

# 22.01 Fall 2016, Problem Set 4 Solutions

October 30, 2016

Complete all the assigned problems, and do make sure to show your intermediate work.

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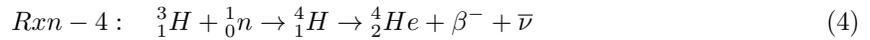
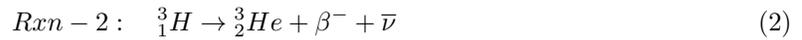
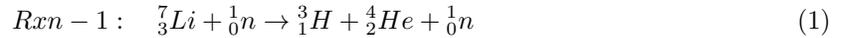
## Part I

# Skill-Building Problems (50 points)

## 1 Successive Decay Chains

The molten salt reactor (MSR) could be used not just for power, but to produce two incredibly valuable gases from its  ${}^7\text{Li}$ -enriched salt coolant: Tritium ( ${}^3\text{H}$ ) and helium-3 ( ${}^3\text{He}$ ). Tritium is extremely useful as fusion reactor fuel, while helium-3 is the most effective gaseous neutron detector available. The latter is valuable enough that [NASA has considered mining it on the Moon](#).

1. Starting with the capture of a neutron by  ${}^7\text{Li}$ , write a complete set of nuclear reactions that describe the production and destruction of  ${}^3\text{H}$  and  ${}^3\text{He}$ . Consider that both gases can also capture neutrons to be destroyed, with microscopic cross sections  $\sigma_{H,n}$  and  $\sigma_{He,n}$ .



2. Develop a set of differential equations, similar to the Bateman equations, describing the production and destruction of  ${}^3\text{H}$  and  ${}^3\text{He}$ .

**First, we set up the equations for the concentration of each isotope, assuming we have some amount of  ${}^7\text{Li}$  to begin with:**

$$\frac{dLi}{dt} = -N_{Li}\sigma_{c,7-Li}\Phi_{reactor} \quad (\text{Rxn} - 1) \quad (5)$$

$$\frac{dH}{dt} = N_{Li}\sigma_{c,7-Li}\Phi_{reactor} - N_H\sigma_{c,H}\Phi_{reactor} - \lambda_H N_H \quad (\text{Rxn} - 1, 2, 4) \quad (6)$$

$$\frac{dHe-3}{dt} = \lambda_H N_H - N_{He-3}\sigma_{c,He-3}\Phi_{reactor} \quad (\text{Rxn} - 2, 3) \quad (7)$$

$$\frac{dHe-4}{dt} = N_{He-3}\sigma_{c,He-3}\Phi_{reactor} + N_H\sigma_{c,H}\Phi_{reactor} \quad (\text{Rxn} - 3, 4) \quad (8)$$

Next, we make an assumption that because  $\sigma_{c,H-3}$  is so small, that it gives a negligible change in concentrations. This reduces the equations to the following:

$$\frac{dLi}{dt} = -N_{Li}\sigma_{c,7-Li}\Phi_{reactor} \quad (9)$$

$$\frac{dH}{dt} = N_{Li}\sigma_{c,7-Li}\Phi_{reactor} - \lambda_H N_H \quad (10)$$

$$\frac{dHe-3}{dt} = \lambda_H N_H - N_{He-3}\sigma_{c,He-3}\Phi_{reactor} \quad (11)$$

$$\frac{dHe-4}{dt} = N_{He-3}\sigma_{c,He-3}\Phi_{reactor} \quad (12)$$

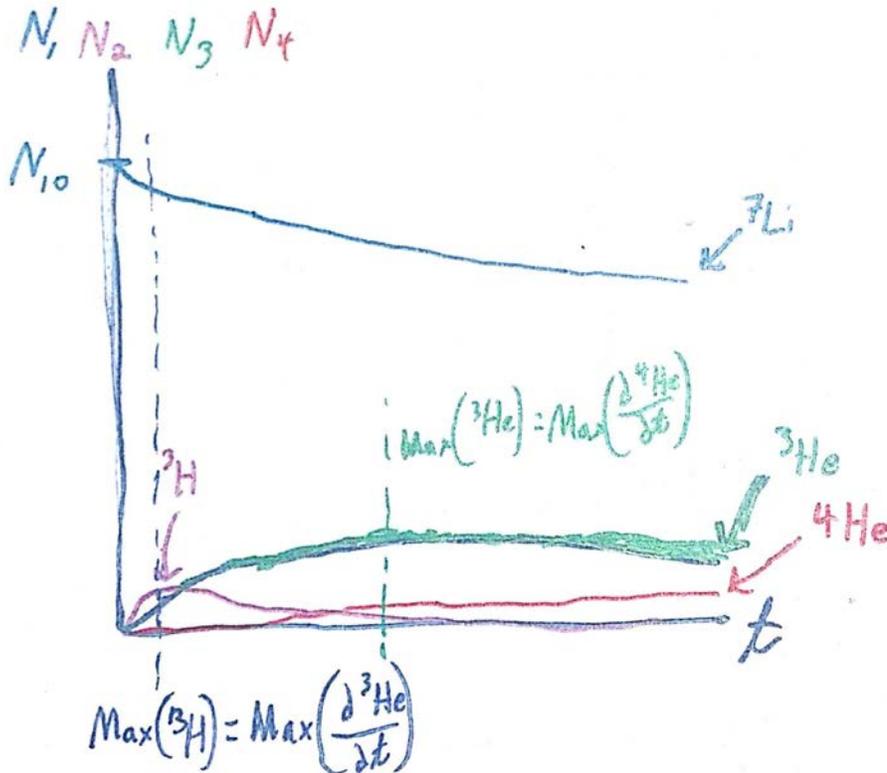
3. Graph the solutions to this set of differential equations, showing the relative levels of  ${}^7\text{Li}$ ,  ${}^3\text{H}$ , and  ${}^3\text{He}$  in the reactor. Use the following data to make your graphs:

$$\sigma_{c,7-Li} = 10^{-4}b \quad \sigma_{c,3-H} = 10^{-10}b \quad \sigma_{c,3-He} = 10^{-5}b \quad \lambda_{3H} = 1.8 \cdot 10^{-9} \left[ \frac{1}{s} \right] \quad \Phi = 1 \cdot 10^{14} \frac{n}{cm^2 - s}$$

Without numerically solving the equations, we do know the following things:

- ${}^7\text{Li}$  proceeds by simple exponential decay, at a pretty slow rate because of the small cross section.
- Because  ${}^3\text{H}$  decays so much faster than it is produced, its level remains very low. For short times, the slope of the  ${}^3\text{H}$  level will be the negative value of  ${}^7\text{Li}$ .
- ${}^3\text{He}$  is burned even more slowly than  ${}^7\text{Li}$ , so its slope will be proportional to the value of  ${}^3\text{H}$  at all times. The maximum production rate of  ${}^3\text{He}$  occurs at the maximum value of  ${}^3\text{H}$ .
- ${}^4\text{He}$  just builds up, very slowly, from both the burning of  ${}^3\text{He}$  and  ${}^7\text{Li}$ .

Using this information, the following graph (showing exaggerated burn/production rates) was generated, which is quite accurate. If I were to draw it a second time, I would “stretch out” the  ${}^3\text{H}$  line, and adjust the slopes of  ${}^3\text{He}$  and  ${}^4\text{He}$  accordingly:



4. Now assume that  ${}^3\text{H}$  costs \$30,000/g and  ${}^3\text{He}$  costs \$53,000/g. Graph the value of the gases in the MSR as a function of time, assuming an initial  ${}^7\text{Li}$  number density of  $10^{22} \frac{\text{atoms}}{\text{cm}^3}$ .

Because we know from (b) above that there is almost no  ${}^3\text{H}$  in the reactor at any given time, the price of the gases in the reactor is simply  $53,000 \cdot {}^3\text{He}$  for most times. Therefore, except for a short blip up during peak  ${}^3\text{H}$  production, it will proportionally follow the  ${}^3\text{He}$  inventory.

## 2 Statistics and Certainty

Since we know that smoking is a major source of radioactivity, smoke shops should also be major sources of radioactivity. How long would you have to count in a smoke shop to be 95% sure that you can distinguish your count from the background? Assume a new background activity of  $A_{\text{Smoke-Shop}}$  inside the store.

For this problem, we must use the relation for *uncertainty in quadrature*, because both the new background counting experiment and the counting experiment will have their own uncertainties associated with their respective count rates. Using Poisson statistics, the standard deviation of a number of counts  $n$  is  $\sigma = \sqrt{n}$ . A count rate  $c$  is defined as  $\frac{n}{t}$ , where  $t$  is the counting time. Therefore, the standard deviation of a count rate is  $\sigma_c = \frac{\sqrt{n}}{t} = \frac{\sqrt{ct}}{t} = \sqrt{\frac{c}{t}}$ . The uncertainties for the background count rate  $c_b$  and the gross count rate in the smoke shop  $c_g$  added together in quadrature become:

$$\sigma_{net} = \sqrt{\sigma_b^2 + \sigma_g^2} = \sqrt{\left(\sqrt{\frac{c_b}{t_b}}\right)^2 + \left(\sqrt{\frac{c_g}{t_g}}\right)^2} = \sqrt{\frac{c_b}{t_b} + \frac{c_g}{t_g}} \quad (13)$$

Assuming we can control the background counting time  $t_b$ , and we measure a known background count rate  $c_b$ , then it's up to us to find an acceptable uncertainty rate that we can determine with 95% confidence. The 95% confidence interval for our acceptable uncertainty (error) means that the following equation must be satisfied:

$$2\sigma_{net} = (\% \text{ Error}) c_{net} \quad (14)$$

We substitute the expression for  $\sigma_{net}$ , and the relation  $c_{net} = c_g - c_b$ , to relate the count rate in the smoke shop to the required counting time inside:

$$2\sqrt{\frac{c_b}{t_b} + \frac{c_g}{t_g}} = (\% \text{ Error}) (c_g - c_b) \quad (15)$$

$$\left[\sqrt{\frac{c_b}{t_b} + \frac{c_g}{t_g}}\right]^2 = \left[\frac{\% \text{ Error}}{2} (c_g - c_b)\right]^2 \quad (16)$$

$$\frac{c_b}{t_b} + \frac{c_g}{t_g} = \frac{(\% \text{ Error})^2}{4} (c_g - c_b)^2 \quad (17)$$

$$\frac{c_b}{t_b} + \frac{c_g}{t_g} = \frac{(\% \text{ Error})^2}{4} (c_g^2 - 2c_g c_b + c_b^2) \quad (18)$$

$$\frac{c_g}{t_g} = \frac{(\% \text{ Error})^2}{4} (c_g^2 - 2c_g c_b + c_b^2) - \frac{c_b}{t_b} \quad (19)$$

$$\frac{t_g}{c_g} = \frac{1}{\frac{(\% \text{ Error})^2}{4} (c_g^2 - 2c_g c_b + c_b^2) - \frac{c_b}{t_b}} \quad (20)$$

$$t_g = \frac{c_g}{\frac{(\% \text{ Error})^2}{4} (c_g^2 - 2c_g c_b + c_b^2) - \frac{c_b}{t_b}} \quad (21)$$

We now have an equation relating the required counting time in the smoke shop  $t_g$  to all the other parameters in the experiment, including the background counting rate and time, the measured counting rate  $c_g$ , and the percent error that we are willing to accept. Finally, we recognize that:

- Our detector will have some efficiency  $\eta_{detector}$  (a number between zero and one) in counting particles that enter the detector,
- Each of the radon decay products may have a number of associated radioactive decays  $n_{part/dis}$  associated with each disintegration, and
- There will be some geometric factor relating the distributed source in the smoke shop, much like an effective solid angle  $\Omega$ .

Therefore, we can relate the net activity to a net count rate as follows:

$$c_n = \frac{A_n}{\eta_{detector} \frac{\Omega_{effective}}{4\pi} n_{part/dis}} \quad (22)$$

Substituting into our final expression yields:

$$t_g = \frac{\left( \frac{A_n}{\eta_{detector} \frac{\Omega_{effective}}{4\pi} n_{part/dis}} - c_b \right)}{\frac{(\%Error)^2}{4} \left( \frac{A_n}{\eta_{detector} \frac{\Omega_{effective}}{4\pi} n_{part/dis}} \right)^2 - \frac{c_b}{t_b}} \quad (23)$$

Now we have a direct relation between the activity of the smoke shop, our background counting experiment, known quantities like detector efficiency, and our acceptable percentage error that can be reported with 95% confidence.

Let's throw in a few example parameters: Suppose we are willing to accept a 25% error (0.25) with 95% confidence, and we took a normal one-hour background reading (60 minutes) yielding a background count rate of 25 CPM (counts per minute). Our detector has an efficiency of 1% (0.01), we'll assume that each disintegration yields two detectable gamma rays on average, and our effective solid angle being surrounded by our source is  $\pi Sr$  (steradians). This yields a much simpler expression:

$$t_g \text{ (minutes)} = \frac{(200A_n - 25)}{625A_n^2 - \frac{25}{60}} \quad (24)$$

A theoretical estimate of the activity of a smoke shop would therefore give you a good guess as to how much time you would have to count for. As an example, let's say you measure a gross count rate ( $c_g$ ) of 35 CPM inside. Equation 21 yields a required counting time of only 30.6 minutes to be 95% confident in your result. This makes it a potentially awkward, but not impossible, task.

1. *Bonus Question (25 points): Go do this. Tell us how radioactive a local smoke shop is, state/calculate your uncertainty, and how long you had to count. You should get the shop owner's permission before doing this, to avoid arousing suspicion.*

### 3 Radioactive Dating with Confidence

For this problem, [consider the methods used to radioactively date the Shroud of Turin](#), thought to be the burial cloth of Jesus of Nazareth.

1. Why did the investigators use carbon as the dating isotope? Consider what other isotopes could have been present, and give at least three reasons that carbon was chosen.  
The investigators used carbon-14 because its half life is 5,730 years, and if the shroud were indeed the burial cloth of Jesus, then a sizeable chunk of radioactivity would be measurable. If the half life were too short or too long, then very little counts would be acquired, greatly increasing the uncertainty of the count rate. The goal here would be to maximize the activity of the isotope being counted. Too short a half life, and it would have all decayed away. Too long, and very few atoms would decay at any given time at all.

2. How did the investigators use statistics to prove beyond a reasonable doubt that the Shroud of Turin was *not* the burial cloth? What is a  $\chi^2$  test, and how did they arrive at the  $\chi^2$  values in the paper?

Samples of the shroud were sent to different laboratories, for their own dating procedures and uncertainty to be measured. Each laboratory used their own procedures, showing that each sample (1 - the shroud, and 2-4, the “control” samples known to be from different time periods) were able to be measured within a certain range of ages. The chi-squared test here is applied to measure the “goodness of fit” of each sample’s ages, and tells us the confidence of a random measurement of that specimen falling within the uncertainty bounds of the expected, or “theoretical,” measured age. The fact that many laboratories were only able to measure the shroud sample with 5% confidence within the specified limits does not mean that they were wrong, as their dating methods showed great confidence for the other, control samples.

3. Why did the investigators send so many unknown control samples to so many laboratories, and why did they use different cleaning procedures?

This was done to avoid any chance of introducing bias into the results. Many, many people encounter significant quantities of feelings when dealing in matters of faith and religion, therefore those who would “want” the shroud to be or not to be the real burial cloth could have tampered with the evidence if they knew which piece they had. Keeping everyone guessing completely removes the incentive to tamper with the specimens. Different cleaning procedures were also used, to ensure that if one cleaning procedure artificially reduced the carbon-14 content of a specimen, then it would be discovered, and that method could be eliminated from altering the actual amount of carbon-14 present.

## 4 Generating Cobalt-60 and Profit

How many days should one irradiate a 100g source of  $^{59}\text{Co}$  in order to maximize profit from the reactor? Assume a fully homogeneous, thermal reactor, with the following parameters:

$$\Phi = 10^{14} \frac{n}{\text{cm}^2\text{s}} \quad \text{Reactor Cost} = \$1000/\text{day} \quad {}^{60}\text{Co} = \$100/\mu\text{Ci}$$

Start by writing down what is physically happening (the nuclear reactions), model them using a system of differential equations, solve the system of equations for the amount of  $^{60}\text{Co}$  as a function of time, and construct & solve an equation to maximize the profit of the reactor. Look up any nuclear data that you need from the JANIS cross section database and the KAERI Table of Nuclides.

Let’s say we have the following nuclear reactions (like on the lecture on October 6th), where we are producing  $^{60}\text{Co}$  from the neutron bombardment of  $^{59}\text{Co}$ , and  $^{60}\text{Co}$  has its own decay constant  $\lambda$  where it decays by  $\beta^-$  decay into  $^{60}\text{Ni}$ :



We also have to account for the fact that both  $^{59}\text{Co}$  and  $^{60}\text{Co}$  are “burned” in the reactor by capturing neutrons. The first reaction produces  $^{60}\text{Co}$ , while the second one depletes it. We therefore define a couple of *neutron capture cross sections*:

$$\sigma_{c_{59\text{Co}}} = \sigma_{59} = 20 b \quad \sigma_{c_{60\text{Co}}} = \sigma_{60} = 2 b \quad 1 b = 10^{-24} \text{cm}^2 \quad (26)$$

where we have looked up the values of the cross sections from the JANIS database, using the ENDF VII library for incident neutron data, and we’ve chosen the values at 0.025eV (the kinetic energy of thermal neutrons). Let’s just pretend that our reactor has only thermal neutrons in it, a “one group” approximation. Oh, let’s also define the neutron flux of our reactor, or the number of neutrons zipping through every square centimeter per second:

$$\Phi = 10^{14} \frac{n}{\text{cm}^2\text{s}} \quad (27)$$

We next look up the half life of  $^{60}\text{Co}$ , and use Equation ?? to get its decay constant:

$$t_{1/2} = 1925.4 \text{ days} = 1.66 \cdot 10^8 \text{ s} \quad \lambda = \frac{\ln(2)}{1.66 \cdot 10^8 \text{ s}} = 4.17 \cdot 10^{-9} \text{ s}^{-1} \quad (28)$$

Finally, we have to define some initial amount of  $^{59}\text{Co}$  that we put into our reactor. Let's just say it was 100g of  $^{59}\text{Co}$ , or 1.69 moles, or  $10^{24}$  atoms:

$$N_{10} = 10^{24} \text{ atoms} \quad (29)$$

Finally we are ready to construct our differential equations to physically model this real system. First, we recognize from the KAERI Table of Nuclides that  $^{59}\text{Co}$  is stable, so it has no production term, and the only way for it to be destroyed is to be "burned" by neutron capture:

$$\frac{dN_1}{dt} = -\sigma_{59}\Phi N_1 \quad (30)$$

Next, the production rate of  $^{60}\text{Co}$  is equal to the destruction rate of  $^{59}\text{Co}$ , while  $^{60}\text{Co}$  can be destroyed both by natural radioactive decay and artificial "burning:"

$$\frac{dN_2}{dt} = \sigma_{59}\Phi N_1 - \sigma_{60}\Phi N_2 - \lambda N_2 \quad \frac{dN_2}{dt} = \sigma_{59}\Phi N_1 - (\lambda + \sigma_{60}\Phi) N_2 \quad (31)$$

Finally, the only way to produce  $^{60}\text{Ni}$  is by radioactive decay of  $^{60}\text{Co}$ . Note that "burning"  $^{60}\text{Co}$  does NOT produce  $^{60}\text{Ni}$ :

$$\frac{dN_3}{dt} = \sigma_{60}\Phi N_2 \quad (32)$$

Here we are ignoring the "burning" of  $^{60}\text{Ni}$ . In fact, let's just ignore  $^{60}\text{Ni}$  altogether, because we don't care about it:

$$\cancel{\frac{dN_3}{dt} = \sigma_{60}\Phi N_2} \quad (33)$$

Now we start with the easy equation for  $N_1(t)$ . Note that the quantity  $\sigma_{59}\Phi$  has the *same units* as  $\lambda$ , so the equation takes the same form:

$$\frac{dN_1}{dt} = (-\sigma_{59}\Phi) N_1 \quad \int \left( \frac{dN_1}{N_1} \right) = \int (-\sigma_{59}\Phi) dt \quad N_1(t=0) = N_{10} \quad N_1(t) = N_{10} e^{-\sigma_{59}\Phi t} \quad (34)$$

Now we take this expression for  $N_1$  and substitute it into Equation 31:

$$\frac{dN_2}{dt} = \sigma_{59}\Phi N_{10} e^{-\sigma_{59}\Phi t} - (\lambda + \sigma_{60}\Phi) N_2 \quad (35)$$

Next we rearrange terms so that all the stuff is on one side of the equation:

$$\frac{dN_2}{dt} + (\lambda + \sigma_{60}\Phi) N_2 - \sigma_{59}\Phi N_{10} e^{-\sigma_{59}\Phi t} = 0 \quad (36)$$

Next we introduce our *integrating factor*,  $\mu$ :

$$\mu = e^{\int (\lambda + \sigma_{60}\Phi) dt} = e^{(\lambda + \sigma_{60}\Phi)t} \quad (37)$$

and we multiply every term in Equation 36 by  $\mu$ :

$$\left( \frac{dN_2}{dt} \right) e^{(\lambda + \sigma_{60}\Phi)t} + ((\lambda + \sigma_{60}\Phi) N_2) e^{(\lambda + \sigma_{60}\Phi)t} - (\sigma_{59}\Phi N_{10} e^{-\sigma_{59}\Phi t}) e^{(\lambda + \sigma_{60}\Phi)t} = \quad (38)$$

We recognize that our first two terms look eerily similar to the end result of the Product Rule:

$$\frac{d(a(t)b(t))}{dt} = a(t) \frac{db(t)}{dt} + \frac{da(t)}{dt} b(t) \quad \text{equivalently} \quad (ab)' = ab' + a'b \quad (39)$$

We then smooch the first two terms of Equation 38 together using the Product Rule in reverse, and combine the exponential parts of the third term:

$$\frac{d(N_2 e^{(\lambda + \sigma_{60}\Phi)t})}{dt} = \sigma_{59}\Phi N_{10} e^{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)t} \quad (40)$$

We then integrate both sides:

$$\int \left( \frac{d(N_2 e^{(\lambda + \sigma_{60}\Phi)t})}{dt} \right) dt = \int \left( \sigma_{59}\Phi N_{10} e^{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)t} \right) dt \quad (41)$$

This just kills the derivative on the left hand side, puts the exponential term in the denominator on the right hand side, and introduces a constant of integration:

$$N_2 e^{(\lambda + \sigma_{60}\Phi)t} = \frac{\sigma_{59}\Phi N_{10}}{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)} e^{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)t} + C \quad (42)$$

We now use the initial condition that we had no  $^{60}\text{Co}$  when we first started producing it at  $t = 0$ :

$$N_2(t=0) = 0 \quad \cancel{N_2} e^{\cancel{(\lambda + \sigma_{60}\Phi) \cdot 0}} = \frac{\sigma_{59}\Phi N_{10}}{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)} e^{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi) \cdot 0} + C \quad (43)$$

$$\cancel{(0)} e^{\cancel{(\lambda + \sigma_{60}\Phi) \cdot (0)}} = \frac{\sigma_{59}\Phi N_{10}}{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)} e^{\cancel{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi) \cdot (0)}} + C \quad (44)$$

$$0 = \frac{\sigma_{59}\Phi N_{10}}{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)} + C \quad C = \frac{-\sigma_{59}\Phi N_{10}}{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)} \quad (45)$$

We finally plug this integration constant back into Equation 42 and do a bit of rearranging:

$$N_2 e^{(\lambda + \sigma_{60}\Phi)t} = \frac{\sigma_{59}\Phi N_{10}}{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)} e^{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)t} - \frac{\sigma_{59}\Phi N_{10}}{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)} \quad (46)$$

$$N_2 e^{(\lambda + \sigma_{60}\Phi)t} = \frac{\sigma_{59}\Phi N_{10}}{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)} \left[ e^{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)t} - 1 \right] \quad (47)$$

$$\frac{\cancel{N_2} e^{\cancel{(\lambda + \sigma_{60}\Phi)t}}}{\cancel{e^{(\lambda + \sigma_{60}\Phi)t}}} = \frac{\sigma_{59}\Phi N_{10}}{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)} \left[ \frac{e^{\cancel{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)t}}}{\cancel{e^{(\lambda + \sigma_{60}\Phi)t}}} - \frac{1}{e^{(\lambda + \sigma_{60}\Phi)t}} \right] \quad (48)$$

$$N_2(t) = \frac{\sigma_{59}\Phi N_{10}}{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)} \left[ e^{(-\sigma_{59}\Phi)t} - \frac{1}{e^{(\lambda + \sigma_{60}\Phi)t}} \right] \quad (49)$$

$$N_2(t) = \frac{\sigma_{59}\Phi N_{10}}{((\lambda + \sigma_{60}\Phi) - \sigma_{59}\Phi)} \left[ e^{(-\sigma_{59}\Phi)t} - e^{-(\lambda + \sigma_{60}\Phi)t} \right] \quad (50)$$

Not surprisingly, this looks *exactly* like Equation 4.40 from the Turner book:

$$N_2(t) = \frac{\lambda_1 N_{10}}{(\lambda_2 - \lambda_1)} [e^{-\lambda_1 t} - e^{-\lambda_2 t}] \quad (51)$$

where we have defined  $\lambda_1 = \sigma_{59}\Phi$  and  $\lambda_2 = \lambda + \sigma_{60}\Phi$ . Now let's start plugging in some of the values:

$$\lambda_1 = \sigma_{59}\Phi = \left( 20 \cdot 10^{-24} \text{cm}^2 \right) \left( 10^{14} \frac{n}{\text{cm}^2 \text{s}} \right) = 2 \cdot 10^{-9} \text{s}^{-1} \quad (52)$$

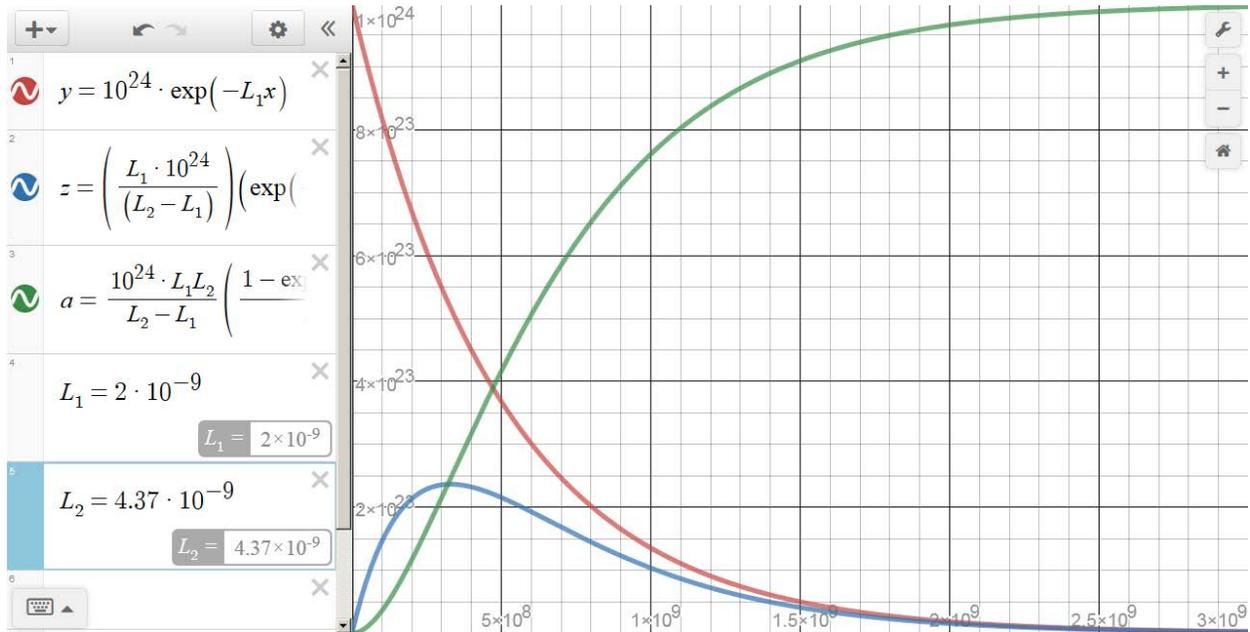
$$\lambda_2 = \lambda + \sigma_{60}\Phi = \left( 4.17 \cdot 10^{-9} \frac{1}{\text{s}} \right) + \left( 2 \cdot 10^{-24} \text{cm}^2 \right) \left( 10^{14} \frac{n}{\text{cm}^2 \text{s}} \right) = 4.37 \cdot 10^{-9} \text{s}^{-1} \quad (53)$$

How we plug these values into Equation 51:

$$N_2(t) = \frac{(2 \cdot 10^{-9} \text{s}^{-1})(10^{24} \text{ atoms})}{(4.37 \cdot 10^{-9} \text{s}^{-1} - 2 \cdot 10^{-9} \text{s}^{-1})} \left[ e^{-(2 \cdot 10^{-9} \text{s}^{-1})t} - e^{-(4.37 \cdot 10^{-9} \text{s}^{-1})t} \right] \quad (54)$$

$$N_2(t) = \frac{(2 \cdot 10^{15} \text{ atoms})}{(2.37 \cdot 10^{-9} \text{s}^{-1})} \left[ e^{-(2 \cdot 10^{-9} \text{s}^{-1})t} - e^{-(4.37 \cdot 10^{-9} \text{s}^{-1})t} \right] \quad (55)$$

We can then graph this situation using Desmos:



Note that the x-axis is in seconds, and the y-axis is in atoms. This shows you that it takes about  $3.204 \cdot 10^8$  seconds, or 10 years, to reach a maximum inventory of  $^{60}\text{Co}$  in the reactor. A very real example, using actual numbers, from a very theoretical derivation!

Finally, we construct an equation to calculate the profit that we can extract from the reactor, remembering that the activity in Bq is written as  $A = \lambda N$ :

$$\$(t) = \left( \frac{\$100}{1 \mu\text{Ci } ^{60}\text{Co}} \right) \left( \frac{1 \mu\text{Ci } ^{60}\text{Co}}{3.7 \cdot 10^4 \text{ Bq } ^{60}\text{Co}} \right) \left( \frac{N_2(t)}{\lambda + \sigma_{60}\Phi} \right) - \left( \frac{\$1,000}{1 \text{ day}} \right) \left( \frac{1 \text{ day}}{86,400 \text{ s}} \right) t(s) \quad (56)$$

$$\$(t) = \left( \frac{\$100}{3.7 \cdot 10^4 \text{ Bq } ^{60}\text{Co}} \right) \left( \frac{N_2(t)}{4.37 \cdot 10^{-9} \text{ s}^{-1}} \right) - \left( \frac{\$1,000}{86,400 \text{ s}} \right) t(\text{s}) \quad (57)$$

$$\$(t) = \left( \frac{\$100}{3.7 \cdot 10^4 \text{ Bq } ^{60}\text{Co}} \right) \left( \frac{N_2(t) (\text{atoms})}{4.37 \cdot 10^{-9} \text{ s}^{-1}} \right) - \left( \frac{\$1,000}{86,400 \text{ s}} \right) t(\text{s}) \quad (58)$$

$$\$(t) = \left( \frac{\$100 N_2(t)}{(3.7 \cdot 10^4)(4.37 \cdot 10^{-9})} \right) - 0.011574t \quad (59)$$

Finding the maximum of this equation graphically yields the same time as the maximum in  $^{60}\text{Co}$  inventory, or  $3.204 \cdot 10^8$  seconds. Clearly this reactor's profit model isn't quite accurate... we haven't accounted for the isotopic isolation costs of extracting cobalt.

## Part II

# Take-Home Lab: Estimating the Radioactivity of One Banana (50 points)

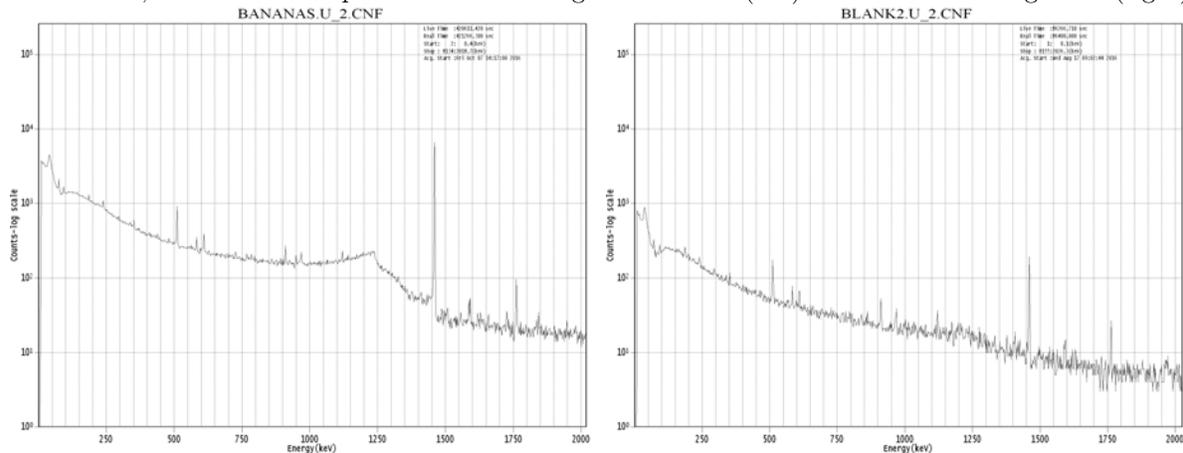
Using the banana ashes accumulated from last year's 22.01 class, estimate the radioactivity of one banana. Make the following assumptions:

1. The ashes were created from 50 pounds of peeled, fresh, ripe bananas.
2. No other sources of contamination are present in the bananas

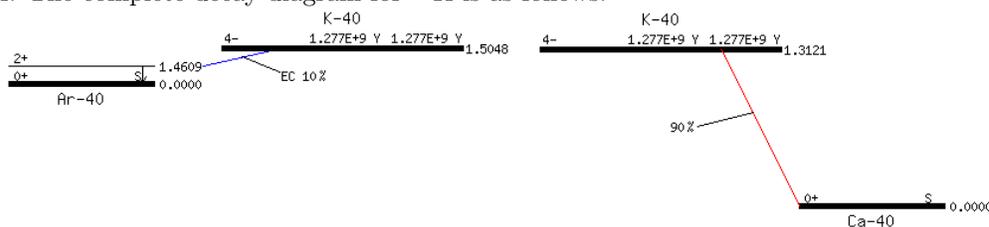
Use the high purity Germanium detector (HPGe) in the Nuclear Reactor Laboratory (NRL) to collect background and banana spectra. Devise a way to determine the total radioactivity of your sample based on any of the available features of the spectrum (peak height, number of counts, area under peak, etc.) of the most appropriate peak of the most appropriate isotope. Compare your answer to any reputable source from the literature, and make sure to cite your source:

- Journal articles need the authors, title, journal, volume, pages, and year.
- Books need the authors, editors, title, publisher, pages used, year, and ISBN number.
- Other articles (like those online) need the author, URL, date that you accessed it, and date of publication. These should be used as a last resort.

Solution: First, we look at the spectra from counting our bananas (left) and from our background (right):



We know that bananas are good sources of potassium, and that 0.011% of all potassium is radioactive  $^{40}\text{K}$ . The complete decay diagram for  $^{40}\text{K}$  is as follows:



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These branching ratios have been slightly rounded from their true values to one significant digit, we will use these. This diagram shows us that 10% of the time, we expect  $^{40}\text{K}$  to decay by electron capture, releasing a 1.461 MeV gamma ray. Looking at our spectra above, both of these have a strong peak at about 1.460 MeV. This is the peak of interest for us, and it is the *photoelectric effect peak* for  $^{40}\text{K}$ . Because we have a large HPGe detector, there are very few single- and double-escape peaks from pair production, so we can assume that this peak contains both the photoelectron interactions *and* most of the pair production



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