22.01 Fall 2021, Quiz 2 Solutions

December 2, 2021

Quiz Instructions: Answers can be given symbolically or graphically, no calculation is necessary aside from mental order-of-magnitude math. ***No devices, or anything else allowed, except for one calculator and one double-sided, 8.5 x 11 inch or A4 sheet of paper.*** Define any intermediate variables or symbols which you need to complete the problems. Generous partial credit will be given for correct methodology, even if the solution is not given.

Please upload a PDF of your answers to the Canvas site at any time in the 24 hour exam period. You can type your answers, draw them on the computer, use paper and take pictures with your phone, or anything else you like.

Many useful formulas are included for you on the last page of the exam.

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1 (70 points) Short Answers, 10 points each

Each of these problems can be solved with one sentence, one equation, or one graph.

1.1 Which will cause more nuclear damage during ion irradiation of pure iron: 1 kg of protons, 1 kg of carbon ions, or 1 kg of iron ions? Why do you say so?

Short answer: Iron ions.

Explanation: Let's look at how the stopping formula scales with the two things that differ here: (1) ion mass and (2) ion atomic number. The useful formulas below give you the electronic stopping power and the electronic/nuclear stopping power ratios, which you'll have to put together:

$$-\frac{dT}{dx}_{ioniz.} = \frac{4\pi k_0^2 N Z_1^2 Z_2 e_c^4}{m_e v^2} ln\left(\frac{2m_e v^2}{\bar{I}}\right); \qquad \frac{-\left(\frac{dT}{dx}\right)_{ioniz.}}{-\left(\frac{dT}{dx}\right)_{nucl.}} = \frac{2M}{m_e Z}\frac{ln\left(\frac{\gamma_e E_i}{\bar{I}}\right)}{ln\left(\frac{\gamma E_i}{E_d}\right)} \tag{1}$$

Let's strip these down to the two things we care about: ion mass and ion atomic number. We can ditch the stuff in the natural log because it doesn't vary very quickly, forget all the constants which are identical, and throw the big-Z on top to get the relative scaling relations:

$$-\frac{dT}{dx}_{nucl.} \propto \frac{Z_2^2}{M} \tag{2}$$

Here you can see that nuclear stopping scales with atomic number squared, but only linearly with ion mass. Therefore the bigger the Z of the ion, the more damaging it will be, even accounting for the fact that 1 kg of iron ions contains a factor of A fewer ions than 1 kg of protons.

1.2 Using the two-group neutron criticality relation, predict what will happen to k_{eff} of a pure ^{239}Pu sphere which, after falling back to Earth upon a failed launch and heating up during re-entry, falls into an ice-cold region of the ocean. Just arrows on terms and justifications will do.

$$k_{eff} = \frac{\left(\bar{\nu}_f \overline{\Sigma}_{f_f} + \bar{\nu}_{th} \overline{\Sigma}_{f_{th}} \left[\frac{\overline{\Sigma}_{s_{f\to th}}}{\overline{D}_{th} B_g^2 + \overline{\Sigma}_{a_{th}}} \right]\right)}{\overline{\Sigma}_{s_{f\to th}} + \overline{\Sigma}_{a_f} + \overline{D}_f B_g^2}$$

Short answer: k_{eff} should go up because of (1) neutron reflection back into the sphere and (2) increased neutron moderation.

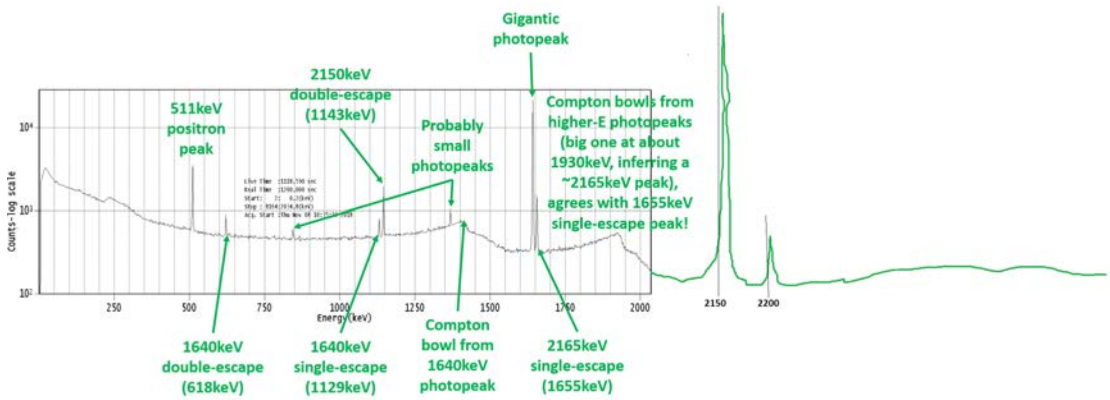
Explanation: Let's see what will change in each term of the equation. First, $\Sigma_{f_{th}}$ will go waaaay up because the neutrons will lose energy from moderation, and most fissile fuels love slow neutrons. That also is a part of absorption, so some of the thermal absorption will also go up, but at the same time thermal leakage $\left(D_{th} B_g^2\right)$ will go down from the water reflection so that stuff in the denominator will partly cancel itself out. Finally, the scattering from the fast to thermal energy group $\Sigma_{s_{f\to th}}$ will go way up due to the water on the outside, further increasing the fission rate. And even though that same $\Sigma_{s_{f\to th}}$ term is on the bottom of the whole equation, it's multiplied by $\nu_{th}\Sigma_{f_{th}}$ on the top so its contribution is much more powerful there. Therefore, the net effect is to increase criticality.

$$[\Uparrow k_{eff}] = \frac{\left(\bar{\nu}_f \overline{\Sigma}_{f_f} + \bar{\nu}_{th} \left[\Uparrow \overline{\Sigma}_{f_{th}}\right] \left[\frac{\Uparrow \overline{\Sigma}_{s_{f\to th}}}{\downarrow \left[\overline{D}_{th} B_g^2\right] + \uparrow \overline{\Sigma}_{a_{th}}} \right]\right)}{\Uparrow \overline{\Sigma}_{s_{f\to th}} + \uparrow \overline{\Sigma}_{a_f} + \downarrow \left[\overline{D}_f B_g^2\right]}$$

1.3 For the following photon spectrum, explain the physical origin(s) of each visible feature in terms of photon interactions.

... and ...

1.4 For the photon spectrum above, continue drawing what you would expect the spectrum to look like between 2000-3000 keV, based on what you can infer about the spectrum from the 0-2000 keV data.



1.5 Set up, but do not solve, a system of equations which quantifies the amount of ^{60}Co one would have as a function of time, by starting with a mass M_{59} of ^{59}Co in a nuclear reactor with a neutron flux Φ. Define any symbols you need to complete the equation. How do you determine the optimum time to harvest your ^{60}Co?

We'll just have to set up the series decay equations, assuming we put in stable ^{59}Co and it absorbs neutrons to make ^{60}Co:

$$\frac{dN_{59}}{dt} = -\sigma_{\gamma_{59}} \Phi N_{59} \tag{3}$$

$$\frac{dN_{60}}{dt} = \sigma_{\gamma_{59}} \Phi N_{59} - \left(\lambda_{60} + \sigma_{\gamma_{60}} \Phi\right) N_{60} \tag{4}$$

We'll also have to convert from a starting mass of ^{59}Co to get our initial value of the number of atoms of ^{59}Co:

$$N_{59}\left(t = 0\right) = \frac{M_{59} N_{Av}}{MM_{59}} \tag{5}$$

where N_{Av} is Avogadro's number in $\frac{atoms}{mol}$ and MM_{59} is the molar mass of ^{59}Co in $\frac{g}{mol}$. Finally we recognize that we'll have a maximum of ^{60}Co when its derivative is zero:

$$\frac{dN_{60}}{dt} = 0 = \sigma_{\gamma_{59}} \Phi N_{59} - \left(\lambda_{60} + \sigma_{\gamma_{60}} \Phi\right) N_{60} \tag{6}$$

$$\sigma_{\gamma_{59}} \Phi N_{59} = \left(\lambda_{60} + \sigma_{\gamma_{60}} \Phi\right) N_{60} \tag{7}$$

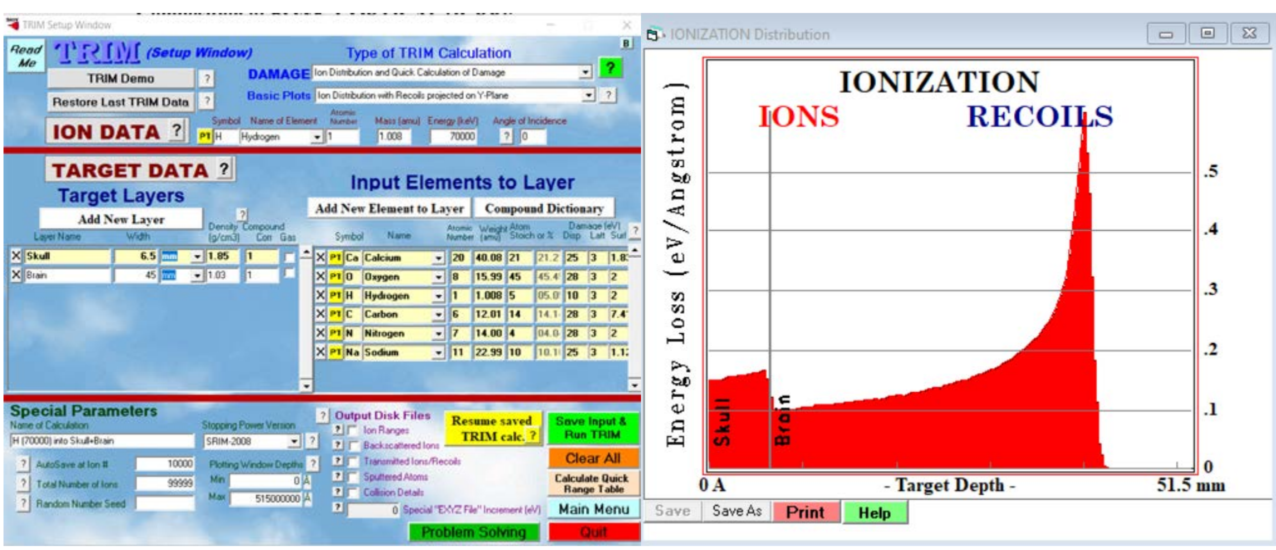
That's as far as you have to go!

1.6 Draw an approximate sketch of the energy deposition as a function of depth, resulting from a proton beam entering a human skull and stopping at a tumor at the center of the brain. Model this as a two-layer system (bone-brain).

I'll use SRIM to make this easier to follow. I'm also including a screenshot of the SRIM setup which produces this result. Any sketch which gets the following main points will get full credit:

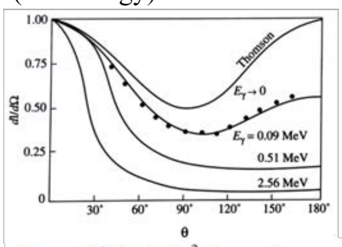
1. Bone is denser and has calcium (the heaviest element in the way), so its stopping power should be higher than the subsequent brain tissue.

2. The energy deposition should have a Bragg peak at the end of range of the tumor, after which the energy deposition should drastically drop off.

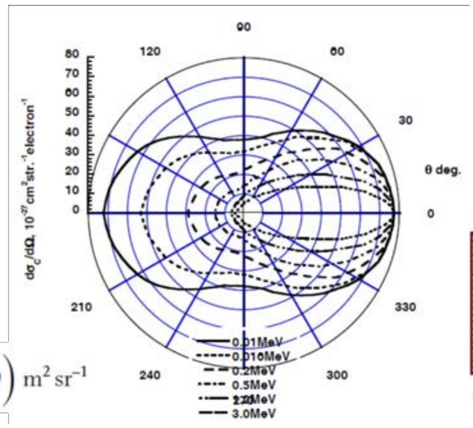


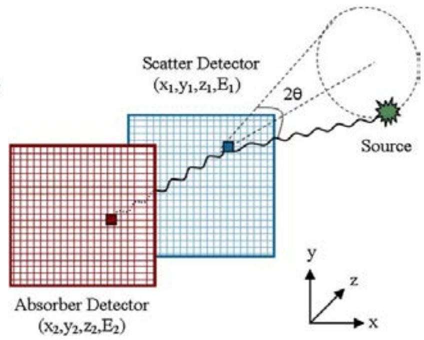
1.7 **A *Compton Camera Swarm* (which I think I just made up) consists of lots of drone-mounted detectors looking for coincident photon detections, using the relationship between Compton scattering energy and angle. Assuming the drone swarm starts as a sphere at the origin (of our arbitrary coordinate system), and it thinks it detects a photon coming somewhere from the +z direction, draw the ideal shape in which the swarm should rearrange itself to maximize the likelihood of further Compton scatter detections coming from the same direction. Use the Klein-Nishina angular photon cross section formula to guide your answer.**

A simplified version… and the Thomson scattering (low energy) limit



$$\frac{d_e\sigma}{d\Omega} = \frac{k_0^2 e^4}{2m^2c^4}\left(\frac{\nu'}{\nu}\right)^2\left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu} - \sin^2\theta\right)\ \mathrm{m^2\,sr^{-1}}$$





Short Answer: Draw a cone pointing in the +z direction.

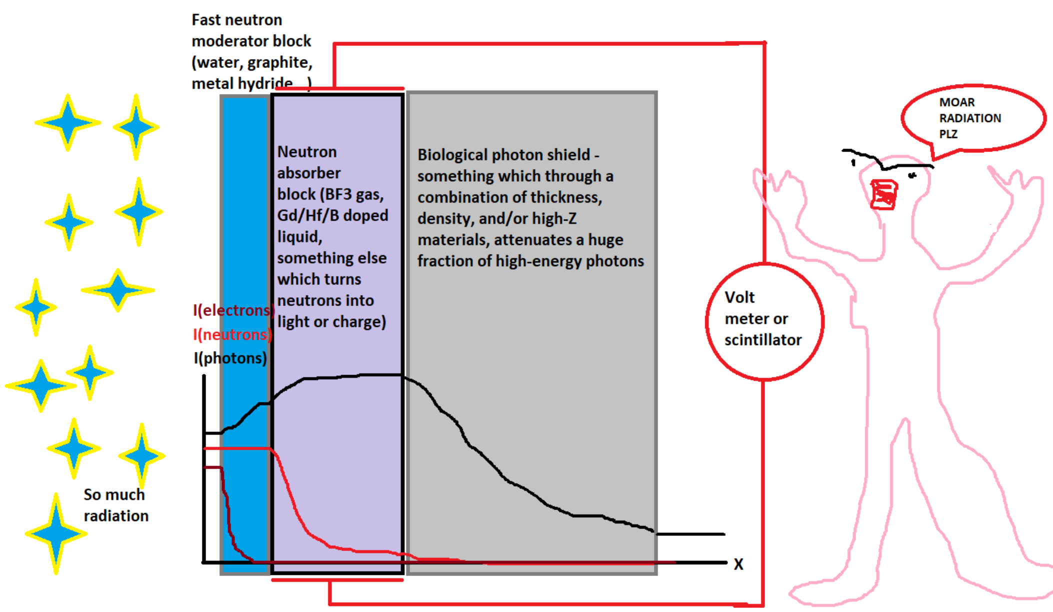
*Explanation: The Klein-Nishina cross section, which describes the likelihood of Compton scattering into different angles as a function of energy, predicts **forward scattering** as by far the most likely possibility. Therefore, if the swarm thinks there is a source in the +z direction, arranging itself into a thin cone with its vertex pointing towards +z will maximize the likelihood of catching those forward-scattered Compton photons.*

2 (15 points) Design a person-sized detector/shield combination, which is excellent at detecting the flux of high energy (fast) neutrons, amidst a radiation field consisting of high and low energy neutrons, electrons, and photons. Also take into account how to shield the detector operator from all forms of radiation present, who would be standing behind the detector. Assume the radiation source is in front of the detector, and the operator is standing behind it as a shield. You can use any combination of materials which you would like.

Fast neutrons are notoriously difficult to shield or catch, because all the cross sections are so low. However, they can be moderated to become slow neutrons, at which point a highly absorbing *thermal* neutron medium could catch them and function as the detector medium. The only other thing to mention is that the detector will require some sort of *voltage* to accelerate the ions created by neutron absorption into a current, which some system of electronics can count as charge proportional to fast neutron flux.

For shielding, the neutrons if properly and fully moderated won't pose an issue themselves, nor will the electrons, as both will be totally stopped in the detector moderator/absorber medium. However, neutrons can create many long-lived activation products which release gamma rays upon decay, and high-energy electrons can release lots of *bremsstrahlung* x-rays which must be shielded, just like the high and low-energy photons, respectively. Therefore, you'll want a thick/dense/high-Z enough shield behind this detector to *attenuate* a large enough fraction of these photons so as not to damage the operator. How much, exactly, we will do in the next section.

The following crude MS Paint sketch demonstrates a minimum drawing to receive full credit on this half of the problem:



(15 points) Next, make it quantitative: Assume that you want to shield the operator from 99% of all the radiation which would be present in front of the detector, assuming the operator is standing behind it. Assume fluxes of $\Phi_{n_{high-E}}$, $\Phi_{n_{low-E}}$, $\Phi_{e^-_{high-E}}$, $\Phi_{e^-_{low-E}}$, $\Phi_{\gamma_{high-E}}$, $\Phi_{\gamma_{low-E}}$. Set up, but do not solve, the equations which would allow you to determine the thicknesses of the detector layers to ensure at least 99% shielding from each and every form of radiation.

Let's start with the easiest one - the electrons! Because they are charged particles, we simply need to make sure that the range of the first layer (the neutron moderating layer) will exceed the range of the electrons in that medium. This will both shield the user from the electrons and stop them from affecting the fast neutron signal. We use the range equation to determine this, which is the integral of the inverse of the total stopping power:

$$Range = \int_0^{E_i} -\left(\frac{dT}{dx}\right)^{-1}_{total}; \qquad x_{moderator} > Range \tag{8}$$

We also note that the total stopping power is the sum of the electronic, nuclear, and radiative contributions:

$$\left(\frac{dT}{dx}\right)_{total} = \left(\frac{dT}{dx}\right)_{elec.} + \left(\frac{dT}{dx}\right)_{nucl.} + \left(\frac{dT}{dx}\right)_{brem.} \tag{9}$$

and the electronic stopping power, plus the ratios between it and the other two types, are in the Useful Formulas section below. Therefore this part is done.

Next, let's do the neutrons. We recall that we can write that neutrons are attenuated using their *macroscopic total cross section* Σ_t, in a very similar way as for photons:

$$\Phi_n(x) = \Phi_{n_0} e^{-\Sigma_t x} \tag{10}$$

However, we do not know whether the first moderating layer will effectively stop all the low-energy neutrons, and we know it will create more via fast neutron moderation. Thus we need to make sure the thickness of the highly absorbing, neutron detecting layer combined with the moderating layer is thick enough to stop at least 99% of all fast neutrons coming in with the following two equations:

$$\frac{\Phi_{n,fast}(x_{moderator} + x_{detector})}{\Phi_{n,fast}(0)} \leq 0.01; \qquad 0.01 = \left(e^{-\Sigma_{t,moderator}(E_{high})x_{moderator}}\right)\left(e^{-\Sigma_{t,detector}(E_{high})x_{detector}}\right) \tag{11}$$

Here we note that we multiply the two attenuating factors, because whatever fraction of neutrons make it through the moderator layer $\left(e^{-\Sigma_{t,moderator}(E_{high})x_{moderator}}\right)$ must be further cut down by the detector layer $\left(e^{-\Sigma_{t,detector}(E_{high})x_{detector}}\right)$ such that the *multiple* of these fractions is less than 0.01, representing 99% attenuation. We also note that moderation (elastic scattering) is part of the total cross section, thus moderation is accounted for. The same equation applies for the slow neutrons... except that we are creating TONS of extra slow neutrons by moderating the fast ones! Therefore, to be on the safe side, we need to make sure that the attenuation of *only the detector layer* will catch at least 99% of all slow neutrons which enter it:

$$\frac{\Phi_{n,slow}(x_{moderator} + x_{detector})}{\Phi_{n,slow}(x_{moderator})} \leq 0.01; \qquad 0.01 = \left(e^{-\Sigma_{t,detector}(E_{low})x_{detector}}\right) \tag{12}$$

This both ensures that the detector acts as a good shield for slow neutrons *and* captures at least 99% of them to turn them into signal. To finish off the neutrons, we need to estimate the fraction of extra slow neutrons created in the moderator layer, which we can do with the *scattering cross section* $\Sigma_s(E_{high})$:

$$\Phi_{n,slow}(x_{moderator}) = \overbrace{\Phi_{n,slow}(0)\left(e^{-\Sigma_{t,moderator}(E_{slow})x_{moderator}}\right)}^{unattenuated\,slow\,neutrons} + \overbrace{\Phi_{n,fast}(0)\left(e^{-\Sigma_{s,moderator}(E_{fast})x_{moderator}}\right)}^{newly\,created\,slow\,neutrons} \tag{13}$$

Let's not worry about neutrons scattering in many directions, as (1) we'll assume this is an 1D problem and (2) that gives us an assumption in the conservative direction.

Finally, let's think about the photons. Neutron capture in the first two layers can create more photons of both energies, and we need to make the final shield thick enough to attenuate 99% of both photons coming from the radiation source and those generated by nuclear reactions in the first two layers. We'll set up equations just like the neutron ones, except with an extra layer where things can be created, both from electron bremsstrahlung and neutron nuclear reactions.

$$\frac{\Phi_{\gamma,fast}\left(x_{moderator}+x_{detector}+x_{shield}\right)}{\Phi_{\gamma,fast}\left(0\right)} \leq 0.01 \tag{14}$$

$$0.01 = \left(e^{-\mu_{t,moderator}(E_{high})x_{moderator}}\right)\left(e^{-\mu_{t,detector}(E_{high})x_{detector}}\right)\left(e^{-\mu_{t,shield}(E_{high})x_{shield}}\right) \tag{15}$$

$$\begin{aligned}\Phi_{\gamma,low}\left(x_{moderator}\right) = {} & \overbrace{\Phi_{\gamma,low}\left(0\right)\left(e^{-\mu_{t,moderator}(E_{low})x_{moderator}}\right)\left(e^{-\mu_{t,detector}(E_{low})x_{detector}}\right)}^{unattenuated\, low-E\, photons} \\ & \overbrace{+\Phi_{n,fast}\left(0\right)\left[\left(e^{-\Sigma_{(n,\gamma),moderator}(E_{fast})x_{moderator}}\right)+\left(e^{-\Sigma_{(n,\gamma),detector}(E_{fast})x_{detector}}\right)\right]}^{newly\, created\, photons\, from\, fast\, neutrons} \\ & \overbrace{+\Phi_{n,slow}\left(0\right)\left[\left(e^{-\Sigma_{(n,\gamma),moderator}(E_{slow})x_{moderator}}\right)+\left(e^{-\Sigma_{(n,\gamma),detector}(E_{slow})x_{detector}}\right)\right]}^{newly\, created\, photons\, from\, slow\, neutrons} \\ & \overbrace{+\Phi_{e^-,fast}\left(0\right)\left[\frac{\frac{dT}{dx}_{brem}}{\frac{dT}{dx}_{total}}\right]\left[\frac{brem.\, photons}{fast\, electron}\right]}^{newly\, created\, photons\, from\, fast\, electrons}\end{aligned} \tag{16}$$

You don't need to give an expression for the last part per se since we didn't give you any formulas, just a unit analysis will do.

Useful Formulas

$$\Delta\lambda = \frac{h}{m_e c}\left(1 - cos\theta\right) \qquad T_{e^-} = h\nu - h\nu' = h\nu\frac{1 - cos\theta}{\frac{m_e c^2}{h\nu} + 1 - cos\theta}$$

$$I = I_0 e^{-\left(\frac{\mu}{\rho}\right)\rho x} = I_0 e^{-\Sigma x} \qquad \frac{d\sigma_C}{d\Omega} = \frac{k_0^2 e^4}{2m_e^2 c^4}\left(\frac{\nu'}{\nu}\right)^2\left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu} - sin^2\theta\right)$$

$$-\frac{dT}{dx} = \frac{4\pi k_0^2 N Z_1^2 Z_2 e_c^4}{m_e v^2} ln\left(\frac{2m_e v^2}{\bar{I}}\right) = \frac{2\pi k_0^2 N Z_1^2 Z_2 e_c^4}{E_i}\frac{m_{ion}}{m_e} ln\left(\frac{\gamma_e E_i}{\bar{I}}\right);\ \gamma_e = \frac{4m_e M}{\left(M + m_e\right)^2}$$

$$\frac{-\left(\frac{dT}{dx}\right)_{ioniz.}}{-\left(\frac{dT}{dx}\right)_{nucl.}} = \frac{2M}{m_e Z}\frac{ln\left(\frac{\gamma_e E_i}{\bar{I}}\right)}{ln\left(\frac{\gamma E_i}{E_d}\right)}; \qquad \gamma = \frac{4mM}{\left(m + M\right)^2} \qquad \frac{-\left(\frac{dT}{dx}\right)_{rad.}}{-\left(\frac{dT}{dx}\right)_{ioniz.}} = \left(\frac{m_e}{M}\right)^2\left(\frac{ZE_i}{1400 m_e c^2}\right) \qquad Range = \int_0^{E_i} -\left(\frac{dT}{dx}\right)^{-1}$$

$$\frac{1}{v}\frac{d\phi\left(\mathbf{r}, E, \mathbf{\Omega}, t\right)}{dt} = \frac{\chi\left(E\right)}{4\pi}\int_V\int_{E'}\int_{\Omega'} d^3 r dE' \boldsymbol{d\Omega'} \nu\left(E'\right)\Sigma_f\left(E'\right)\phi\left(\mathbf{r}, E', \mathbf{\Omega'}, t\right)$$

$$+S_0\left(\mathbf{r}, E, \mathbf{\Omega}, t\right) + \int_V\int_{E'}\int_{\Omega'} d^3 r dE d\mathbf{\Omega'}\Sigma_s\left(E'\right)\phi\left(\mathbf{r}, E', \mathbf{\Omega'}, t\right) F\left(E' \to E, \mathbf{\Omega'} \to \mathbf{\Omega}\right)$$

$$-\int_V d^3 r dE d\mathbf{\Omega}\Sigma_t\left(E\right)\phi\left(\mathbf{r}, E, \mathbf{\Omega}, t\right) - \int_V d^3 r dE \boldsymbol{d\Omega}\mathbf{\Omega}\cdot\boldsymbol{\nabla}\phi\left(\mathbf{r}, E, \mathbf{\Omega}, t\right)$$

$$\bar{\Sigma} = \frac{\int_0^{E_{max}} \Sigma\left(E\right)\Phi\left(E\right) dE}{\int_0^{E_{max}} \Phi\left(E\right) dE} \qquad B^2_{material} = \frac{\nu\Sigma_f - \Sigma_a}{D} \qquad B_g^2 = \left(\frac{\pi}{a_{ex}}\right)^2\ or\ \left(\left(\frac{\nu_1}{R_{ex}}\right)^2 + \left(\frac{\pi}{H}\right)^2\right)\ or\ \left(\frac{\pi}{R_{ex}}\right)^2$$

$$\Sigma_a\Phi - D\nabla^2\Phi = \frac{\nu\Sigma_f\Phi}{k_{eff}} \implies \Sigma_a\Phi - DB^2\Phi = \frac{\nu\Sigma_f\Phi}{k_{eff}} \implies k_{eff} = \frac{\nu\Sigma_f}{\Sigma_a + DB^2}$$

$$\frac{-\nabla^2\Phi}{\Phi} = B^2_{geometry}; \quad ,\Phi\left(x\right) = A\,cos\left(B_g x\right) + G\,sin\left(B_g x\right)\ \ in\,Cartesian\,coordinates$$

$$k_{eff} = \frac{\left(\bar{\nu}_f\overline{\Sigma}_{f_f} + \bar{\nu}_{th}\overline{\Sigma}_{f_{th}}\left[\frac{\overline{\Sigma}_{s_{f\to th}}}{\overline{D}_{th}B_g^2 + \overline{\Sigma}_{a_{th}}}\right]\right)}{\overline{\Sigma}_{s_{f\to th}} + \overline{\Sigma}_{a_f} + \overline{D}_f B_g^2} \qquad Q = 0 = T_f\left(1 + \frac{1}{A}\right) - T_i\left(1 - \frac{1}{A}\right) - \frac{2}{A}\sqrt{T_i T_f}cos\theta \quad (17)$$