22.01 Fall 2016, Problem Set 2 Solutions

October 8, 2016

Complete all the assigned problems, and do make sure to show your intermediate work.

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1 Predicting Nuclear Stability

Using the Table of Nuclides, answer the following questions about sodium (Z=11):

1. Using the excess mass from the table of nuclides, calculate the ***binding energy*** and the ***binding energy per nucleon*** for each isotope of sodium listed on the KAERI table, starting with A=18. Use the table of nuclides to check your answer.
The excess mass, binding energy, mass number (A), and binding energy per nucleon are calculated as follows:

$$\Delta\,[amu] = A\,[amu] - M\,(A,Z)\,[amu]\,; \qquad M\,(A,Z) = A - \Delta \tag{1}$$

$$\Delta\,[MeV] = (A\cancel{[amu]} - M\,(A,Z)\,\cancel{[amu]})\left(\frac{931.49\,\frac{MeV}{\cancel{c^2}}}{1\,\cancel{amu}}\right)\cancel{c^2} \tag{2}$$

$$BE\,(A,Z)\,[amu] = (ZM_H + (A-Z)\,M_n - M\,(A,Z))\,[amu] \tag{3}$$

$$BE\,(A,Z)\,[MeV] = \ (ZM_H + (A-Z)\,M_n - A + \Delta)\,\cancel{[amu - c^2]}\left(\frac{931.49\,\frac{MeV}{\cancel{c^2}}}{1\,\cancel{amu}}\right)\cancel{c^2} \tag{4}$$

$$M_H = 1.0078\,amu; \qquad M_n = 1.0087\,amu \tag{5}$$

Combining equations 4 and 5:

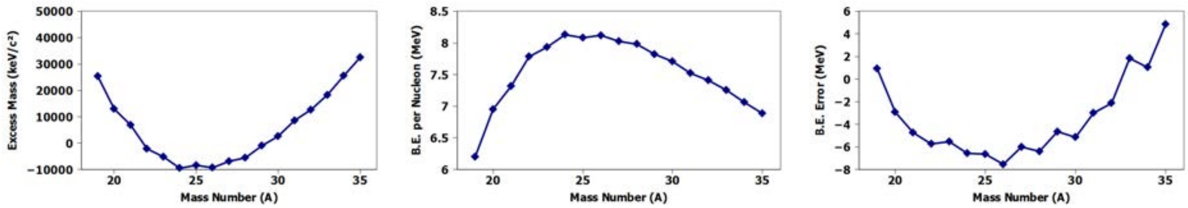
$$BE\,(A,Z)\,[MeV] = \ ((1.0078-1.0084)\,Z + (1.0084-1)\,A)\,\cancel{[amu - c^2]}\left(\frac{931.49\,\frac{MeV}{\cancel{c^2}}}{\cancel{amu}}\right)\cancel{c^2} + \Delta\,[MeV] \tag{6}$$

$$BE\,(A,Z)\,[MeV] = ((0.0084A - 0.0006Z)\,[amu])\,931.49\,MeV + \Delta\,[MeV] \tag{7}$$

Then, the binding energy per nucleon is just $\frac{\mathbf{BE(A,Z)}}{\mathbf{A}}$.

The data are shown on the attached spreadsheet for your comparison.

2. Graph the excess mass of each isotope of sodium as a function of mass number (A). Also graph the difference between the semi-empirical and measured nuclear binding energies. What trends do you see?
The three graphs, plotted separately, are shown here:



The first graph shows the expected "bowl" shape in the excess mass vs. mass number curve, as the stable isotopes in the middle will have had more mass convert to binding energy, and therefore a lower excess mass from Equation 2. The second graph shows the binding energy per nucleon, showing a maximum (and a minimum in mass per nucleon, therefore higher stability) at the stable isotopes of sodium. One important trend in the first and second graphs is that even nuclei tend to have lower excess masses, and therefore higher stabilities, as would be expected from our liquid drop model of the nucleus. This effect is most pronounced for the more stable nuclei, but is harder to see in the first graph because the total, rather than the nucleon-specific, binding energy is shown. The third graph shows a relatively small error of no more than a few MeV between the semi-empirical and measured values of binding energy, with the former calculated from Equation 8 and the latter taken from the KAERI table. It seems the pairing term needs a bit better calibration for sodium (Z=11).

3. For each region where an increase in excess mass is seen from the most stable isotope(s) (left third, middle third, right third), briefly say why the nuclei in each region are most unstable. (Hint: What do you know about the relative number of protons and neutrons in a nucleus, and how does that help determine stability?)
On the left third of the first graph, nuclei are "proton-rich," meaning that the asymmetry reduction in binding energy is very strong. In addition, the Coulomb repulsion term, which remains constant at constant Z, plays a stronger role in reducing the A-dependent volumetric binding energy.
In the middle of the graph, it is the pairing term that seems to change the nuclear stability the most, as evidenced by the oscillatory motion in the curve.
On the right third of the first graph, the asymmetry term dominates in reducing binding energy (increasing excess mass), as the nuclei are "neutron-rich."

2 Liquid-Drop Nuclear Models

For these questions, consider the liquid drop model of nuclear mass, which states that the mass of a nucleus can be empirically calculated as in Eq. 4.10 (p. 59) of *Nuclear Radiation Interactions* by S. Yip.

1. Explain the origin of each additive term in this expression. Pay particular attention to the exponents in each one, and explain why they are what they are.
The semi-empirical mass formula, expressed as the binding energy, is as follows:

$$BE\,(A,Z) = a_v A - a_s A^{\frac{2}{3}} - a_c \frac{Z\,(Z-1)}{A^{\frac{1}{3}}} - a_a \frac{((A-Z)-Z)^2}{A} + a_p \delta; \qquad \delta = \begin{Bmatrix} \frac{1}{\sqrt{A}}\; even-even \\ 0\; even-odd \\ -\frac{1}{\sqrt{A}}\; odd-odd \end{Bmatrix} \quad (8)$$

The origins of the five terms are as follows:
1) Volumetric binding energy, as the number of nuclear bonds increases proportionally to the number of nucleons
2) Surface energy reduction, because the nucleons on the free surface aren't bound to as many neighboring nuclei. The exponent comes from the relation between the formulas for the volume and surface area of a sphere.
3) Coulomb repulsive energy reduction, because all Z protons each feel the repulsive force by the other $(Z-1)$ protons. The repulsive forces are in one direction, or 1/3rd the

dimensionality of a sphere.
4) Asymmetric energy reduction, stemming from a strong decrease in stability from unequal numbers of protons and neutrons. Nucleons have energy shell levels just like electrons in an atom, and they are most stable when corresponding shell levels are filled.
5) Pairing energy change, which increases stability for even-even nuclei (paired, filled energy levels), and decreases stability for odd-odd nuclei (more unpaired, unfilled energy levels.)

2. Why does the δ term in this expression change sign for odd/even nuclei?
   See (5) in the answer to problem 2.1

3. Modify equation 4.10 to empirically calculate the total rest mass of a given nucleus.
   The equation for the binding energy of a nucleus in terms of its rest mass (here Equation 3 in these solutions) can be combined with Equation 8 in these solutions to yield the following:

$$M\left(A,Z\right)=ZM_H+\left(A-Z\right)M_n-\frac{a_vA+a_sA^{\frac{2}{3}}+a_c\frac{Z(Z-1)}{A^{\frac{1}{3}}}+a_a\frac{(N-Z)^2}{A}-a_p\delta}{c^2};\quad \delta=\begin{Bmatrix}\frac{1}{\sqrt{A}}\ even-even\\ 0\ even-odd\\ -\frac{1}{\sqrt{A}}\ odd-odd\end{Bmatrix} \tag{9}$$

where all masses are expressed in amu, and are converted to MeV using the relation $\mathrm{E}=\mathrm{mc}^2$.

Part I

Noodle-Scratchers (50 points)

3 Recasting the Semi-Empirical Mass Formula (with answer)

Derive an expression for the most stable number of neutrons for a given nucleus with Z protons. Graph this expression as a function of Z. How does your prediction compare with the isotopes of sodium?

Answer:

$$0=-\frac{a_s}{3}A^{\frac{-4}{3}}-\frac{4a_c}{3}Z\left(Z-1\right)A^{\frac{-7}{3}}+4Za_aA^{-2}-8Z^2a_aA^{-3} \tag{10}$$

For this question, we can use Equation 9's full expression for the mass of a nucleus given A and Z. Then, assuming Z is held constant, we can take the derivative of M(A,Z) per nucleon (M/A) with respect to A and set it equal to zero to find the minimum mass. Taking a nucleus with a constant Z protons and a variable N neutrons would yield an always-increasing mass. Therefore, to make an equal comparison one must divide the mass by the total number of nucleons (A). Here we will forget about the pairing term, as it's piecewise and doesn't have a smooth derivative:

$$\frac{M\left(A,Z\right)}{A}=\frac{Z}{A}M_H+\left(1-\frac{Z}{A}\right)M_n-a_v+a_sA^{\frac{-1}{3}}+a_c\frac{Z\left(Z-1\right)}{A^{\frac{4}{3}}}+a_a\frac{\left(N-Z\right)^2}{A^2} \tag{11}$$

The expression for the most stable number of nucleons can be found by setting the derivative $\frac{\partial\left(\frac{\mathrm{M}}{\mathrm{A}}\right)}{\partial \mathrm{A}}=0$ and solving for A:

$$\frac{\partial\left(\frac{M}{A}\right)}{\partial A}=0=\frac{-Z}{A^2}M_p+\frac{Z}{A^2}M_n-\frac{a_s}{3}A^{\frac{-4}{3}}-\frac{4a_c}{3}Z\left(Z-1\right)A^{\frac{-7}{3}}+\frac{\partial}{\partial A}\left[a_a\frac{\left(A-2Z\right)^2}{A^2}\right] \tag{12}$$

Next we recognize that $(N-Z)=(A-2Z)$:

$$\frac{\partial\left(\frac{M}{A}\right)}{\partial A}=0=\frac{-Z}{A^2}M_p+\frac{Z}{A^2}M_n-\frac{a_s}{3}A^{\frac{-4}{3}}-\frac{4a_c}{3}Z\left(Z-1\right)A^{\frac{-7}{3}}+\frac{\partial}{\partial A}\left[a_a\frac{A^2-4AZ+4Z^2}{A^2}\right] \tag{13}$$

$$\frac{\partial\left(\frac{M}{A}\right)}{\partial A}=0=\frac{-Z}{A^2}M_p+\frac{Z}{A^2}M_n-\frac{a_s}{3}A^{\frac{-4}{3}}-\frac{4a_c}{3}Z\left(Z-1\right)A^{\frac{-7}{3}}+4Za_aA^{-2}-8Z^2a_aA^{-3} \tag{14}$$

Next we recognize that $(M_n - M_p)$ is approximately zero:

$$0=-\frac{a_s}{3}A^{\frac{-4}{3}}-\frac{4a_c}{3}Z\left(Z-1\right)A^{\frac{-7}{3}}+4Za_aA^{-2}-8Z^2a_aA^{-3} \tag{15}$$

Taking Z=11 for sodium, and substituting the semi-empirical constants from the reading, the expression is as follows:

$$0=-6A^{\frac{-4}{3}}-116.2A^{\frac{-7}{3}}+1034A^{-2}-22748A^{-3};\quad A_{min\,M}=24.1 \tag{16}$$

This is remarkably close to one of sodium's stable isotopes, with A=24.

4 Q-Values and Nuclear Power (part open-ended, part with answer)

For these questions, consider equations 4.2 - 4.6 (pp. 54-55) in *Nuclear Radiation Interactions*.

1. Show that the Q-value for a reaction can be expressed solely in terms of nuclear binding energies (derive equation 4.6).
   The full Q-equation for a nuclear reaction of this general form:

$$i+I\rightarrow f+F+Q \tag{17}$$

   can be expressed solely in terms of rest mass energies:

$$Q=\left(m_i+m_I-m_f-m_F\right)c^2 \tag{18}$$

   and the equation defining the binding energy is the difference between the masses of a given nucleus and its constituent nucleons:

$$BE\left(A,Z\right)=\left(ZM_p+NM_n\right)C^2-M\left(A,Z\right) \tag{19}$$

   This can be rearranged to give the following:

$$M\left(A,Z\right)=\left(ZM_p+NM_n\right)c^2-BE\left(A,Z\right) \tag{20}$$

   Substituting this expression into Equation 17, we arrive at:

$$Q=\left(\left(Z_i+Z_I-Z_f-Z_F\right)M_p+\left(N_i+N_I-N_f-N_F\right)M_n\right)c^2-BE_i\left(A,Z\right)+BE_I\left(A,Z\right)-BE_f\left(A,Z\right)-BE_F\left(A,Z\right) \tag{21}$$

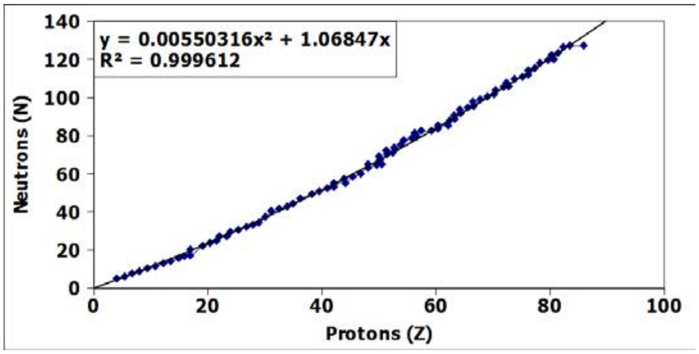
   where in this equation, all quantities, including binding energies, are expressed in mass units. Because the total number of nucleons is unchanged in a nuclear reaction, the Z and N terms are either exactly or approximately conserved (the first case for simple rearrangement of nuclei, like alpha decay, the second case for things like beta decay)). That leaves only the binding energy terms, which when propagating the signs becomes:

$$Q=BE_f\left(A,Z\right)+BE_F\left(A,Z\right)-BE_i\left(A,Z\right)-BE_I\left(A,Z\right) \tag{22}$$

   where in this last equation, the binding energies are expressed in MeV.

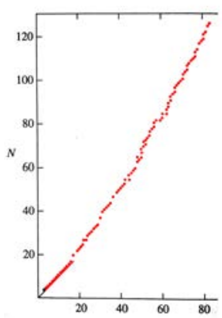
2. Using a data extraction program like the Web Plot Digitizer to get data points from Figure 4.4 (p. 58), produce an empirical expression for the optimum number of neutrons (N) for a given number of protons (Z) in a nucleus. Comment on the quality of that fit, and explain in which regions the fit is the best, and in which the fit is the worst.
   Answer: Something close to $N=0.0055Z^2+Z$

Using the Web Plot Digitizer, the following graph and line of best fit was found for the odd nuclei:



The fit is extraordinarily good, except in regions around Z=32-35, 45-49, and 52-60. Still, for all the theoretical stuff, a simple polynomial fit does seem to work surprisingly well. The origin of this quadratic dependence likely lies in the Coulomb term, which slowly gets stronger with increasing nuclear size. It then requires more and more neutrons to screen neighboring neutrons.

Note that in order to most easily extract these points in an automated fashion, Figure 4.4 from Yip was photographed using a cell phone, and the points of the graph highlighted using GIMP (a free version of Photoshop-like software) to make them red, and more easily recognized. <u>Remember, if you find yourself doing anything tedious (like looking for points one by one), look for another way!</u> The graph used to produce this data looks as follows:



5 Predicting the Island of Stability (open-ended)

Does the semi-empirical formula predict the "island of stability" containing the superheavy elements (SHEs)? If so, graphically or mathematically explain how. If not, read through the article from Physics Today and suggest a missing term to the semi-empirical mass formula, which would account for the more stable SHEs. Justify your extra term by checking the improved fit with the known elements.

(Actual answers may vary, we will judge your solutions based on (1) Your creativity, (2) how you justify your answer, and (3) how you numerically show how well your answer corrects the semi-empirical mass formula to include reality.)

The whole point of the Physics Today article is that superheavy elements (SHEs) exist due to nuclear "magic numbers," some of which we already know, and do exhibit an unusually large number of stable isotopes:

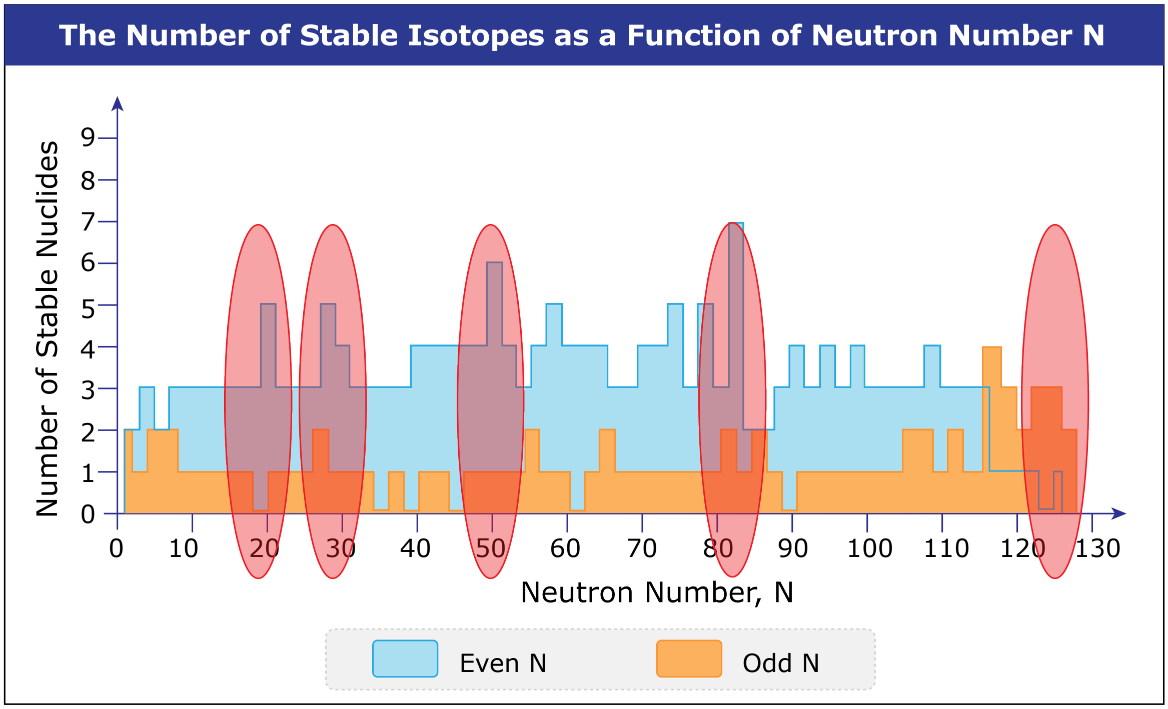
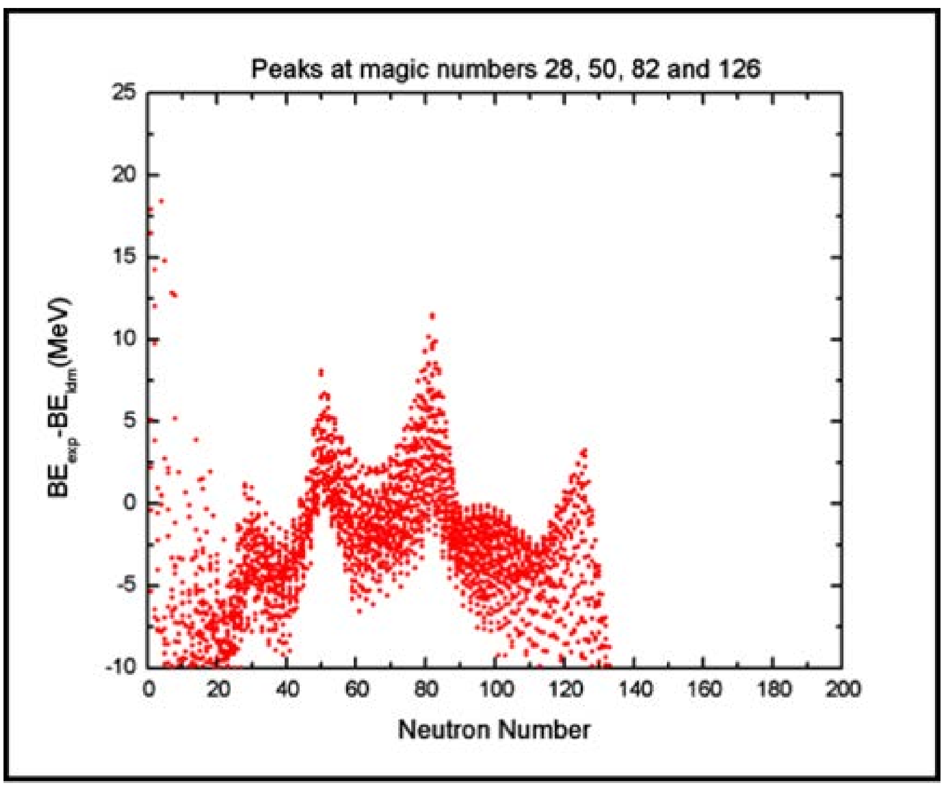


Image by MIT OpenCourseWare.

In fact, one of the in-class slides shows this very regular pattern in the error between the theoretical and the measured binding energies of all known isotopes:



It seems that the real (experimental) binding energy should be up to 10MeV higher when magic numbers are hit, and up to 10MeV lower halfway between the magic numbers. So, let's say that the new binding energy (BE_{new}) should be the old binding energy (BE_{old}) plus some correction factor. As a first guess, let's say that the binding energy should increase with proximity to any magic number, from either side. A Gaussian function should fit this very well, which takes on the following form:

$$y = a_m e^{\frac{-(x-n)^2}{2w_n^2}} \tag{23}$$

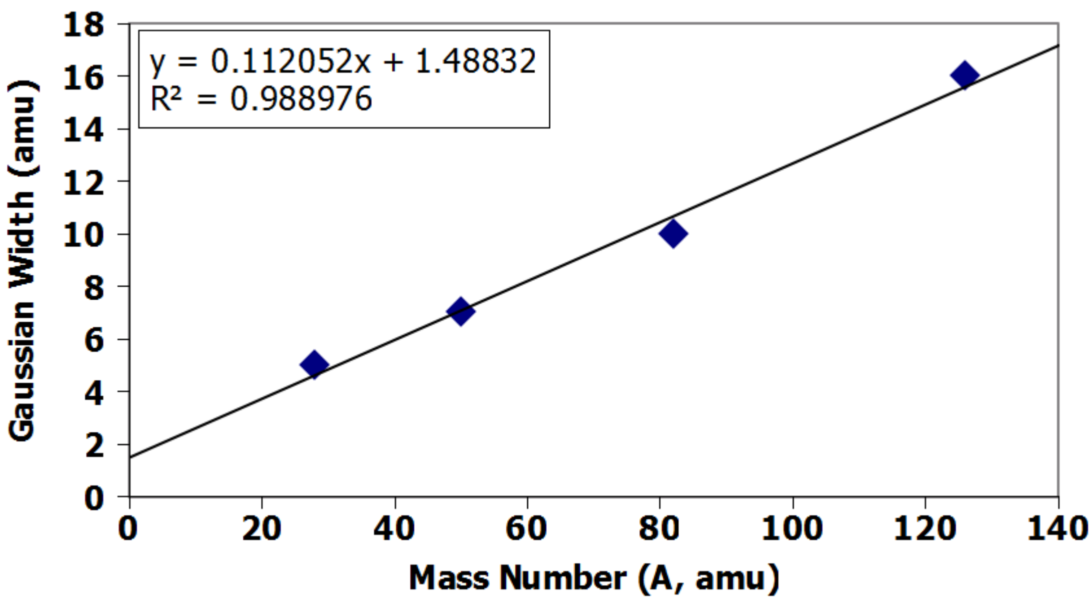
where we are denoting a_m as a new semi-empirical fitting factor constant ('m' stands for 'magic'), n is the particular magic number in question (which simply determines the location of the center of the Gaussian), and w_n represents the width of the Gaussian for magic number n. To determine a_m, we simply observe that all the Gaussian-like parts of the error in the figure above seem to have a total height of 20MeV, and that protons & neutrons can have their own magic numbers. Therefore we assume that we'll have two of these Gaussians multiplied to give a total correction of $\pm 20MeV$, so we set $a_m = \sqrt{20}\ MeV$. We should also make sure to subtract 10 MeV from the overall binding energy, so that the average error neither near nor far from a magic number will be zero. Then we can construct our correction term by noting that every magic number will impart its own little "island of stability," so we need a summation of similar Gaussian terms of the same form, with different locations and widths depending on the magic numbers, ***noting that this effect should work for both protons and neutrons separately***:

$$BE_{new} - BE_{old} = a_m \sum_{n=2,8,20,28,50,82,126...}^{all\,magic\,numbers} e^{\frac{-(x-n)^2}{2w_n^2}} \tag{24}$$

Now all that remains is to empirically choose the Gaussian half-widths w_n to make our data fit. It's very hard to see what the widths for the tiny magic numbers are, though starting at n=28, the widths appear to be:

n	28	50	82	126
w_n	5	7	10	16

The width for n=126 was gleaned from just guessing the left side width, as we don't have data on nuclides far to the right yet. Finally, we can empirically fit a function to these (scarce) data, to predict what the next Gaussian width will be. The graph below shows just how well a linear function approximates this data, with an equation of roughly $w_n = 0.11n + 1.5$:



Finally, we substitute this empirical equation into our binding energy correction term, to allow the semi-empirical mass formula to predict the binding energies of nuclides close to or far from any magic number. We also ensure that should we reach a "double magic" nucleus, :

$$BE_{new} - BE_{old} = a_m \left[\sum_{n_Z=Z}^{magic} e^{\frac{-(Z-n)^2}{2(0.11n_Z+1.5)^2}} \right] \left[\sum_{n_N=N}^{magic} e^{\frac{-(N-n)^2}{2(0.11n_N+1.5)^2}} \right] - 10; \quad a_m = \sqrt{20}\,MeV \tag{25}$$