Decision Making Under Uncertainty

14.123 Microeconomic Theory III
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Risk v. uncertainty

1. Risk = DM has to choose from alternatives
   - whose consequences are unknown
   - But the probability of each consequence is given
2. Uncertainty = DM has to choose from alternatives
   - whose consequences are unknown
   - the probability of consequences is not given
   - DM has to form his own beliefs
3. Von Neumann-Morgenstern: Risk
4. Goal:
   1. Convert uncertainty to risk by formalizing and eliciting beliefs
   2. Apply Von Neumann Morgenstern analysis

Decision Making Under Risk – Summary

- C = Finite set of consequences
- $X = P$ = lotteries (prob. distributions on C)
- Expected Utility Representation:

$$p \succeq q \Leftrightarrow \sum_{c \in C} u(c)\,p(c) \geq \sum_{c \in C} u(c)q(c)$$

- Theorem: EU Representation $\Leftrightarrow$ continuous preference relation with Independence Axiom:

$$ap+(1-a)r \succcurlyeq aq+(1-a)r \quad \Leftrightarrow \quad p \succcurlyeq q.$$

Road map

1. Acts, States, Consequences
2. Expected Utility Maximization – Representation
3. Sure-Thing Principle
4. Conditional Preferences
5. Eliciting Qualitative Beliefs
6. Representing Qualitative Beliefs with Probability
7. Expected Utility Maximization – Characterization
8. Anscombe & Aumann trick: use indifference between uncertain and risky events

Model

- C = Finite set of consequences
- S = A set of states (uncountable)
- Act: A mapping $f : S \to C$
- $X = F := C^S$
- DM cares about consequences, chooses an act, without knowing the state
- Example: Should I take my umbrella?
- Example: A game from a player's point of view

Expected-Utility Representation

- $\succcurlyeq$ = a relation on F
- Expected-Utility Representation:
  - A probability distribution p on S with expectation E
  - A VNM utility function $u : C \to \mathbb{R}$ such that
  - $f \succcurlyeq g \Leftrightarrow U(f) \equiv E[u \circ f] \geq E[u \circ g] \equiv U(g)$
- Necessary Conditions:

P1: $\succcurlyeq$ is a preference relation

Sure-Thing Principle

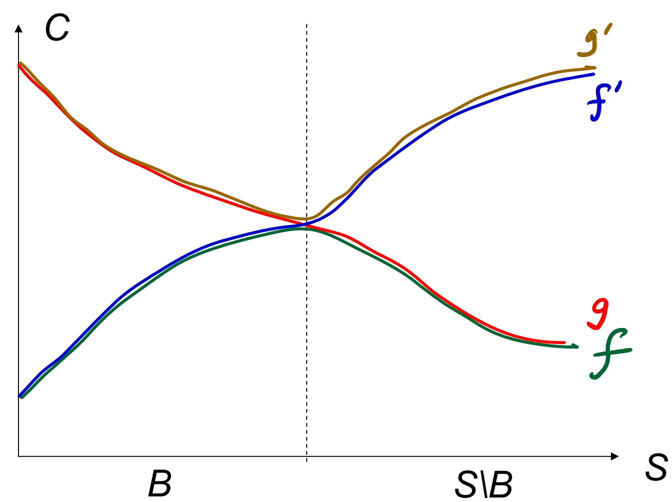
- If
  - $f \succcurlyeq g$ when DM knows $B \subseteq S$ occurs,
  - $f \succcurlyeq g$ when DM knows $S \backslash B$ occurs,
- Then $f \succcurlyeq g$
- when DM doesn't know whether B occurs or not.

P2: Let f, f', g, g' and B be such that

- $f(s) = f'(s)$ and $g(s) = g'(s)$ at each $s \in B$
- $f(s) = g(s)$ and $f'(s) = g'(s)$ at each $s \in S \backslash B$.

Then, $f \succcurlyeq g \Leftrightarrow f' \succcurlyeq g'$.

Sure-Thing Principle – Picture



STP

Conditional Preference

- For any acts f and h and event B,

$$f_{|B}^{h}(s)=\begin{cases} f(s) & \text{if } s\in B \\ h(s) & \text{otherwise} \end{cases}$$

- Definition: $f \succcurlyeq g$ given B iff $f_{|B}{}^{h} \succcurlyeq g_{|B}{}^{h}$.
- Sure-Thing Principle = conditional preference is well-defined
- Informal Sure-Thing Principle, formally:
  - $f \succcurlyeq g$ given B: $f_{|B}{}^{f} \succcurlyeq g_{|B}{}^{f}$.
  - $f \succcurlyeq g$ given $S\backslash B$: $f_{|S\backslash B}{}^{g} \succcurlyeq g_{|S\backslash B}{}^{g}$.
  - Transitivity: $f = f_{|B}{}^{f} \succcurlyeq g_{|B}{}^{f} = f_{|S\backslash B}{}^{g} \succcurlyeq g_{|S\backslash B}{}^{g} = g$.
- B is null $\Leftrightarrow$ $f \sim g$ given B for all $f,g \in$ F.

P3: For any $x,x' \in C$, $f,f' \in F$ with $f \equiv x$ and $f' \equiv x'$, and any non-null B,

$$f \succcurlyeq f' \text{ given } B \Leftrightarrow x \succcurlyeq x'.$$

STPP

Eliciting Beliefs

- For any $A \subseteq S$ and x, $x' \in C$, define $f_A{}^{x,x'}$ by

$$f_{A}^{x,x'}(s)=\begin{cases} x & \text{if } s\in A \\ x' & \text{otherwise} \end{cases}$$

Definition: For any $A,B \subseteq S$,

$$A \succcurlyeq B \Leftrightarrow f_A{}^{x,x'} \succcurlyeq f_B{}^{x,x'}$$

for some $x,x' \in C$ with $x \succ x'$.

- $A \succcurlyeq B$ means A is at least as likely as B.

P4: There exist x, $x' \in C$ such that $x \succ x'$.

P5: For all $A,B \subseteq S$, $x,x',y,y' \in C$ with $x \succ x'$ and $y \succ y'$,

$$f_A{}^{x,x'} \succcurlyeq f_B{}^{x,x'} \Leftrightarrow f_A{}^{y,y'} \succcurlyeq f_B{}^{y,y'}.$$

Qualitative Probability

Definition: A relation $\succcurlyeq$ between the events is said to be a qualitative probability iff

1. $\succcurlyeq$ is complete and transitive;
2. for any $B,C,D \subseteq S$ with $B \cap D = C \cap D = \varnothing$,
   $B \succcurlyeq C \Leftrightarrow B \cup D \succcurlyeq C \cup D$;
3. $B \succcurlyeq \varnothing$ for each $B \subseteq S$, and $S \succ \varnothing$.

Fact: "At least as likely as" relation above is a qualitative probability relation.

Quantifying qualitative probability

- For any probability measure p and relation $\succcurlyeq$ on events, p is a probability representation of $\succcurlyeq$ iff
  $$B \succcurlyeq C \Leftrightarrow p(B) \geq p(C) \quad \forall B,C \subseteq S.$$
- If $\succcurlyeq$ has a probability representation, then $\succcurlyeq$ is a qualitative probability.
- S is infinitely divisible under $\succcurlyeq$ iff $\forall n$, S has a partition $\{D_1^1, \ldots, D_n^n\}$ such that $D_1^1 \sim \ldots \sim D_n^n$.

P6: For any $x \in C$, $g,h \in F$ with $g \succ h$, S has a partition $\{D^1,\ldots,D^n\}$ s.t.
$$g \succ h_i^x \text{ and } g_i^x \succ h$$
for all i≤n where $h_i^x(s) = x$ if $s \in D^i$ and $h(s)$ otherwise.

- P6 implies that S is infinitely divisible under $\succcurlyeq$.

Probability Representation

Theorem: Under P1-P6, $\succcurlyeq$ has a unique probability representation p.

Proof:

- For any event B and n, define
  $$k(n,B) = \max \{r \mid B \succcurlyeq D_1^{1} \cup \ldots \cup D_n^{r}\}$$
- Define $p(B) = \lim_n k(n,B)/2^n$.
- $B \succcurlyeq C \Rightarrow k(n,B) \geq k(n,C)\ \forall n \Rightarrow p(B) \geq p(C)$.
- P6': If $B \succ C$, S has a partition $\{D^1,\ldots,D^n\}$ s.t. $B \succ C \cup D^i$ for each $i \leq n$.
- $B \succ C \Rightarrow p(B) > p(C)$.
- Uniqueness: $k(n,B)/2^n \leq p'(B) < (k(n,B)+1)/2^n$

Expected Utility Maximization – Characterization

Theorem: Assume that C is finite. Under P1-P6, there exist a utility function $u : C \to R$ and a probability measure p on S such that $\forall f,g \in F$,

$$f \succeq g \iff \sum_{c \in C} p(\{s \mid f(s) = c\})\, u(c) \geq \sum_{c \in C} p(\{s \mid g(s) = c\})\, u(c)$$