Chapter 1

Theory of Choice

In these notes, I will summarize the basic ideas in choice theory, which you must be familiar with from 14.121. I will describe three ways of modeling individual choice, namely choice function, preference, and utility maximization. I will present the conditions under which one can use each model. One can always use choice functions in modeling a decision maker's choice at a given situation. In order to represent a choice function by a complete and transitive preference relation, one must have a non-empty choice function that satisfies the weak axiom of revealed preference. Finally, a complete and transferable preference relation can be represented by a utility function, as long as it continuous.

1.1 Alternatives

Consider a set X of alternatives. Alternatives are mutually exclusive in the sense that one cannot choose two distinct alternatives at the same time. Take also the set of feasible alternatives exhaustive so that a decision maker's choices will always be defined.[1]

[1] Note that this is a matter of modeling. For instance, if we have options Coffee and Tea, we define alternatives as C = Coffee but no Tea, T = Tea but no Coffee, CT = Coffee and Tea, and NT = no Coffee and no Tea.

1.2 Choice

While X consists of all possible alternatives, some of these alternatives may not be feasible for the decision maker. He is constrained to choose from a set $A \subset X$. A choice function describes what a decision maker would have chosen under various hypothetical constraints.

Definition 1 *A* choice function *is a mapping* $c : 2^X \setminus \{\varnothing\} \to 2^X \setminus \{\varnothing\}$ *such that* $c(A) \subseteq A$ *for all* $A \subseteq X$.

Here, $c(A)$ is meant to be the set of all alternatives that the decision maker *may* choose from A. His actual choice will be a single alternative within $c(A)$. Note that $c(A)$ is non-empty by definition. In canonical models, it is also assumed that the choice function satisfies the following assumption.

Axiom 1 (Weak Axiom of Revealed Preferences) *For any* $A, B \subseteq X$ *and any* $x, y \in A \cap B$, *if* $x \in c(A)$ *and* $y \in c(B)$, *then* $x \in c(B)$.

The Weak Axiom of Revealed Preferences states that if x is chosen in the presence of y (so that it is revealed that x is at least as good as y), then whenever y is chosen in the presence of x, x could have been chosen, too. This axiom embodies two assumptions. First, the choice is a result of binary comparison. Second, the underlying preference used in the comparison is not affected by the set A of available alternatives. (For example, the decision maker does not learn from the available choices.)

1.3 Preference

A *relation* (on X) is a subset of $X \times X$. A relation $\succeq$ is said to be *complete* if and only if, given any $x, y \in X$, either $x \succeq y$ or $y \succeq x$. A relation $\succeq$ is said to be *transitive* if and only if, given any $x, y, z \in X$,

$$[x \succeq y \text{ and } y \succeq z] \Rightarrow x \succeq z.$$

Definition 2 *A relation is a* preference relation *if and only if it is complete and transitive.*

Given any preference relation $\succeq$, the *strict preference* $\succ$ is defined by

$$x \succ y \iff [x \succeq y \text{ and } y \not\succeq x],$$

and the *indifference* $\sim$ is defined by

$$x \sim y \iff [x \succeq y \text{ and } y \succeq x].$$

Here, $x \succeq y$ means that the decision maker finds x at least as good as y; $x \succ y$ means that the decision maker finds x strictly better than y, and $x \sim y$ means that the decision maker is indifferent between x and y.

Now, consider a decision maker who chooses a best alternative according to a preference relation $\succeq$ within each set $A \subseteq X$ of available alternatives. His choice function $c_{\succeq}$ is given by

$$c_{\succeq}(A) = \{x \in A | x \succeq y \quad \forall y \in A\} \qquad \left(\forall A \in 2^X \setminus \{\varnothing\}\right).$$

An important question is which choice functions can be thought of coming from such a decision maker. This is formulated in the following definition.

Definition 3 *A choice function c is* represented by $\succeq$ *iff $c = c_{\succeq}$.*

Representation by a preference relation $\succeq$ means that decision maker's choices are made as if he tries to choose a best available element according to $\succeq$. It turns out that the weak axiom of revealed preferences is equivalent to such a representation.

Theorem 1 *Assume that X is finite. A choice function c is represented by some preference relation $\succeq$ if and only if c satisfies weak axiom of revealed preferences.*

It is a useful exercise to show that if c is represented by some preference relation $\succeq$, then it satisfies Axiom 1. For the converse, define $\succeq_c$ by $x \succeq_c y \iff x \in c(\{x, y\})$. Under Axiom 1, it is another useful exercise to show that $c = c_{\succeq_c}$.

1.4 Utility

A relation $\succeq$ can be *represented* by a utility function $U : X \to \mathbb{R}$ in the following sense:

$$x \succeq y \iff U(x) \geq U(y) \qquad \forall x, y \in X. \tag{OR}$$

The following theorem states that a relation needs to be a preference relation in order to be represented by a utility function.

Theorem 2 *Assume that X is finite (or countable). A relation can be presented by a utility function in the sense of (OR) if and only if it is complete and transitive. Moreover, if $U : X \to \mathbb{R}$ represents $\succeq$, and if $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then $f \circ U$ also represents $\succeq$.*

By the last statement, we call such utility representations *ordinal*. To prove this result for finite X, define $U(x) = \#\{y \in X | x \succeq y\}$ and check that U represents $\succeq$ when $\succeq$ is complete and transitive. We are mainly interested in decision under uncertainty. In that case, the natural set of alternatives (e.g. the set of all possible lotteries) is infinite. When X is infinite, one also needs to impose a continuity assumption.

Definition 4 *A preference relation $\succeq$ is said to be* continuous *if and only if the upper- and lower-contour sets $\{y|y \succeq x\}$ and $\{y|x \succeq y\}$ are closed for every $x \in X$.*

Continuity can also be defined as: for any two sequences (x_n) and (y_n) with $x_n \to x$ and $y_n \to y$,

$$[x_n \succeq y_n \quad \forall n] \Longrightarrow x \succeq y.$$

That is, the weak preference is always preserved in the limit. The main result in this lecture is that continuous preference relations can be represented by (continuous) utility functions:

Theorem 3 *Assume that X is a compact, convex subset of a separable metric space. A preference relation $\succeq$ can be represented by a continuous utility function $U : X \to \mathbb{R}$ in the sense of (OR) if and only if $\succeq$ is continuous.*

This result is a generalization of well-known results by Wold (1943), Debreu (1954), and Arrow-Hahn (1971). One can easily (i.e. you should) check that if $\succeq$ is represented by a continuous utility function $U : X \to \mathbb{R}$ in the sense of (OR), then $\succeq$ is continuous. You must have seen the proof of the converse for the special case considered by Debreu.

As an exercise, show that lexicographic preference relation cannot be represented by any utility function (if you don't remember from 14.121). Find also a discontinuous preference relation that is represented by a discontinuous utility function. Hence, continuity of preferences is not superfluous for ordinal representation, but it is not necessary, either.[2]

Two properties of continuous preferences will be useful in the sequel. For any $x \in X$, define the indifference set by

$$I(x) = \{y|x \sim y\}.$$

The first property is that $I(x)$ is a closed set (because $I(x) = \{y|y \succeq x\} \cap \{y|x \succeq y\}$). The second property is that $I(x)$ intersects any continuous path that connects a superior alternative to an inferior one:

Lemma 1 *Take any $x', x'' \in X$ with $x' \succ x \succ x''$ and any continuous mapping $\phi : [0,1] \to X$ with $\phi(1) = x'$ and $\phi(0) = x''$. Then, there exists $t \in [0,1]$ such that $\phi(t) \in I(x)$.*

This immediately follows from Theorem 3 and the intermediate value theorem. (By Theorem 3, $U \circ \phi$ is continuous, and $U(\phi(1)) = U(x') > U(x) > U(x'') = U(\phi(0))$.) Normally, this fact is proved as a main step towards proving Theorem 3, as you may remember from the earlier classes.

[2] Some form of countability/continuity is necessary for representability. X must be separable with respect to the order topology of $\succeq$, i.e., it must contain a countable subset that is dense with respect to the order topology. (See Theorem 3.5 in Kreps, 1988.)

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Spring 2010