# Lecture 7
# Rationalizability

14.123 Microeconomic Theory III

Muhamet Yildiz

## A Game

| | L | R |
|---|---|---|
| T | (2,0) | (-1,1) |
| M | (0,10) | (0,0) |
| B | (-1,-6) | (2,0) |

## Assume

| | L | R |
|---|---|---|
| T | (2,0) | (-1,1) |
| M | (0,10) | (0,0) |
| B | (-1,-6) | (2,0) |

Player 1 is rational

Player 2 is rational

Player 2 is rational and
Knows that Player 1 is rational

Player 1 is rational,
knows that 2 is rational
knows that 2 knows that
1 is rational

## Assume

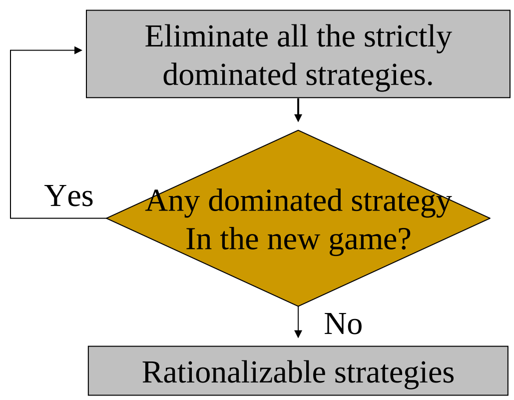
| 1 \ 2 | L | m | R |
|---|---|---|---|
| T | (3,0) | (1,1) | (0,3) |
| M | (1,0) | (0,10) | (1,0) |
| B | (0,3) | (1,1) | (3,0) |

P1 is rational

P2 is rational and
knows that P1 is rational

P1 is rational and
knows all these

## Rationalizability




The play is rationalizable, provided that …

## Formally,

- Game G = $(N, S_1,\ldots,S_n; u_1,\ldots,u_n)$, where
  - $N$ = set of players
  - $S_i$ = set of all strategies of player $i$,
  - $u_i : S_1 \times \ldots \times S_n \rightarrow R$ is i's vNM utility function.
- Belief = a probability distribution $\mu_{-i}$ on $S_{-i}$
- Mixed strategy = a probability distribution $\sigma_i$ on $S_i$
- Notation: $u_i(s_i,\mu_{-i})$, $u_i(\sigma_i,s_{-i})$, etc.
- $s_i$ is a best response to $\mu_{-i}$ $\Leftrightarrow$ $u_i(s_i,\mu_{-i}) \geq u_i(s_i',\mu_{-i})$ $\forall s_i'$.
- $B_i(\mu_{-i})$ = set of best responses to $\mu_{-i}$
- $\sigma_i$ strictly dominates $s_i$ $\Leftrightarrow$ $u_i(\sigma_i,s_{-i}) > u_i(s_i,s_{-i})$ for all $s_{-i}$.
- $s_i$ is strictly dominated $\Leftrightarrow$ some $\sigma_i$ strictly dominates $s_i$

## Rationality & Dominance

**Theorem:** $s_i^*$ is never a best reply to a belief $\mu_{-i}$ $\Leftrightarrow$ $s_i^*$ is strictly dominated.

Proof:

- (=>) Assume $s_i^* \in B_i(\mu_{-i})$.
  - $\Rightarrow$ $\forall s_i$, $u_i(s_i^*,\mu_{-i}) \geq u_i(s_i,\mu_{-i})$
  - $\Rightarrow$ $\forall \sigma_i$, $u_i(s_i^*,\mu_{-i}) \geq u_i(\sigma_i,\mu_{-i})$
  - $\Rightarrow$ No $\sigma_i$ strictly dominates $s_i^*$.
- Separating-Hyperplane Theorem: For any convex, non-empty and disjoint $C$ and $D$ with $C$ closed, $\exists r$: $\forall x \in \mathrm{cl}(D)$ $\forall y \in C$, $r \cdot x \geq r \cdot y$.
- (<=) Assume $s_i^*$ is not strictly dominated.
- Define

$$C = \{u_i(\sigma_i,.) | \sigma_i \text{ is a mixed strategy of } i\}$$

$$D = \{x \,|x_k > u_i(s_i^*,s_{-i}^k)\ \forall k\}.$$

- $C$ and $D$ are disjoint, convex and non-empty with $C$ closed.
- By SHT, $\exists \mu_{-i}$: $\forall \sigma_i$, $u_i(s_i^*,\mu_{-i}) \geq u_i(\sigma_i,\mu_{-i})$

## Iterated strict dominance & Rationalizability

- $S^0 = S$
- $S_i^m = B_i(\Delta(S_{-i}^{m-1}))$
  - where $\Delta(S_{-i}^{m-1})$ = beliefs with support on $S_{-i}^{m-1}$
- Previous Theorem:

$$S_i^m = S_i^m \setminus \{s_i \mid \exists \sigma_i\text{: } u_i(\sigma_i,s_{-i}) > u_i(s_i,s_{-i})\ \forall s_{-i} \in S_{-i}^{m-1}\}$$

- (Correlated) Rationalizable strategies:

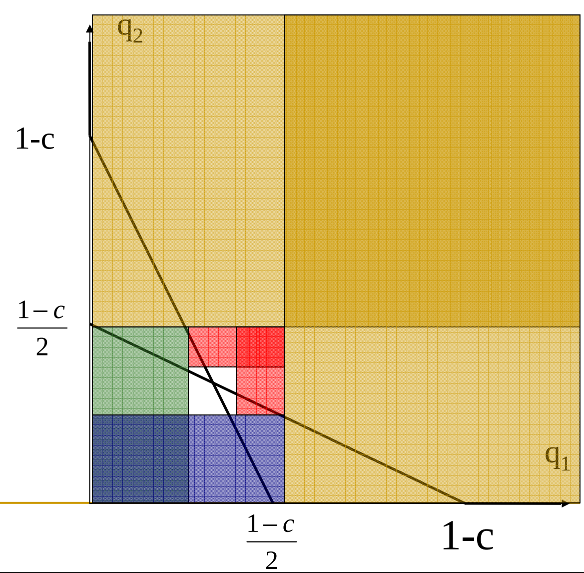
$$S_i^\infty = \bigcap_{k=0}^{\infty} S_i^k$$

## Foundations of rationalizability

- If the game and rationality are common knowledge, then each player plays a rationalizable strategy.
- Each rationalizable strategy profile is the outcome of a situation in which the game and rationality are common knowledge.
- In any "adaptive" learning model the ratio of players who play a non-rationalizable strategy goes to zero as the system evolves.

## Rationalizability in Cournot Duopoly

Simultaneously, each firm $i \in \{1,2\}$ produces $q_i$ units at marginal cost $c$, and sells it at price $P = \max\{0, 1-q_1-q_2\}$.



## Rationalizability in Cournot duopoly

- If i knows that $q_j \le q$, then $q_i \ge (1-c-q)/2$.
- If i knows that $q_j \ge q$, then $q_i \le (1-c-q)/2$.
- We know that $q_j \ge q^0 = 0$.
- Then, $q_i \le q^1 = (1-c-q^0)/2 = (1-c)/2$ for each i;
- Then, $q_i \ge q^2 = (1-c-q^1)/2 = (1-c)(1-1/2)/2$ for each i;
- …
- Then, $q^n \le q_i \le q^{n+1}$ or $q^{n+1} \le q_i \le q^n$ where
  $q^{n+1} = (1-c-q^n)/2 = (1-c)(1-1/2+1/4-\ldots+(-1/2)^n)/2$.
- As $n \to \infty$, $q^n \to (1-c)/3$.