizability. Note that the original definitions of Bernheim (1985) and Pearce (1985) impose independence, and these concepts are identical in two player games.

Definition 27 *The* expected payoff *from a strategy s_i against a belief μ_{-i} is*

$$u_i\left(s_i, \mu_{-i}\right) = \sum_{s_{-i} \in S_{-i}} u_i\left(s_i, s_{-i}\right) \mu_{-i}\left(s_{-i}\right).$$

Definition 28 *For any player i, a strategy s_i^* is a* best response *to a belief μ_{-i} if and only if*

$$u_i(s_i^*, \mu_{-i}) \geq u_i(s_i, \mu_{-i}), \forall s_i \in S_i.$$

Here I use the notion of a *weak best reply*, requiring that there is no other strategy that yields a strictly higher payoff against the belief. A notion of strict best reply would require that s^* yields a strictly higher expected payoff than any other strategy.

Definition 29 *For any player i, playing a strategy s_i is said to be* rational *if and only if s_i is a best response to some belief μ_{-i}.*

Playing a strategy is not rational if and only if it is never a weak best reply. This idea of rationality is closely related to the following notion of dominance.

Definition 30 *A strategy s_i^** strictly dominates *s_i if and only if*

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}.$$

Similarly, a mixed strategy σ_i strictly dominates s_i if and only if $u_i(\sigma_i, s_{-i}) \equiv \sum_{s_i' \in S_i} \sigma_i(s_i') u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}$.

That is, no matter what the other players play, playing s_i^* is strictly better than playing s_i for player i. In that case, if i is rational, he would never play the strictly dominated strategy s_i. That is, there is no belief under which he would play s_i, for s_i^* would always yield a higher expected payoff than s_i no matter what player i believes about the other players.[1]

Definition 31 *A strategy s_i is said to be* strictly dominated *if and only if there exists a pure or mixed strategy that strictly dominates s_i.*

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[1] As a simple exercise, prove this statement.

A Notice that neither of the pure strategies T, M, and B dominates any strategy. Nevertheless, M is dominated by the mixed strategy that σ_1 that puts probability $1/2$ on each of T and B. For each p, the payoff from σ_1 is

$$U_{\sigma_1} = \frac{1}{2}(3p - 1) + \frac{1}{2}(2 - 3p) = \frac{1}{2},$$

which is larger than 0, the payoff from M. Recall that in our example there is no belief (p) under which M is a best response. This is indeed a general result:

Theorem 9 *Playing a strategy s_i is not rational for i (i.e. s_i is never a weak best response to a belief μ_{-i}) if and only if s_i is strictly dominated.*

Proof. I will only show that if s_i is not strictly dominated it is a weak best response to some belief. (The converse is straightforward.) For each mixed strategy σ_i, consider the utility vector

$$\vec{u}_i(\sigma_i) = (u_i(\sigma_i, s_{-i}))_{s_{-i} \in S_{-i}},$$

and let U_i be the set of all such vectors. Clearly, U_i is convex. Take any s_i that is not strictly dominated, and define

$$V_i = \left\{ v \in \mathbb{R}^{S_{-i}} \mid v \gg \vec{u}_i(s_i) \right\}.$$

Clearly, V_i is also convex, and since s_i is not strictly dominated, $U_i \cap V_i = \varnothing$. Hence, by the separating-hyperplane theorem there exists $\mu_{-i} \in \mathbb{R}^{S_{-i}}$ such that $\mu_{-i} \cdot (\vec{u}_i(\sigma_i) - v_i) \le 0$ for all $\vec{u}_i(\sigma_i) \in U_i$ and $v \in V_i$. By definition of V_i, $\mu_{-i} \ge 0$. Since $\vec{u}_i(s_i)$ is on the boundary of V_i, it is also true that for all $\vec{u}_i(\sigma_i) \in U_i$, $\mu_{-i} \cdot (\vec{u}_i(\sigma_i) - \vec{u}_i(s_i)) \le 0$, showing that

$$u_i(\sigma_i, \mu_{-i}) = \mu_{-i} \cdot \vec{u}_i(\sigma_i) \le \mu_{-i} \cdot \vec{u}_i(s_i) = u_i(\sigma_i, \mu_{-i}).$$

∎

(In this proof, one can allow S_{-i} to be infinite.)

Theorem 9 states that *if we assume that players are rational (and that the game is as described), then we conclude that no player plays a strategy that is strictly dominated (by some mixed or pure strategy), and this is all we can conclude.*

Let us write

$$S_i^1 = \{s_i \in S_i \mid s_i \text{ is not strictly dominated}\}.$$

By Theorem 9, S_i^1 is the set of all strategies that are best response to some belief.

Let us now explore the implications of the assumption that player i is rational and knows that the other players are rational. To this end, we consider the strategies s_i that are best response to a belief μ_{-i} of i on S_{-i} such that for each $s_{-i} = (s_j)_{j \neq i}$ with $\mu_{-i}(s_{-i}) > 0$ and for each j, there exists a belief μ_j of j on S_{-j} such that s_j is a best response to μ_j. Here, the first part (i.e. s_i is a best response to a belief μ_{-i}) corresponds to rationality of i and the second part (i.e. if $\mu_{-i}(s_{-i}) > 0$, then s_j is a best response to a belief μ_j) corresponds to the assumption that i knows that j is rational. By Theorem 9, each such s_j is not strictly dominated, i.e., $s_j \in S_j^1$. Hence, by another application of Theorem 9, s_i is not *strictly dominated given* S_{-i}^1, i.e., there does not exist a (possibly mixed) strategy σ_i such that

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \qquad \forall s_{-i} \in S_{-i}^1.$$

Of course, by Theorem 9, the converse of the last statement is also true. Therefore, the set of strategies that are rationally played by player i knowing that the other players is also rational is

$$S_i^2 = \left\{ s_i \in S_i \mid s_i \text{ is not strictly dominated given } S_{-i}^1 \right\}.$$

By iterating this logic, one obtains the following iterative elimination procedure, called *iterative elimination of strictly-dominated strategies.*

Definition 32 (Iterative Elimination of Strictly-Dominated Strategies) *Set $S^0 = S$, and for any $m > 0$ and set*

$$S_i^m = \left\{ s_i \in S_i \mid s_i \text{ is not strictly dominated given } S_{-i}^{m-1} \right\},$$

i.e., $s_i \in S_i^m$ iff there does not exist any σ_i such that

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \qquad \forall s_{-i} \in S_{-i}^{m-1}.$$

Caution: Two points are crucial:

1. We eliminate only the *strictly* dominated strategies. We do not eliminate a strategy if it is weakly dominated but not strictly dominated. For example, we do not

eliminate any strategy in

	L	R
T	1,1	0,0
B	0,0	0,0

although (T,L) is a dominant strategy equilibrium.

2. We do eliminate the strategies that are strictly dominated by mixed strategies (but not necessarily by pure strategies). For example, in the game in (8.1), we do eliminate M although neither T nor B dominates M.

Notice that when there are only finitely many strategies, this elimination process must stop at some m, i.e., there will be no dominated strategy to eliminate after a round.

Note that, for any m, a strategy s_i is in S_i^m if and only if it is rationally played by i in a situation in which (1) i is rational, (2) i knows that every player is rational, i knows that everybody knows that every body is rational, and ... (m) i know that every body knows that ... everybody knows that everybody is rational. That is, s_i is a best response to a belief μ_{-i}^1 such that every s_j^1 in the support of μ_{-i}^1 is a best response to some belief μ_{-j}^2 such that every every s_k^2 in the support of μ_{-j}^2 is a best response to some belief μ_{-k}^3 ... up to order m. It is in that sense S^m is the set of strategy profiles that are consistent with mth-order mutual knowledge of rationality.

Exercise 12 *Using Theorem 9, prove the claim in the previous paragraph.*

Rationalizability corresponds to the limit of the iterative elimination of strictly-dominated strategies.

Definition 33 (Rationalizability) *For any player i, a strategy is said to be* rationalizable *if and only if $s_i \in S_i^\infty$ where*

$$S_i^\infty = \bigcap_{m \geq 0} S_i^m.$$

Rationalizability corresponds to the set of strategies that are rationally played in situations in which it is common knowledge that everybody is rational, as defined at the beginning of the lecture. When a strategy s_i is rationalizable it can be justified/rationalized

by an indefinite chain of beliefs μ_{-i} as above. On the other hand, if a strategy is not rationalizable, it must have been eliminated at some stage m, and such a strategy cannot be rationalized by a chain of beliefs longer than m.

We call the elimination process that keeps iteratively eliminating all strictly dominated strategies until there is no strictly dominated strategy *Iterated Elimination of Strictly Dominated Strategies*; we eliminate indefinitely if the process does not stop. We call a strategy *rationalizable* if and only if it survives iterated elimination of strictly dominated strategies.

Nash Equilibrium v. Rationalizability   Every Nash equilibrium is rationalizable, which can be shown as a straightforward exercise. Converse is not true. For example, in The Battle of The Sexes, (Opera, Ballet) is not a Nash equilibrium, but both Opera and Ballet are rationalizable strategies. Of course, these strategies correspond to some Nash equilibria, but one can easily construct a game in which some rationalizable strategies do not correspond to any Nash equilibrium.

Exercise 13 *Show that if $\sigma_i^* (s_i) > 0$ for some Nash equilibrium σ^*, then $s_i \in S_i^\infty$. Find a game $(N, S, u_1, \ldots, u_n)$ and a strategy $s_i \in S_i^\infty$ such that no Nash equilibrium assigns positive probability on s_i.*

Example: (Beauty Contest)   Consider an n-player game in which each player i has strategies $x_i \in [0, 100]$, and payoff

$$u_i (x_1, \ldots, x_n) = - \left(x_i - \frac{2}{3} \frac{x_1 + \cdots x_n}{n} \right)^2 .$$

Notice that, in this game, each player tries to play a strategy that is equal to two thirds of the average strategy, which is also affected by his own strategy. Each person is therefore interested guessing the other players' average strategies, which depends on the other players' estimate of the average strategy. Let's apply our procedure to this game.

First, since each strategy must be less than or equal to 100, the average cannot exceed 100, and hence any strategy $x_i > 200/3$ is strictly dominated by $200/3$. Indeed,

any strategy $x_i > x^1$ is strictly dominated by x^1 where[2]

$$x^1 = \frac{2(n-1)}{3n-2}100.$$

To show that $x_i > x^1$ is strictly dominated by x^1, we fix any $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n)$ and show that

$$u_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots x_n) < u_i(x_1, \ldots, x_{i-1}, x^1, x_{i+1}, \ldots x_n). \qquad (8.2)$$

By taking the derivative of u_i with respect to x_i, we obtain

$$\frac{\partial u_i}{\partial x_i} = -2\left(1 - \frac{2}{3n}\right)\left(x_i - \frac{2}{3}\frac{x_1 + \cdots x_n}{n}\right).$$

Clearly, $\partial u_i / \partial x_i < 0$ if

$$\left(x_i - \frac{2}{3}\frac{x_1 + \cdots x_n}{n}\right) > 0,$$

which would be the case if

$$x_i > \frac{2}{3n-2}\sum_{j \neq i} x_j. \qquad (8.3)$$

But since each $x_j \leq 100$, the sum $\sum_{j \neq i} x_j$ is less than or equal to $(n-1)100$. Hence, it suffices that

$$x_i > \frac{2}{3n-2}(n-1)100 = x^1.$$

Therefore, in the region $x_i > x^1$, u_i is a strictly decreasing function of x_i and (8.2) is satisfied. This shows that all the strategies $x_i > x^1$ are eliminated in the first round.

On the other hand, each $x_i \leq x^1$ is a best response to some $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n)$ with

$$x_i = \frac{2}{3n-2}\sum_{j \neq i} x_j.$$

Therefore, at the end of the first round the set of surviving strategies is $[0, x^1]$.

Now, suppose that at the end of round m, the set of surviving strategies is $[0, x^m]$ for some number x^m. By repeating the same analysis above with x^m instead of 100, we can conclude that at the end of round $m + 1$, the set of surviving strategies is $[0, x^{m+1}]$ where

$$x^{m+1} = \frac{2(n-1)}{3n-2}x^m.$$

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[2]Here x^1 is just a real number, where superscript 1 indicates that we are in Round 1.

The solution to this equation with $x^0 = 100$ is

$$x^m = \left[\frac{2(n-1)}{3n-2}\right]^m 100.$$

Therefore, for each m, at the end of round m, a strategy x_i survives if and only if

$$0 \leq x_i \leq \left[\frac{2(n-1)}{3n-2}\right]^m 100.$$

Since

$$\lim_{m \to \infty} \left[\frac{2(n-1)}{3n-2}\right]^m 100 = 0,$$

the only rationalizable strategy is $x_i = 0$.

Notice that the speed at which x^m goes to zero determines how fast we eliminate the strategies. If the elimination is slow (e.g. when $2(n-1)/(3n-2)$ is large), then many strategies are eliminated at very high iterations. In that case, predictions based on rationalizability will heavily rely on strong assumptions about rationality, i.e., everybody knows that everybody knows that ... everybody is rational. For example, if n is small or the ratio $2/3$ is replaced by a small number, the elimination is fast and the predictions of rationalizability are more reliable. If the n is large or the ratio $2/3$ is replaced by a number close to 1, the elimination is slow and the predictions of rationalizability are less reliable. In particular, the predictions of rationalizability for this game is more robust in a small group than a larger group.

A general problem with rationalizability is that there are usually too many rationalizable strategies; the elimination process usually stops too early. In that case a researcher cannot make much prediction based on such analysis. For example, in the Matching Pennies game

1\2	Head	Tail
Head	-1,1	1,-1
Tail	1,-1	-1,1

every strategy is rationalizable, and we cannot say what the players will do.

8.3 Applications

8.3.1 Cournot Competition

Consider n firms. Each firm i produces $q_i \geq 0$ units of a good at marginal cost $c \geq 0$ and sell it at price

$$P = \max\{1 - Q, 0\} \tag{8.4}$$

where

$$Q = q_1 + \cdots q_n \tag{8.5}$$

is the total supply. Each firm maximizes the expected profit. Hence, the payoff of firm i is

$$\pi_i = q_i (P - c). \tag{8.6}$$

Assuming all of the above is commonly known, we can write this as a game in normal form, by setting

- $N = \{1, 2, \ldots, n\}$ as the set of players

- $S_i = [0, \infty)$ as the strategy space of player i, where a typical strategy is the quantity q_i produced by firm i, and

- $\pi_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ as the payoff function.

Best Response In our analysis, and in the rest of the course, it will be useful to know the best response of a firm i to the production levels of the other firms. Let us write

$$Q_{-i} = \sum_{j \neq i} q_j \tag{8.7}$$

for the total supply of the firms other than firm i. If $Q_{-i} > 1$, then the price $P = 0$ and the best firm i can do is to produce zero and obtain zero profit. Now assume $Q_{-i} \leq 1$. For any $q_i \in (0, 1 - Q_{-i})$, the profit of the firm i is

$$\pi_i (q_i, Q_{-i}) = q_i (1 - q_i - Q_{-i} - c). \tag{8.8}$$

(The profit is negative if $q_i > 0$.) By setting the derivative of π_i with respect to q_i to zero,[3] we obtain the best production level

$$q_i^B(Q_{-i}) = \frac{1 - Q_{-i} - c}{2}. \tag{8.9}$$

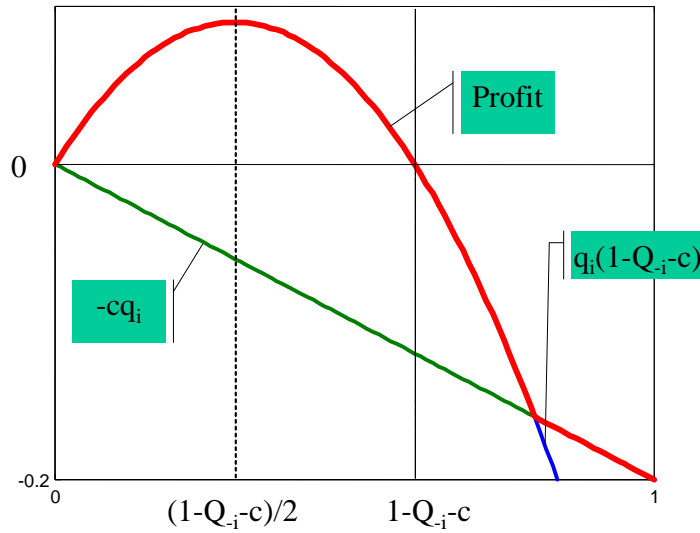The profit function is plotted in Figure 8.1. The best response function is plotted in Figure 8.2.

Figure 8.1:

Cournot Duopoly

Now, consider the case of two firms. In that case, for $i \neq j$, we have $Q_{-i} = q_j$.

Nash Equilibrium   In order to have a Nash equilibrium, we must have

$$q_1 = q_1^B(q_2) \equiv \frac{1 - q_2 - c}{2}$$

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[3]I.e.

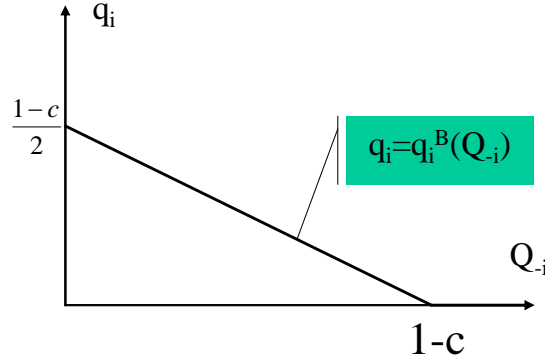$$\frac{\partial \pi_i}{\partial q_i} = 1 - 2q_i - Q_{-i} - c = 0.$$

Figure 8.2:

and

$$q_2 = q_2^B(q_1) \equiv \frac{1 - q_1 - c}{2}.$$

Solving these two equations simultaneously, we obtain

$$q_1^* = q_2^* = \frac{1 - c}{3}$$

as the only Nash equilibrium. Graphically, as in Figure 8.3, we plot the best response functions of each firm and identify the intersections of the graphs of these functions as Nash equilibria. In this case, there is a unique intersection, and therefore there is a unique Nash equilibrium.

Rationalizability   The (linear) Cournot duopoly game we consider here is "dominance solvable" i.e. there is a unique rationalizable strategy. Let us first consider the first couple rounds of elimination to see this intuitively; we will then show mathematically that this is indeed the case.

Round 1   Notice that a strategy $\hat{q}_i > (1 - c)/2$ is strictly dominated by $(1 - c)/2$. To see this, consider any q_j. As in Figure 8.1, $\pi_i(q_i, q_j)$ is strictly increasing until $q_i = (1 - c - q_j)/2$ and strictly decreasing thereafter. In particular,

$$\pi_i((1 - c - q_j)/2, q_j) \geq \pi_i((1 - c)/2, q_j) > \pi_i(\hat{q}_i, q_j),$$

showing that $\hat{q}_i$ is strictly dominated by $(1 - c)/2$. We therefore eliminate all $\hat{q}_i > (1 - c)/2$ for each player i. The resulting strategies are as follows, where the shaded
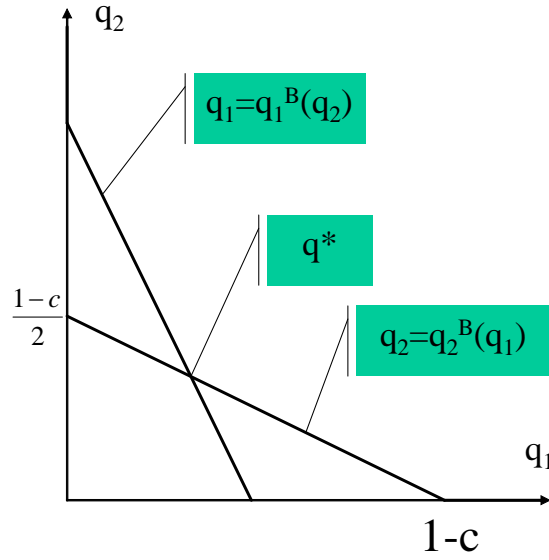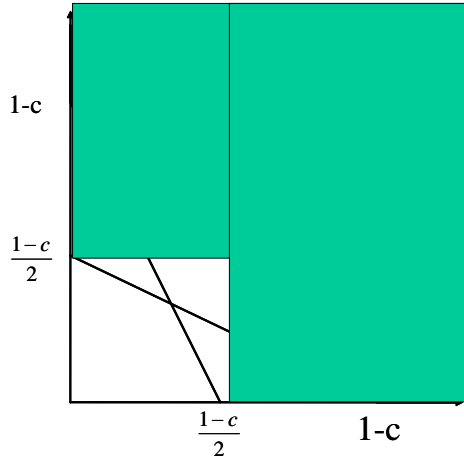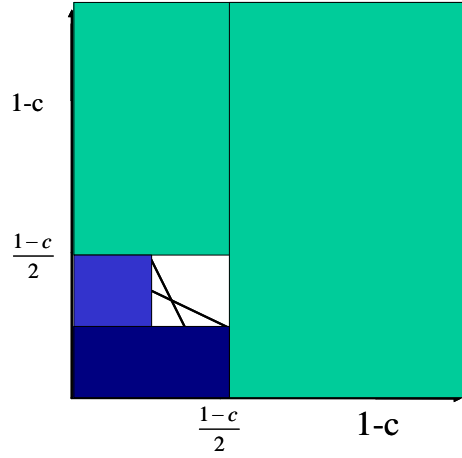
Figure 8.3:

area is eliminated:

Round 2 In the remaining game $q_j \leq (1-c)/2$. Consequently, any strategy $\hat{q}_i < (1-c)/4$ is strictly dominated by $(1-c)/4$. To see this, take any $q_j \leq (1-c)/2$ and recall from Figure 8.1 that π_i is strictly increasing until $q_i = (1-c-q_j)/2$, which is greater than or equal to $(1-c)/4$. Hence,

$$\pi_i(\hat{q}_i, q_j) < \pi_i((1-c)/4, q_j) \leq \pi_i((1-c-q_j)/2, q_j),$$

showing that $\hat{q}_i$ is strictly dominated by $(1-c)/4$. We will therefore eliminate all $\hat{q}_i$ with $\hat{q}_i < (1-c)/4$. The remaining strategies are as follows:

Notice that the remaining game is a smaller replica of the original game. Applying the same procedure repeatedly we eliminate all strategies except for the Nash equilibrium. (After every two rounds, we obtain a smaller replica.) Therefore, the only rationalizable strategy is the unique Nash equilibrium strategy:

$$q_i^* = (1-c)/3.$$

A more formal treatment   We can prove this more formally by invoking the following lemma repeatedly:

Lemma 2 *Given that $q_j \leq \bar{q}$, every strategy $\hat{q}_i$ with $\hat{q}_i < q_i^B(\bar{q})$ is strictly dominated by $q_i^B(\bar{q}) \equiv (1-\bar{q}-c)/2$. Given that $q_j \geq \bar{q}$, every strategy $\hat{q}_i$ with $\hat{q}_i > q_i^B(\bar{q})$ is strictly dominated by $q_i^B(\bar{q}) \equiv (1-\bar{q}-c)/2$.*

Proof.   Let's first prove the first statement. Take any $q_j \leq \bar{q}$. Note that $\pi_i(q_i; q_j)$ is strictly increasing in q_i at any $q_i < q_i^B(q_j)$. Since $\hat{q}_i < q_i^B(\bar{q}) \leq q_i^B(q_j)$,[4] this implies that

$$\pi_i(\hat{q}_i, q_j) < \pi_i(q_i^B(\bar{q}), q_j).$$

That is, $\hat{q}_i$ is strictly dominated by $q_i^B(\bar{q})$.

To prove the second statement, take any $q_j \leq \bar{q}$. Note that $\pi_i(q_i; q_j)$ is strictly decreasing in q_i at any $q_i > q_i^B(q_j)$. Since $q_i^B(q_j) \leq q_i^B(\bar{q}) < \hat{q}_i$, this implies that

$$\pi_i(\hat{q}_i, q_j) < \pi_i(q_i^B(\bar{q}), q_j).$$

---
[4]This is because q_i^B is decreasing.

That is, $\hat{q}_i$ is strictly dominated by $q_i^B(\bar{q})$. ∎

Now, define a sequence $q^0, q^1, q^2, \ldots$ by $q^0 = 0$ and

$$q^m = q_i^B(q^{m-1}) \equiv (1 - q^{m-1} - c)/2 = (1 - c)/2 - q^{m-1}/2$$

for all $m > 0$. That is,

$$
\begin{aligned}
q^0 &= 0 \\
q^1 &= \frac{1-c}{2} \\
q^2 &= \frac{1-c}{2} - \frac{1-c}{4} \\
q^3 &= \frac{1-c}{2} - \frac{1-c}{4} + \frac{1-c}{8} \\
&\quad \cdots \\
q^m &= \frac{1-c}{2} - \frac{1-c}{4} + \frac{1-c}{8} - \cdots - (-1)^m \frac{1-c}{2^m} \\
&\quad \cdots
\end{aligned}
$$

Theorem 10 *The set of remaining strategies after any odd round m $(m = 1, 3, \ldots)$ is $[q^{m-1}, q^m]$. The set of remaining strategies after any even round m $(m = 2, 4, \ldots)$ is $[q^m, q^{m-1}]$. The set of rationalizable strategies is $\{(1-c)/3\}$.*

Proof. We use mathematical induction on m. For $m = 1$, we have already proven the statement. Assume that the statement is true for some odd m. Then, for any q_j available at even round $m+1$, we have $q^{m-1} \leq q_j \leq q^m$. Hence, by Lemma 2, any $\hat{q}_i < q_i^B(q^m) = q^{m+1}$ is strictly dominated by q^{m+1} and eliminated. That is, if q_i survives round $m+1$, then $q^{m+1} \leq q_i \leq q^m$. On the other hand, every $q_i \in [q^{m+1}, q^m] = [q_i^B(q^m), q_i^B(q^{m-1})]$ is a best response to some q_j with $q^{m-1} \leq q_j \leq q^m$, and it is not eliminated. Therefore, the set of strategies that survive the even round $m+1$ is $[q^{m+1}, q^m]$.

Now, assume that the statement is true for some even m. Then, for any q_j available at odd round $m+1$, we have $q^m \leq q_j \leq q^{m-1}$. Hence, by Lemma 2, any $\hat{q}_i > q_i^B(q^m) = q^{m+1}$ is strictly dominated by q^{m+1} and eliminated. Moreover, every $q_i \in [q^m, q^{m+1}] = [q_i^B(q^{m-1}), q_i^B(q^m)]$ is a best response to some q_j with $q^m \leq q_j \leq q^{m-1}$, and it is not eliminated. Therefore, the set of strategies that survive the odd round $m+1$ is $[q^m, q^{m+1}]$.

Finally, notice that

$$\lim_{m \to \infty} q^m = (1-c)/3.$$

Therefore the intersections of the above intervals is $\{(1-c)/3\}$, which is the set of rationalizable strategies. ∎

Cournot Oligopoly

We will now consider the case of three or more firms. When there are three or more firms, rationalizability does not help, i.e., we cannot eliminate any strategy less than the monopoly production $q^1 = (1-c)/2$.

In the first round we eliminate any strategy $q_i > (1-c)/2$, using the same argument in the case of duopoly. But in the second round, the maximum possible total supply by the other firms is

$$(n-1)(1-c)/2 \geq 1-c,$$

where n is the number of firms. The best response to this aggregate supply level is 0. Hence, we cannot eliminate any strategy in round 2. The elimination process stops, yielding $[0,(1-c)/2]$ as the set of rationalizable strategies.

Of course, Cournot oligopoly has a unique Nash equilibrium as in the Cournot duopoly. While the Nash equilibrium remains to make strong predictions as we introduce new firms, the predictions of rationalizability become rather weak. (In equilibrium analysis the weak predictions of rationalizability reappears as instability of equilibrium, making equilibrium behavior highly sensitive to the specification of beliefs.)

8.3.2   Bertrand Competition

Consider two firms. Simultaneously, each firm i sets a price p_i. The firm i with the lower price $p_i < p_j$ sells $1-p_i$ units and the other firm cannot sell any. If the firms set the same price, the demand is divided between them equally. That is, the amount of sales for firm i is

$$Q_i(p_1,p_2) = \begin{cases} 1-p_i & \text{if } p_i < p_j \\ \frac{1-p_i}{2} & \text{if } p_i = p_j \\ 0 & \text{otherwise.} \end{cases}$$

We assume that it costs noting to produce the good (i.e. $c = 0$). Therefore, the profit of a firm i is

$$\pi_i(p_1, p_2) = p_i Q_i(p_1, p_2) = \begin{cases} (1 - p_i)\, p_i & \text{if } p_i < p_j \\ \frac{(1 - p_i) p_i}{2} & \text{if } p_i = p_j \\ 0 & \text{otherwise.} \end{cases}$$

Assuming all of the above is commonly known, we can write this formally as a game in normal form by setting

- $N = \{1, 2\}$ as the set of players

- $S_i = [0, \infty)$ as the set of strategies for each i, with price p_i a typical strategy,

- π_i as the utility function.

Observe that when $p_j = 0$, $\pi_i(p_1, p_2) = 0$ for every p_i, and hence every p_i is a best response to $p_j = 0$. This has two important implications:

1. Every strategy is rationalizable (we cannot eliminate any strategy because each of them is a best reply to zero).

2. $p_1^* = p_2^* = 0$ is a Nash equilibrium.

In the rest of the notes, we will first show that this is indeed the only Nash equilibrium. In other words, even with two firms, when the firms compete by setting prices, the competitive equilibrium will emerge. We will then show that if we modify the game slightly by discretizing the set of allowable prices and putting a minimum price, then the game becomes dominance-solvable, i.e., only one strategy remains rationalizable. In the modified game, the minimum price is the only rationalizable strategy, as in competitive equilibrium. Finally we will introduce small search costs on the part of consumers, who are not modeled as players, we will see the equilibrium behavior is dramatically different from the equilibrium behavior in the original game and competitive equilibrium.

Nash Equilibrium

Theorem 11 *The only Nash equilibrium is $p^* = (0, 0)$.*

Proof. We have seen that $p^* = (0,0)$ is a Nash equilibrium. We will show that if (p_1, p_2) is a Nash equilibrium, then $p_1 = p_2 = 0$. To do this, take any Nash equilibrium (p_1, p_2). We first show that $p_1 = p_2$. Towards a contradiction, suppose that $p_i > p_j$. If $p_j = 0$, then $\pi_j (p_i, p_j) = 0$, while $\pi_j (p_i, p_i) = (1 - p_i) p_i / 2 > 0$. That is, choosing p_i is a profitable deviation for firm j, showing that $p_i > p_j = 0$ is not a Nash equilibrium. Therefore, in order $p_i > p_j$ to be an equilibrium, we must have $p_j > 0$. But then, firm i has a profitable deviation: $\pi_i (p_i, p_j) = 0$ while $\pi_i (p_j, p_{ij}) = (1 - p_j) p_j / 2 > 0$. All in all, this shows that we cannot have $p_i > p_j$ in equilibrium. Therefore, we must have $p_1 = p_2$. But if $p_1 = p_2$ in a Nash equilibrium, then it must be that $p_1 = p_2 = 0$. This is because if $p_1 = p_2 > 0$, then firm 1 would have a profitable deviation: $\pi_1 (p_1, p_2) = (1 - p_1) p_1 / 2$ while $\pi_1 (p_1 - \varepsilon, p_2) = (1 - p_1 + \varepsilon) (p_1 - \varepsilon)$, which is close to $(1 - p_1) p_1$ when ε is close to zero. ∎

Rationalizability with discrete prices

Now suppose that the firms have to set prices as multiples of pennies, and they cannot charge zero price. That is, the set of allowable prices is

$$P = \{0.01, 0.02, 0.03, \dots\}.$$

The important assumption here is that the minimum allowable price $p^{\min} = 0.01$ yields a positive profit. We will now see that the game is "dominance-solvable" under this assumption. In particular $p^{\min}$ is the only rationalizable strategy, and it is the only Nash equilibrium strategy. Let us start with the first step.

Step 1: *any price p greater than the monopoly price $p^{mon} = 0.5$ is strictly dominated by some strategy that assigns some probability $\epsilon > 0$ to the price $p^{\min} = 0.01$ and probability $1 - \epsilon$ to the price $p^{mon} = 0.5$.*

Proof. Take any player i and any price $p_i > p^{mon}$. We want to show that the mixed strategy σ^ϵ with $\sigma^\epsilon (p^{mon}) = 1 - \epsilon$ and $\sigma^\epsilon (p^{\min}) = \epsilon$ strictly dominates p_i for some $\epsilon > 0$.

Take any strategy $p_j > p^{mon}$ of the other player j. We have

$$\pi_i (p_i, p_j) \le p_i Q (p_i) = p_i (1 - p_i) \le 0.51 \cdot 0.49 = 0.2499,$$

where the first inequality is by definition and the last inequality is due to the fact that

$p_i \geq 0.51$. On the other hand,

$$
\begin{aligned}
\pi_i \left(\sigma^\epsilon, p_j \right) &= (1 - \epsilon) p^{mon} \left(1 - p^{mon} \right) + \epsilon p^{\min} \left(1 - p^{\min} \right) \\
&> (1 - \epsilon) p^{mon} \left(1 - p^{mon} \right) \\
&= 0.25 \left(1 - \epsilon \right).
\end{aligned}
$$

Thus, $\pi_i \left(\sigma^\epsilon, p_j \right) > 0.2499 \geq \pi_i \left(p_i, p_j \right)$ whenever $0 < \epsilon \leq 0.0004$. Choose $\epsilon = 0.0004$.

Now, pick any $p_j \leq p^{mon}$. Since $p_i > p^{mon}$, we now have $\pi_i \left(p_i, p_j \right) = 0$. But

$$
\pi_i \left(\sigma^\epsilon, p_j \right) = (1 - \epsilon) p^{mon} \left(1 - p^{mon} \right) + \epsilon p^{\min} \left(1 - p^{\min} \right) \geq \epsilon p^{\min} \left(1 - p^{\min} \right) > 0.
$$

That is, $\pi_i \left(\sigma^\epsilon, p_j \right) > \pi_i \left(p_i, p_j \right)$. Therefore, σ^ϵ strictly dominates p_i. ∎

Step 1 yields the eliminations in the first round 1.

Round 1   By Step 1, all strategies p_i with $p_i > p^{mon} = 0.5$ are eliminated. Moreover, each $p_i \leq p^{mon}$ is a best reply to $p_j = p_i + 1$, and is not eliminated. Therefore, the set of remaining strategies is

$$
P^2 = \{0.01, 0.02, \ldots, 0.5\}.
$$

Round m   Suppose that the set of remaining strategies to round m is

$$
P^m = \{0.01, 0.02, \ldots, \bar{p}\}.
$$

Then, the strategy $\bar{p}$ is strictly dominated by a mixed strictly dominated by the mixed strategy σ^ϵ with $\sigma^\epsilon \left(\bar{p} - 0.01 \right) = 1 - \epsilon$ and $\sigma^\epsilon \left(p^{\min} \right) = \epsilon$, as we will see momentarily. We then eliminate the strategy $\bar{p}$. There will be no more elimination because each $p_i < \bar{p}$ is a best reply to $p_j = p_i + 0.01$.

To prove that $\bar{p}$ is strictly dominated by σ^ϵ, note that the profit from $\bar{p}$ for player i is

$$
\pi_i \left(\bar{p}, p_j \right) = \begin{cases} \bar{p} \left(1 - \bar{p} \right) / 2 & \text{if } p_j = \bar{p}, \\ 0 & \text{otherwise.} \end{cases}
$$

On the other hand,

$$
\begin{aligned}
\pi_i \left(\sigma_{\bar{p}}^\epsilon, \bar{p} \right) &= (1 - \epsilon) \left(\bar{p} - 0.01 \right) \left(1 - \bar{p} + 0.01 \right) + \epsilon p^{\min} \left(1 - p^{\min} \right) \\
&> (1 - \epsilon) \left(\bar{p} - 0.01 \right) \left(1 - \bar{p} + 0.01 \right) \\
&= (1 - \epsilon) \left[\bar{p} \left(1 - \bar{p} \right) - 0.01 \left(1 - 2\bar{p} \right) \right].
\end{aligned}
$$

Then, $\pi_i\left(\sigma^\epsilon, \bar{p}\right) > \pi_i\left(\bar{p}, p_j\right)$ whenever

$$\epsilon \leq 1 - \frac{\bar{p}\left(1-\bar{p}\right)/2}{\bar{p}\left(1-\bar{p}\right) - 0.01\left(1-2\bar{p}\right)}.$$

But $\bar{p} \geq 0.02$, hence $0.01\left(1-2\bar{p}\right) < \bar{p}\left(1-\bar{p}\right)/2$, thus the right hand side is greater than 0. Choose

$$\epsilon = 1 - \frac{\bar{p}\left(1-\bar{p}\right)/2}{\bar{p}\left(1-\bar{p}\right) - 0.01\left(1-2\bar{p}\right)} > 0$$

so that $\pi_i\left(\sigma_{\bar{p}}^\epsilon, \bar{p}\right) > \pi_i\left(\bar{p}, p_j\right)$. Moreover, for any $p_j < \bar{p}$,

$$
\begin{aligned}
\pi_i\left(\sigma_{\bar{p}}^\epsilon, p_j\right) &= (1-\epsilon)\left(\bar{p}-0.01\right)\left(1-\bar{p}+0.01\right) + \epsilon p^{\min}\left(1-p^{\min}\right) \\
&\geq \epsilon p^{\min}\left(1-p^{\min}\right) > 0 = \pi_i\left(\bar{p}, p_j\right),
\end{aligned}
$$

showing that $\sigma_{\bar{p}}^\epsilon$ strictly dominates $\bar{p}$, and completing the proof.

Therefore, the process continues until the set of remaining strategies is $\left\{p^{\min}\right\}$ and it stops there. Therefore, $p^{\min}$ is the only rationalizable strategy.

Since players can put positive probability only on rationalizable strategies in a Nash equilibrium, the only possible Nash equilibrium is $\left(p^{\min}, p^{\min}\right)$, which is clearly a Nash equilibrium.

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