Decision Making Under Risk

14.123 Microeconomic Theory III
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Road map

1. Choice Theory – summary
2. Basic Concepts:
   1. Consequences
   2. Lotteries
3. Expected Utility Maximization
   1. Representation
   2. Characterization
4. Indifference Sets under Expected Utility Maximization

Choice Theory – Summary

1. X = set of alternatives
2. Ordinal Representation: $U : X \rightarrow \mathbb{R}$ is an ordinal representation of $\succcurlyeq$ iff:
$$x \succcurlyeq y \Leftrightarrow U(x) \geq U(y) \quad \forall x,y \in X.$$
3. If $\succcurlyeq$ has an ordinal representation, then $\succcurlyeq$ is complete and transitive.
4. Assume X is a compact, convex subset of a separable metric space. A preference relation has a continuous ordinal representation if and only if it is continuous.
5. Let $\succcurlyeq$ be continuous and $x' \succ x \succ x''$. For any continuous $\varphi:[0,1] \rightarrow X$ with $\varphi(1)=x'$ and $\varphi(0)=x''$, there exists t such that $\varphi(t) \sim x$.

Model

- DM = Decision Maker
- DM cares only about consequences
  - C = Finite set of consequences
- Risk = DM has to choose from alternatives
  - whose consequences are unknown
  - But the probability of each consequence is known
- Lottery: a probability distribution on C
- P = set of all lotteries p,q,r
- $X = P$
- Compounding lotteries are reduced to simple lotteries!

Expected Utility Maximization
Von Neumann-Morgenstern representation

A lottery (in P)

Expected value of u under p

$$p \succeq q \Leftrightarrow \underbrace{\sum_{c \in C} u(c)\, p(c)}_{U(p)} \geq \underbrace{\sum_{c \in C} u(c) q(c)}_{U(q)}$$

$$U(p) \quad \geq \quad U(q)$$

- $U : P \to \mathbb{R}$ is an ordinal representation of $\succcurlyeq$.
- $U(p)$ is the expected value of u under p.
- U is linear and hence continuous.

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Expected Utility Maximization
Characterization (VNM Axioms)

Axiom A1: $\succcurlyeq$ is complete and transitive.

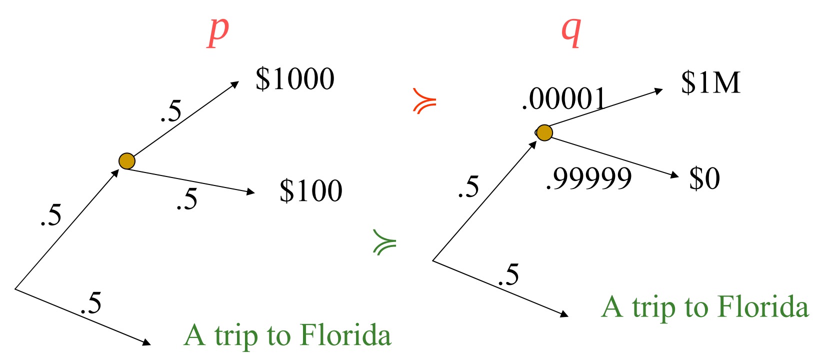
Axiom A2 (Continuity): $\succcurlyeq$ is continuous.

VNM

Independence Axiom

Axiom A3: For any $p,q,r \in P$, $a \in (0,1]$,

$$ap+(1-a)r \succcurlyeq aq+(1-a)r \quad \Leftrightarrow \quad p \succcurlyeq q.$$



Expected Utility Maximization Characterization Theorem

- $\succcurlyeq$ has a von Neumann – Morgenstern representation iff $\succcurlyeq$ satisfies Axioms A1-A3;
- i.e. $\succcurlyeq$ is a continuous preference relation with Independence Axiom.
- u and v represent $\succcurlyeq$ iff $v = au + b$ for some $a > 0$ and any b.

Exercise

- Consider a relation $\succcurlyeq$ among positive real numbers represented by VNM utility function u with $u(x) = x^2$.

  Can this relation be represented by VNM utility function $u^*(x) = x^{1/2}$?

  What about $u^{**}(x) = 1/x$?

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Implications of Independence Axiom (Exercise)

- For any p,q,r,r' with $r \sim r'$ and any a in (0,1],
  $$ap+(1-a)r \succcurlyeq aq+(1-a)r' \Leftrightarrow p \succcurlyeq q.$$
- Betweenness: For any p,q,r and any a,
  $$p \sim q \Rightarrow ap+(1-a)r \sim aq+(1-a)r.$$
- Monotonicity: If $p \succ q$ and $a > b$, then
  $$ap + (1-a)q \succ bp + (1-b)q.$$
- Extreme Consequences: $\exists c^B, c^W \in C$: $\forall p \in P$,
  $$c^B \succcurlyeq p \succcurlyeq c^W.$$

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Proof of Characterization Theorem

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- $c^B \sim c^W$ trivial. Assume $c^B \succ c^W$.
- Define $\varphi : [0,1] \to P$ by $\varphi(t)=tc^W+(1-t)c^B$.
- Monotonicity: $\varphi(t) \succcurlyeq \varphi(t') \Leftrightarrow t \geq t'$.
- Continuity: $\forall p \in P$, $\exists$ unique $U(p) \in [0,1]$ s.t.

$$p \sim \varphi(U(p)).$$

- Check Ordinal Representation:

$$p \succcurlyeq q \Leftrightarrow \varphi(U(p)) \succcurlyeq \varphi(U(q)) \Leftrightarrow U(p) \geq U(q)$$

- U is linear: $U(ap+(1-a)q)=aU(p)+(1-a)U(q)$
- because $ap+(1-a)q \sim a\varphi(U(p))+(1-a)\varphi(U(q))$

$$= \varphi(aU(p)+(1-a)U(q)),$$

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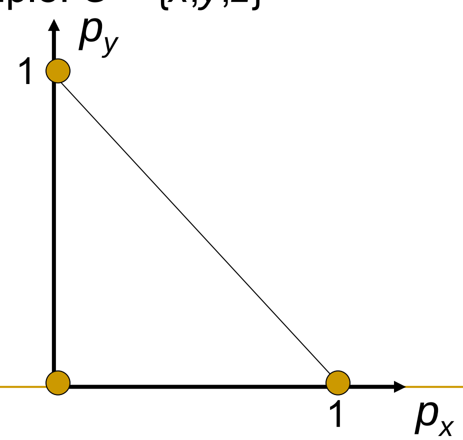
Indifference Sets under Independence Axiom

1. Indifference sets are straight lines
2. … and parallel to each other.

Example: $C = \{x,y,z\}$



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