14.123 Microeconomic Theory III
Final Make Up Exam

March 16, 2010
(80 Minutes)

1. (30 points) This question assesses your understanding of expected utility theory.

   (a) Show that there exists a preference relation on preferences that satisfies the independence axiom but is discontinuous. (For an example, you can take the set of consequences as $\{x, y, z\}$ and consider lexicographic preferences.)

   Answer: Denote the lotteries by $(p_x, p_y, 1 - p_x - p_y)$. Consider the lexicographic relation

   $$p \succeq q \iff [(p_x > q_x) \text{ or } [p_x = q_x \text{ and } p_y > q_y]].$$

   This is a discontinuous preference relation because the upper counter set for p is $\{q|q_x > p_x\} \cup \{q|q_x = p_x, q_y \geq p_y\}$, which is clearly not a closed set when p is in the interior. To check that it satisfies the independence axiom, take any $p, q, r \in P$ and $a \in (0, 1]$. If $p \sim q$, then $p = q$ (as the indifference sets are singletons), and hence $ap + (1-a)r = aq + (1-a)r$, showing that $ap + (1-a)r \sim aq + (1-a)r$. If $p \succ q$, then either $p_x > q_x$, in which case $ap_x + (1-a)r_x > aq_x + (1-a)r_x$, showing that $ap + (1-a)r \succ aq + (1-a)r$, **or** $p_x = q_x$ and $p_y > q_y$, in which case $ap_x + (1-a)r_x = aq_x + (1-a)r_x$ and $ap_y + (1-a)r_y > aq_y + (1-a)r_y$, showing once again that $ap + (1-a)r \succ aq + (1-a)r$. This shows that if $p \succeq q \Longrightarrow ap + (1-a)r \succeq aq + (1-a)r$. Conversely, if $p \not\succeq q$, then $q \succ p$ (by completeness) and as we just shows this imples that $aq + (1-a)r \succ ap + (1-a)r$, showing that $ap + (1-a)r \not\succeq aq + (1-a)r$.

   (b) Under Postulates P1-5 of Savage, consider the as likely as relation $\dot{\sim}$ between events, derived from betting preferences as in the class. Consider events A and B such that $A \dot{\sim} S \backslash A$ and $B \dot{\sim} S \backslash B$, where S is the state space. Show that $A \dot{\sim} B$.

   Answer: Recall from the class that under P1-5, $\dot{\succeq}$ is a qualitative probability. In particular, if C, D, E are disjoint events,

   $$C \dot{\succ} D \iff C \cup E \dot{\succ} D \cup E. \tag{1}$$

   Now, for simplicity of notation, let $W = A \cap B$, $X = B \backslash C$, $Y = A \backslash B$, and $Z = S \backslash (A \cup B)$. Suppose for a contradiction that $A \dot{\succ} B$. Then, by (1), $Y \dot{\succ} X$. Moreover, since $A \dot{\succ} B$ and $B \dot{\sim} S \backslash B$, we also have $A \dot{\succ} S \backslash B$, showing by (1) that $W \dot{\succ} Z$. Now,

   $$A = Y \cup W \dot{\succ} X \cup W \dot{\succ} X \cup Z = S \backslash A,$$

   a contradiction. [The strict preferences are obtained by (1).]

2. (40 points) There are two dates $t \in \{0, 1\}$ and two players $i \in N = \{1, 2\}$. Each player i has an asset that pays X_i at date $t = 1$ where X_1 and X_2 are independently and identically distributed with $N(\mu, \sigma^2)$. The players consume only at date $t = 1$, and the von-Neumann and Morgenstern utility function of player i is $u_i(x_i) = -e^{-\alpha x_i}$ where x_i his consumption and $\alpha > 0$. (Each player cares only about his final consumption.) A

feasible allocation is a pair (x_1, x_2) of random variables with $x_1 + x_2 \leq X_1 + X_2$ (state by state). Consider the following perfect-information game. At $t = 0$, Player 1 offers an allocation $(x_{1,0}, x_{2,0})$, and Player 2 decides whether to accept the offer or reject it. If the offer is accepted, the game ends and players consume $(x_{1,0}, x_{2,0})$ at $t = 1$. If the offer is rejected, then the dividends X_1 and X_2 become publicly observable, and we proceed to $t = 1$. At $t = 1$, Player 1 offers an allocation $(x_{1,1}, x_{2,1})$. As in the previous round, if Player 2 accepts the offer, they consume the offered allocation; each consumes his own asset otherwise.

(a) Compute a sequential equilibrium of this game. (The equilibrium allocation is unique. The only uncertainty the players face is the values of dividends, for which the beliefs are already given in the question.) How does the equilibrium payoffs changes as we vary the risk-aversion parameters α_1 and α_2?

Answer: Note that if players do not agree at $t = 0$, in any sequential equilibrium, they must consume their own asset at $t = 1$. Player 2 accepts an offer only if $x_{2,1} \geq X_2$, and by feasibility, this implies that $x_{1,1} \leq X_1 + X_2 - x_{2,1} \leq X_1$ for an acceptable offer. It is part of a sequential equilibrium at $t = 1$ that Player 2 accepts an offer $(x_{1,1}, x_{2,1})$ iff $x_{2,1} \geq X_2$, and Player 1 offers $(x_{1,1}, x_{2,1}) = (X_1, X_2)$. Hence, in terms of certainty equivalence, the continuation value of Player 2 at the end of $t = 0$ is

$$CE_{2,0} = \mu - \frac{1}{2}\alpha_2\sigma^2.$$

Therefore, at $t = 0$, Player 2 accepts an offer $(x_{1,1}, x_{2,1})$ iff $CE_2[x_{2,0}] \geq CE_{2,0}$. Player 1 therefore offers $(x_{1,0}, x_{2,0})$ such that

$$x_{1,0} = \arg \max_{\substack{x_1 + x_2 \leq X_1 + X_2 \\ CE_2[x_{2,0}] \geq CE_{2,0}}} CE_1[x_1].$$

Since this is a decision theory course you are expected to solve this optimization problem. Since $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$, recall from lecture notes that under CARA utilities we have transferable utilities in terms of certainty equivalence and any optimal allocation is of the form

$$\begin{aligned} x_1 &= \frac{\alpha_2}{\alpha_1 + \alpha_2}(X_1 + X_2) + \tau \\ x_2 &= \frac{\alpha_1}{\alpha_1 + \alpha_2}(X_1 + X_2) - \tau \end{aligned}$$

for some constant τ. Hence, Player 1 offers such an optimal allocation with $CE_2[x_2] = CE_{2,0}$. That is,

$$CE_2[x_2] = \frac{\alpha_1}{\alpha_1 + \alpha_2} 2\mu - \frac{1}{2}\left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right)^2 \alpha_2 2\sigma^2 - \tau = \mu - \frac{1}{2}\alpha_2\sigma^2$$

yielding

$$\tau_0 = \left(\frac{2\alpha_1}{\alpha_1 + \alpha_2} - 1\right)\mu - \frac{1}{2}\left(2\left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right)^2 - 1\right)\alpha_2\sigma^2.$$

The equilibrium offer is

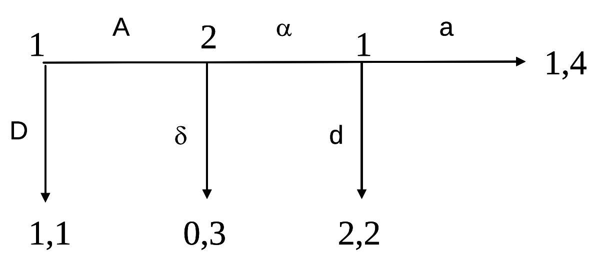
$$x_{1,0} = \frac{\alpha_2}{\alpha_1+\alpha_2}(X_1+X_2)+\tau_0$$
$$x_{2,0} = \frac{\alpha_1}{\alpha_1+\alpha_2}(X_1+X_2)-\tau_0.$$

Since Player 2 is indifferent between accepting or rejecting the offer, his payoff is $CE_{2,0}$, which decreases with α_2 and is invariant to α_1. Player 1's payoff is decreasing with α_1 but may decrease or increase with α_2, depending on who bears the most of the risk.

(b) Suppose that Player 1 observes the values X_1 and X_2 of the dividends at the beginning of the game, before making his offer. Compute the set of all sequential equilibria in pure strategies. (Use Bayes' formula for densities. Show your result.)

Answer: The only sequential equilibrium strategy profile is Player 1 offers $(x_{1,0},x_{2,0}) = (X_1,X_2)$, and Player 2 accepts an offer $(x_{1,0},x_{2,0})$ iff $x_{2,0}(\omega) \geq X_2(\omega)$ at every state ω. The beliefs are as follows. If the offer is $(x_{1,0},x_{2,0}) = (X_1,X_2)$, Player 2 keeps his prior, which is clearly consistent with the strategy of player 1. If $(x_{1,0},x_{2,0}) \neq (X_1,X_2)$, then he conditions on the event $\{\omega | x_{2,0}(\omega) \leq X_2(\omega)\}$, which is consistent with the strategy of Player 1 with the perturbation that the types trembles only to the beneficial trades. His strategy is clearly a best response to these beliefs. To see that there is no other sequential equilibrium strategy profile, suppose Player 2 accepts an offer $(x_{1,0},x_{2,0})$ such that $x_{2,0} < X_2$ for some realization (X_1,X_2). There then exists such an allocation $\left(x^*_{1,0},x^*_{2,0}\right)$ offered by some type (X_1,X_2) with $x^*_{1,0} > X_1$ and $x^*_{2,0} < X_2$ for that realization. Otherwise all such types would give up beneficial trade and consume their own asset. The sequential rationality implies that if type (X_1,X_2) offers $\left(x^*_{1,0},x^*_{2,0}\right)$, then $x^*_{1,0} \geq X_1$ and thus $x^*_{2,0} \leq X_1+X_2-x^*_{1,0} \leq X_2$ at the realized value. Hence, since the offer is on the path, consistency implies that Player 2 assigns zero probability on $\left\{\omega | x^*_{2,0}(\omega) > X_2(\omega)\right\}$ and positive probability on $\left\{\omega | x^*_{2,0}(\omega) < X_2(\omega)\right\}$. He must then reject the offer, a contradiction.

3. (30 points) Consider the reduced normal form of the following game, in which the equivalent strategies Dd and Da are represented by a single strategy D.



(a) Compute the set of rationalizable strategies. (Show your result.)

Answer: Clearly Aa is strictly dominated by a mixture of D and Ad. The remaining game in reduced form is

	δ	α
D	1,1	1,1
Ad	0,3	2,2

Clearly no other strategy is eliminated, and $S^\infty = \{D, Ad\} \times \{\alpha, \delta\}$.

(b) Compute the set correlated equilibria. (Show your result.)

Answer: As we have seen, a correlated equilibrium assigns positive probability only on S^∞. Moreover, for any correlated equilibrium p, it must be that $p(Ad, \alpha) = 0$. This is because if $p(Ad, \alpha) > 0$, when Player 2 is asked to play α, it is a strictly better response to play δ. Given that $p(Ad, \alpha) = 0$, it must also be the case that $p(Ad, \delta) = 0$; otherwise when Player 1 is asked to play Ad, he would know that player 2 plays δ, and it is a better response to play D. Therefore, Nash equilibria are the only correlated equilibria. Player 1 plays D and player 2 plays δ with probability of at least $1/2$.

(c) Suppose that in addition to the type with the payoff function above, with probability 0.1, Player 1 has a "crazy" type who gets -1 if plays D or d and 0 otherwise. Compute a sequential equilibrium.

Answer: By now, you should be able to do this (and I have to go to dinner).