22.01 Fall 2021, Quiz 1 Solutions

October 18, 2021

<u>Quiz Instructions:</u> Answers can be given symbolically or graphically, no calculation is necessary aside from order-of-magnitude math.

No devices, or anything else allowed, except for one calculator (graphing is OK, or calculator apps on your phone/tablet - I'm trusting you!) and one double-sided, 8.5 x 11 inch or A4 sheet of paper, or electronic equivalent.

Define any intermediate variables or symbols which you need to complete the problems. Generous partial credit will be given for correct methodology, even if the solution is not given.

You will have 24 hours of your choosing within a 48 hour window to complete the exam. The exam will be available starting 14:00 Tuesday Oct. 19, until 14:00 Thursday Oct. 21.

Please upload a PDF of your answers to the Canvas site at any time in the 48 hour exam window. You can type your answers, draw them on the computer, use paper and take pictures with your phone, or anything else you like.

---

1 (70 points) Short Answers, 10 points each

Each of these problems can be solved with one sentence, one equation, or one graph.

1.1 In which region would you look on the Table of Nuclides to find isotopes most likely to undergo positron decay? Choose from the following, and explain your reasoning: *slightly neutron-rich ($N > Z$), very neutron-rich ($N \gg Z$), slightly proton-rich ($Z > N$), very proton-rich ($Z \gg N$), heavy (high A), light (low A).*

Very proton-rich ($Z \gg N$)

Positron decay consumes protons and creates neutrons (and neutrinos), bringing the nucleus closer to proton-neutron symmetry. To get really technical (not required for full credit), this minimizes the asymmetry term in the semi-empirical mass formula, or maximizes the same term in the binding energy formulation of the same equation:

$$BE\left(A,Z\right) = ... - a_a \frac{\left(N-Z\right)^2}{A} + ... \tag{1}$$

*Finally, the reason it is **very** proton-rich and not **slightly** proton-rich is that more asymmetry (and lower binding energy) leads to a higher Q-value, and positron decay requires at least $Q \geq 1.022\ MeV$ to occur.*

1.2 Estimate, to an order of magnitude, how much heavier/lighter the meta-state of Technetium (^{99m}Tc) is compared to its ground state ^{99}Tc. The meta-state energy of ^{99m}Tc is at 0.143 MeV. Call it 0.1 MeV for simplicity.

Simple answer: One part per million.

The mass-energy of ^{99}Tc is roughly 99 nucleons times our conversion factor of 931.94937 $\frac{MeV}{amu-c^2}$. Let's call it 100 nucleons and 1000 $\frac{MeV}{amu-c^2}$. That gives us a rest mass energy of about $10^2 * 10^3 = 10^5 MeV$, and *an excited state energy of* $10^{-1}\,MeV$*. Divide these two for our answer:*

$$\frac{10^{-1}\,MeV}{10^{5}\,MeV} = 10^{-6} = 1\,ppm \tag{2}$$

The excited state of $^{99\mathrm{m}}\mathrm{Tc}$ *is about 1 part per million (ppm) heavier than its ground state of* $^{99}\mathrm{Tc}$*!*

1.3 Under what physical/mathematical condition(s) could an isotope theoretically decay by direct emission of a ^{14}C nucleus?

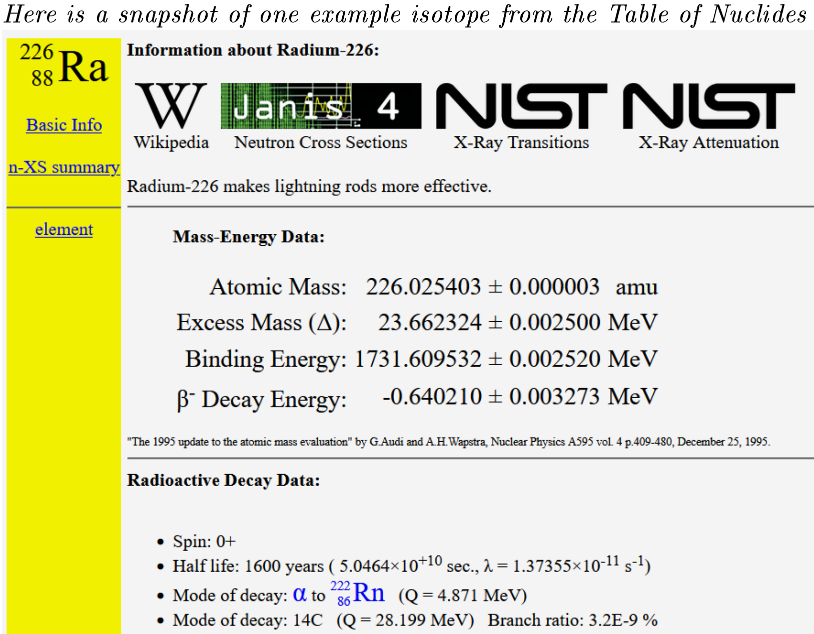
Same as everything: Q>0!

As long as the nuclear reaction has a positive Q-value (exothermic), literally anything is possible. There would probably be some activation energy to overcome for this to happen, akin to fission but not quite as strong, but you don't need to say so for full credit here.

Here is the equation which would describe this reaction:

$${}^{A}_{Z}P \rightarrow {}^{A-14}_{Z-6}D + {}^{14}_{6}C + Q \tag{3}$$

Here is a snapshot of one example isotope from the Table of Nuclides which actually does this!


$^{226}_{88}$Ra

Basic Info

n-XS summary

element

Information about Radium-226:

Wikipedia | Neutron Cross Sections | X-Ray Transitions | X-Ray Attenuation

Radium-226 makes lightning rods more effective.

Mass-Energy Data:

Atomic Mass: 226.025403 ± 0.000003 amu

Excess Mass (Δ): 23.662324 ± 0.002500 MeV

Binding Energy: 1731.609532 ± 0.002520 MeV

β⁻ Decay Energy: -0.640210 ± 0.003273 MeV

"The 1995 update to the atomic mass evaluation" by G.Audi and A.H.Wapstra, Nuclear Physics A595 vol. 4 p.409-480, December 25, 1995.

Radioactive Decay Data:

- Spin: 0+
- Half life: 1600 years ($5.0464\times10^{+10}$ sec., $\lambda = 1.37355\times10^{-11}$ s^{-1})
- Mode of decay: α to $^{222}_{86}$Rn (Q = 4.871 MeV)
- Mode of decay: 14C (Q = 28.199 MeV) Branch ratio: 3.2E-9 %

And here is a description of this phenomenon in general, known as "cluster radioactivity:" https://inis.iaea.org/collection/NCL

1.4 Why is "inelastic scattering (n, n')" not truly a scattering reaction? Use aspects of the generalized Q-equation (see Eq. 23) for your answer.

Simply put, $Q \neq 0$*.*

Scattering is an energy-conserving transfer of momentum from one particle to another, with that transfer dependent on scattering angle. Neutron inelastic "scattering" is actually neutron absorption, followed by the formation of a compound nucleus, and emission of a more bound neutron, leading to net energy absorption $(Q < 0)$ *by the nucleus.*

1.5 Why shouldn't we round nuclear masses/binding energies when identifying gamma peaks on a high-purity Germanium detector (HPGe) spectrum, like that of our Chernobyl honey? Assume these detectors have energy resolutions of ~0.5 keV.

The simple answer: Rounding errors and energy resolution

If you round even the sixth decimal place, because of the conversion factor of $\frac{931.49437\,MeV}{amu-c^2}$, your answer could be off by at least 1 keV, or twice the resolution of the detector. That could lead to mis-identification of gamma peaks!

1.6 Fukushima released about 25 PBq of radiation (2.5×10^{19} Bq), which is a HUGE amount and on par with that released from Chernobyl. By what math/logic would you explain to someone that fish in the ocean are still safe to eat?

The two-sentence answer: Specific activity.

To get a bit more nuanced, Chernobyl released lots of radiation which deposited on the ground and stayed relatively put, causing massive contamination of a smaller area of topsoil and rendering plants/animals nearby far too radioactive for safe consumption. By contrast, almost all of Fukushima's radiation release was into the ocean, either by direct water discharge or atmospheric discharge settling onto the water, where it got diluted by the very large volume of the ocean to levels far below the natural radioactivity of the ocean.

To get quantitative (not required for full credit), the ocean naturally contains about 11,000 $\frac{Bq}{m^3}$ of radioactivity, mostly from ^{40}K. The release from Fukushima added about 0.1% to this - a large absolute release, but a relatively small rise in specific activity.

1.7 Under what criterion could one theoretically observe alpha particle emission at an energy of $E_\alpha = Q$?

The simple answer: $M\,(A, Z) \to \infty$

Because of conservation of kinetic energy and momentum, one could only observe an alpha particle with kinetic energy equal to the total Q-value if the parent (and therefore daughter) nucleus had nearly infinite mass. This would allow the two particles to have equal momenta, while the infinite mass of the daughter nucleus would allow its velocity to approach zero, giving all the kinetic energy to the alpha particle.

2 **(30 points): Set up (do not solve) equations for, and sketch an approximate graph of, the amount of each of three isotopes (N_1, N_2, N_3) in a series decay chain inside a nuclear reactor with neutron flux Φ, where $\boldsymbol{N_3}$ is stable, and $\Phi\sigma_1 = \Phi\sigma_2 = \Phi\sigma_3 = \lambda_1 = \lambda_2$. Pay attention to amounts and rates of change of the isotopes at key points ($t = 0$, $t = \infty$, places where $\frac{dN_i}{dt} = 0$). Why can't you calculate it so easily from Equation 26 below, and why doesn't it *appear* that the total mass of the isotopes is conserved on this graph?**

Overall Philosophy: The big difference between the graphs we developed in class and this problem is that the extra neutron burning ($\Phi\sigma_i$) means that the rates of "decay" of N_1 and N_2 are twice as fast as they would be from natural decay, and that the stable isotope N_1 will still undergo "artificial decay" at half the rates of N_1 and N_2. This is because the removal rate of each isotope in the reactor is equal to the rate of natural, radioactive decay of N_1 and N_2.

Setting Up the Equations: The following equations would apply here, which will help us be more quantitative and exact in drawing our approximate solutions:

$$\frac{dN_1}{dt} = -\left(\lambda_1 + \Phi\sigma_1\right) N_1; \qquad N_1\left(t = 0\right) = N_0 \tag{4}$$

$$\frac{dN_2}{dt} = +\lambda_1 N_1 - \left(\lambda_2 + \Phi\sigma_2\right) N_2; \qquad N_2\left(t = 0\right) = 0 \tag{5}$$

$$\frac{dN_3}{dt} = +\lambda_2 N_2 - \lambda_3 N_3; \qquad N_3\left(t = 0\right) = 0 \tag{6}$$

Now let's use the simplifying assumption from the problem statement ($\lambda_1 = \lambda_2 = \Phi\sigma_1 = \Phi\sigma_2 = \Phi\sigma_3$), and set each equal to some simple value λ:

$$\frac{dN_1}{dt} = -2\lambda N_1; \qquad N_1\left(t = 0\right) = N_0 \tag{7}$$

$$\frac{dN_2}{dt} = \lambda N_1 - 2\lambda N_2; \qquad N_2\left(t = 0\right) = 0 \tag{8}$$

$$\frac{dN_3}{dt} = \lambda N_2 - \lambda N_3; \qquad N_3\left(t = 0\right) = 0 \tag{9}$$

Using these equations, we can start to set slopes equal to zero to find relative maxima of each curve without solving anything!

Getting Quantitative: It's important here to draw the starting slope of $\boldsymbol{N_2}$ as equal to half the negative of that of $\boldsymbol{N_1}$, and the starting slope of $\boldsymbol{N_3}$ should still be zero. This comes from plugging in $t = 0$ to the three equations above and reading off slopes directly:

$$\frac{dN_1\left(t = 0\right)}{dt} = -2\lambda N_0 \tag{10}$$

$$\frac{dN_2\left(t = 0\right)}{dt} = \lambda N_0 - 2\lambda\left(0\right) = \lambda N_0 \tag{11}$$

$$\frac{dN_3\left(t = 0\right)}{dt} = \lambda\left(0\right) - \lambda\left(0\right) = 0 \tag{12}$$

Isotope N_1 still follows a standard exponential decay solution:

$$N_1\left(t\right) = N_0 e^{-2\lambda t} \tag{13}$$

Next, let's plot when the amount of isotope N_2 reaches its maximum, or in other words, when $\frac{dN_2}{dt} = 0$:

$$\frac{dN_2}{dt} = 0 = \lambda N_1 - 2\lambda N_2 \tag{14}$$

$$\cancel{\lambda} N_1 = 2\cancel{\lambda} N_2 \tag{15}$$

Therefore the curve for isotope N_2 reaches its maximum when its value is half that of N_1 at the same time. It will then decay steadily afterwards, trending towards zero.

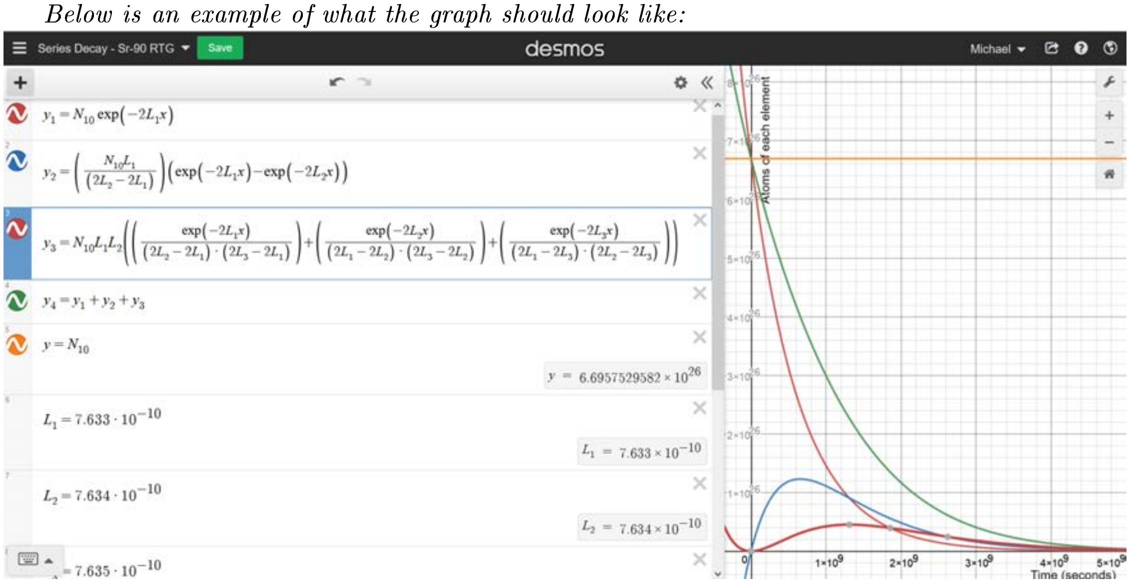
Finally, let's try seeing when N_3 reaches its peak (zero slope):

$$\frac{dN_3}{dt} = 0 = \lambda N_2 - \lambda N_3; N_2 = N_3 \tag{16}$$

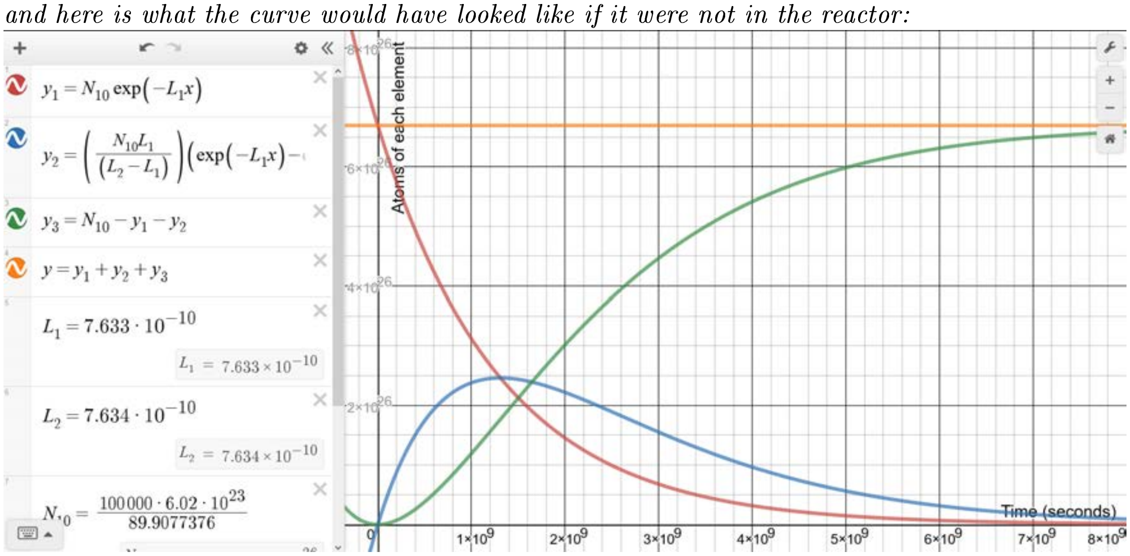
This means that N_3 will peak as it intersects the curve for N_2.

You should also draw that N_2 and N_3 should be allowed to build up somewhat, as this system of equations is not in either limiting case of $\lambda_2 \ll \lambda_1$ or $\lambda_2 \gg \lambda_1$.

Below is an example of what the graph should look like:



and here is what the curve would have looked like if it were not in the reactor:



Finally, mass doesn't <u>appear</u> to be conserved because the neutron burning going on in the reactor is transmuting our isotopes into others which we're not tracking. Energy is <u>always</u> conserved, just not in the form of only these three isotopes in this case!

For reference, here is the generalized form of the solutions to these equations as compactly as possible, as described by the original Harry Bateman himself back in 1910:

$$N_n\left(t\right) = N_{10}\left(\prod_{i=1}^{n-1}\lambda_i\right)\left(\sum_{i=1}^{n}\left(\frac{e^{-\lambda_i t}}{\prod\limits_{j=1;j\neq i}^{n}\left(\lambda_j-\lambda_i\right)}\right)\right) \tag{17}$$

Bonus Question (15 points) - The sun has a power of roughly 4×10^{26} W, powered by a chain of hydrogen fusion reactions, simplified as follows

$$6^1_1H \Longrightarrow 2^1_1H +^4_2He + 2\beta^+ + 2\nu + 2\gamma; \qquad Q = 20\,MeV \tag{18}$$

Estimate, to an order of magnitude, how many neutrinos enter you per second coming from the sun. Assume you subtend a solid angle of 2×10^{-24} Sr (steradians) from the center of the sun.

This is a two-step problem, both involving units. First we convert the sun's power to the number of nuclear fusion reactions happening per second:

$$\left[\frac{\nu}{s}\right] = \left(\frac{4\times 10^{26}\,\cancel{Joules}}{second}\right)\left(\frac{1\,\cancel{eV}}{\cancelto{2}{1.6}\times 10^{-19}\,\cancel{Joule}}\right)\left(\frac{1\,\cancel{MeV}}{10^6\,\cancel{eV}}\right)\left(\frac{\cancel{reaction}}{20\,\cancel{MeV}}\right)\left(\frac{2\,\nu}{\cancel{reaction}}\right) \tag{19}$$

Now that we know the units check out, we next accumulate numbers and orders of magnitude separately, to help make our mental math easier:

$$\left[\frac{\nu}{s}\right] = \left(\frac{4*\cancel{2}}{\cancel{2}*20}\right)\left(\frac{10^{26}}{10^{-19}*10^6}\right) = \left(\frac{1}{5}\right)\times 10^{39} = 2\times 10^{38}\,\frac{\nu}{s} \tag{20}$$

Now we have the number of anti-neutrinos emitted by the sun per second. Next, we take the solid angle subtended by us, and divide by the unit surface area of the sun. We assume (rightfully so) that the sun is spherical, so it has a unit surface area of 4π:

$$\frac{\nu\,through\,you}{second} = \left[\frac{\nu\,\cancel{from\,sun}}{second}\right]\left[\frac{\cancel{Sr}\,through\,you}{4\pi\,\cancel{Sr}\,\cancel{from\,sun}}\right] \tag{21}$$

Once again the units check out, so let's put in our values and do a bit of mental math:

$$\frac{\nu\,through\,you}{second} = \left[2\times 10^{38}\,\frac{\nu}{s}\right]\left[\frac{2\times 10^{-24}}{4\pi}\right] = \frac{1}{\pi}\times 10^{14} \tag{22}$$

Let's call it either 10^{13} or 10^{13}, either is acceptable since we're right on the edge. This agrees well with the often-cited value of 100 trillion $\left(100\times 10^{12}\right)$ neutrinos per second passing through each of us!

Useful Formulas

$$Q = T_3\left(1+\frac{M_3}{M_4}\right) - T_1\left(1-\frac{M_1}{M_4}\right) - \frac{2}{M_4}\sqrt{M_1M_3T_1T_3}cos\,\theta; \qquad \alpha = \left(\frac{A-1}{A+1}\right)^2 \tag{23}$$

$$A = \lambda N; \qquad BE\left(A,Z\right) = \left(ZM_H + \left(A-Z\right)M_n - M\left(A,Z\right)\right)c^2 \tag{24}$$

$$BE\left(A,Z\right) = a_vA - a_sA^{\frac{2}{3}} - a_c\frac{Z\left(Z-1\right)}{A^{\frac{1}{3}}} - a_a\frac{\left(N-Z\right)^2}{A} + \delta; \quad \delta = \left\{\begin{array}{c}\frac{a_p}{\sqrt{A}}\;even-even\\ 0\;even-odd\\ -\frac{a_p}{\sqrt{A}}\;odd-odd\end{array}\right\} \tag{25}$$

$$N_2\left(t\right) = \frac{N_0\lambda_1}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \qquad M = A + \Delta \qquad 1\,amu = 931.49\,\frac{MeV}{c^2} \tag{26}$$