

# 22.02 Intro to Applied Nuclear Physics

## Mid-Term Exam

Thursday March 18, 2010

### Problem 1: Short Questions

40 points

1. *What is an observable in quantum mechanics and by which mathematical object is represented?*  
An observable in QM is any physical quantity that can be measured. Observables are represented by hermitian operators.
2. *How are the possible values of measurement outcomes of an observable determined?*  
When measuring an observable, the possible outcomes of the measurement are given by the eigenvalues of the operator associated with the observable, thus one has to solve the eigenvalue equation for the observable's operator. Since the observable is an hermitian operator, these eigenvalues are real.
3. *What is the probability of finding a particle described by the wavefunction  $\psi(x, y, z)$  in a small volume  $dV = dx dy dz$  around the position  $\vec{r} = \{\tilde{x}, \tilde{y}, 0\}$ ?*  
The probability of finding the particle in a small volume  $dV = dx dy dz$  around the position  $\vec{r} = \{\tilde{x}, \tilde{y}, 0\}$  is  $|\psi(\tilde{x}, \tilde{y}, 0)|^2 dx dy dz$ .
4. *Is the evolution of a quantum mechanical system a stochastic (random) process?*  
No, the evolution of a QM system is deterministic (and governed by the time-dependent Schrödinger equation). Only the result of a measurement is a stochastic variable.
5. *For a system described by the wavefunction  $\psi(x, t = t_1) = a_1 w_1(x) + a_{91} w_{91}(x)$  (where  $w_n$  are energy eigenfunctions with eigenvalues  $E_n$ ) what is the probability of measuring an energy  $E = E_{91}$  at time  $t = t_1$ ? What about at time  $t = 2t_1$ ?*  
If we express a wavefunction as a linear combination of an observable eigenfunctions  $\psi = \sum_n a_n u_n$ , the probability of obtaining the  $n^{\text{th}}$  eigenvalue is given by  $|a_n|^2$ . Here then we have  $p(E_{91}, t = t_1) = |a_{91}|^2$ . The probability of a particular energy does not change with time. More precisely, the wavefunction at a later time would be given by  $\psi(x, t) = a_1 w_1(x) e^{-iE_1(t-t_1)/\hbar} + a_{91} w_{91}(x) e^{-iE_{91}(t-t_1)/\hbar}$ , but the probability  $p(E_{91}, t = t_2) = |a_{91} e^{-iE_{91}(t_2-t_1)/\hbar}|^2 = |a_{91}|^2$  does not change
6. *Consider a particle described by the wavefunction  $\varphi(x)$  and an Hamiltonian with eigenvalues  $E_n$  and corresponding eigenfunctions  $w_n(x)$ . Is the probability of obtaining the energy  $E_n$  given by  $p(E_n) = \int_{-\infty}^{\infty} dx w_n^*(x) \varphi(x)$ ?*  
No, the probability is given by the modulus square:  $p(E_n) = |\int_{-\infty}^{\infty} dx w_n^*(x) \varphi(x)|^2$  (the inner product corresponds to calculating the coefficient  $a_n$  in the expansion  $\varphi(x) = \sum_n a_n w_n$ ).
7. *A particle is in the quantum state  $\psi = B \cos(\sqrt{2}x)$ .*  
a) *What are the possible results of a momentum measurement?*  
b) *What are the probabilities of each possible momentum measurement?*  
The possible outcomes of a momentum measurement are all the momentum eigenvalues  $p = \hbar k$ . The probability of each outcome is found as above by first calculating  $c_n = \int_{-\infty}^{\infty} dx u_n^*(x) \psi(x)$ , where  $u_n(x) = e^{\pm i k_n x}$  are the eigenfunctions of the momentum and then taking the modulus square. Instead of calculating the integral, it is easier to realize that  $\psi$  can be directly be written as a linear combination of momentum eigenfunctions:  $\psi = \frac{B}{2} (e^{i\sqrt{2}x} + e^{-i\sqrt{2}x})$ . Then there are only two outcomes with non-zero probability,  $\sqrt{2}\hbar$  and  $-\sqrt{2}\hbar$ , both with probability  $\frac{1}{2}$ .
8. *A particle is in the quantum state,  $\psi = A e^{-i79kx}$ .*  
a) *What are the possible results of a momentum measurement?*  
b) *What are the probabilities of each possible momentum measurement?*  
c) *What physical situation is represented by this quantum state?*  
As in the previous question, we obtain here that the only measurement outcome with non-zero probability is  $p = -79\hbar k$ , with probability 1. Notice that this wavefunction cannot be normalized. This state represents a traveling wave (from right to left).

9. A particle is in the angular momentum eigenstate,  $\psi = |l, m_z\rangle = |5, -4\rangle$ .
- a) What would a measurement of the total angular momentum,  $L^2$ , yield?  
The eigenvalues of  $L^2$  are  $\hbar^2 l(l+1)$ , thus we would measure  $30\hbar^2$
- b) What would a measurement of the z-component of angular momentum,  $L_z$ , yield?  
The eigenvalues of  $L_z$  are  $\hbar m_z$ , thus we would measure  $-4\hbar$
- c) What would a measurement of the x-component of angular momentum,  $L_x$ , yield?  
Since the state is not in an eigenfunction of the  $L_x$  angular momentum, the result is a priori unknown. Because  $L_z$  is known with absolute certainty, there is a maximum uncertainty in  $L_x$ . However, since the length of the angular momentum and its projection in the z direction are given, the possible values of  $L_x$  are limited. From  $L^2 = L_z^2 + L_y^2 + L_x^2$ , and assuming  $L_y^2 = 0$  to find the maximum value for  $L_x^2$  (in absolute value) we obtain:  $L_x^2 \leq L^2 - L_z^2 = \hbar^2[l(l+1) - m_z^2] = \hbar^2(30 - 16) = 14\hbar^2$ . Then  $-\sqrt{14}\hbar \leq L_x \leq \sqrt{14}\hbar$ , with possible values of  $L_x$  given by  $\hbar m_x$ , with  $m_x$  integers. Finally we have then  $m_x = \{-3, -2, -1, 0, 1, 2, 3\}$ .
10. A nucleus consists of two spin 1/2 nucleons,  $s_1 = \frac{1}{2}$ , and,  $s_2 = \frac{1}{2}$ . Both nucleons are in the orbital angular momentum  $l = 0$ .
- a) How many spin states are there for each nucleon?  
Each nucleon can be in two states  $|\frac{1}{2}, +\frac{1}{2}\rangle = |\uparrow\rangle$  and  $|\frac{1}{2}, -\frac{1}{2}\rangle = |\downarrow\rangle$ , since  $-s \leq m_s \leq s$ .
- b) How many spin states does the system have (based on the uncoupled representation)?  
In the uncoupled representation we can list four possible states:  $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ .
- c) Which quantum numbers would you use to label the coupled representation states? In the coupled representation a complete set of commuting observables is given by  $S^2, S_z, S_1^2$  and  $S_2^2$ , thus good quantum numbers are  $s, m_s, s_1, s_2$  (in this case, we can omit  $s_1$  and  $s_2$  since they're always  $\frac{1}{2}$  and write a state as  $|s, m_s\rangle$ ).

## Problem 2: Eigenvalue problem

15 points

The quantum mechanical observable Grades on the 22.02 Mid-Term has the eigenvalue problem,

$$\hat{G}\varphi_k = g_k\varphi_k, \quad k = 1, 2, 3, 4$$

with  $g_1 = A, g_2 = B, g_3 = C, g_4 = D$  and where  $\varphi_k$  are normalized eigenfunctions. If the state of the system is

$$\psi_{\text{class}} = c_1\varphi_1 + \frac{1}{\sqrt{5}}\varphi_2 + \frac{2}{5}\varphi_3 + \frac{1}{5}\varphi_4$$

and assuming that the usual rules of Quantum Mechanics apply and that there are only 4 possible grades on the exam,

a) What is the probability of an A grade outcome?

The probability is given by  $|c_1|^2$ . Since the state needs to be normalized, we have  $|c_1|^2 = 1 - (\frac{1}{5} + \frac{4}{25} + \frac{1}{25}) = 1 - \frac{10}{25} = \frac{3}{5}$ .

b) What is the average grade of the exam? (you can take  $A=5, B=4, C=3, D=2$ ).

The average grade (or expectation value) is  $5\frac{3}{5} + 4\frac{1}{5} + 3\frac{4}{25} + 2\frac{1}{25} = \frac{109}{25} \approx B^+$ .

c) Consider now the operator  $\widehat{PF}$  (Pass/Fail), that obeys the following rules:

$$\widehat{PF}\varphi_k = P\varphi_k, \quad \text{for } k = 1, 2, 3$$

$$\widehat{PF}\varphi_4 = F\varphi_4$$

where  $P$  and  $F$  are real number. Do the operators  $\hat{G}$  and  $\widehat{PF}$  commute? Give a mathematical reason for your answer. Since the two operators (Grade and Pass/Fail) share common eigenfunctions, they commute.

More precisely: since the two observable share common eigenfunctions, we can write

$$\widehat{PF}\hat{G}\varphi_k = g_k\widehat{PF}\varphi_k = g_k(PF)_k\varphi_k$$

(where  $(PF)_k = P$  for  $k = 1, 2, 3$  and  $= F$  for  $k = 4$ . We also have

$$\hat{G}\widehat{PF}\varphi_k = \hat{G}(PF)_k\varphi_k = (PF)_k\hat{G}\varphi_k = (PF)_kg_k\varphi_k.$$

Thus we proved that for the 4 eigenfunctions:  $[\widehat{PF}, \hat{G}]\varphi_k = 0$ . Now, to state that  $[\widehat{PF}, \hat{G}] = 0$ , we need to prove that  $[\widehat{PF}, \hat{G}]f = 0$  for any function  $f$ . However, the eigenfunctions form a basis, so that any other function in the space can be written as a linear combination of  $\varphi_k$ 's:  $f = \sum_k c_k\varphi_k$ . It's then easy to prove that the previous relation  $\widehat{PF}\hat{G}f = \hat{G}\widehat{PF}f$  is valid for any function  $f$ .

### Problem 3: Scattering

25 points

**a)** (15 points) In the following figures you can see for a given potential profile a set of eigenfunctions (left column) and a set of possible total energy values (right column) describing different scenarios of scattering for a particle incoming from the left side (here I plot the real part of the eigenfunction). You should match each figure in the left column with the corresponding one in the right column (that is, indicate what energy of the particle gives rise to which scattering behavior).

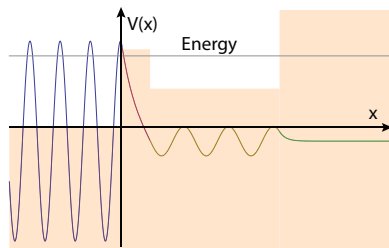
There are four regions of different potential heights. The behavior of the particle wavefunction will be different in every region depending if the total energy is larger or smaller than the potential.

**A-4:** The energy is  $E < V$  in region II and IV. Thus we expect the wave function to be an exponential decay in these regions and a traveling wave in region I and III. The particle is a wave traveling from left to right in region I, that reflects back and get transmitted into region III at the boundary. Its amplitude decreases in region II since there  $E < V$ , yielding a negative kinetic energy and an imaginary wave-number. After emerging into region III, the particle is again represented by a wave, with a reduced amplitude and with a smaller wave-number (hence a larger wavelength:  $\lambda_{III} > \lambda_I$ ). Finally the wavefunction penetrates in region IV, were it exponentially decays since again  $E < V$  there, with  $\kappa_{IV} > \kappa_{II}$ .

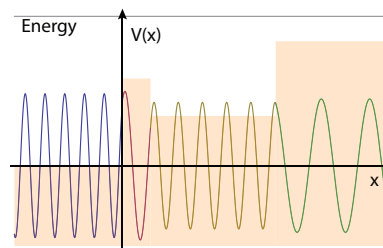
**B-3:** Here the energy is larger than the potential in all regions. We thus expect a traveling wave everywhere. The amplitude is slightly larger in region I because here there is both an incoming and a reflected component (exact amplitude ratios can be calculated by solving the full problem). The wavelengths in the four regions reflect the variation in the difference  $E - V$ : we have  $\lambda_{IV} > \lambda_{II} > \lambda_{III} > \lambda_I$ . At each step in the potential we will have a priori reflection and transmission of the wave.

**C-2:** As  $E < V$  only in region IV we have a traveling wave (with different wavelengths) in the first three regions followed by an exponential decay in the 4th. Again we have  $\lambda_{II} > \lambda_{III} > \lambda_I$ , with  $k_{IV}$  an imaginary number. The exact amplitudes could be calculated by solving the full eigenvalue problem with the given boundary conditions, as well as setting some “initial conditions” (for example taking a situation that describes a particle coming from the left). At each step in the potential we will have a priori reflection and transmission of the wave.

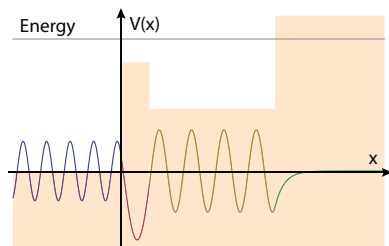
**D-1:** In this case the energy is larger than the potential only in region I, thus the particle, initially a wave incoming from the left, get reflected at the first barrier and penetrates in region II, with an exponential decay. The rate of the decay varies in the three regions, (with  $\kappa = \sqrt{2m(V - E)}/\hbar$  we have:  $\kappa_{IV} > \kappa_{II} > \kappa_{III}$ ) although the decay is so fast that the drawing does not show the changes.



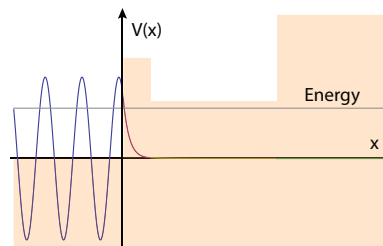
A-4



B-3



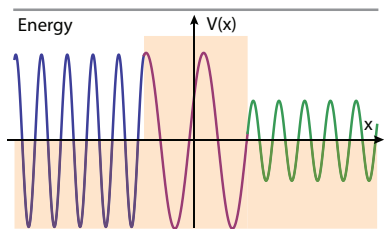
C-2



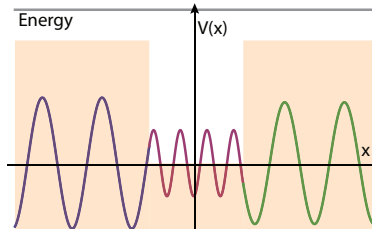
D-1

**b)** (10 points) Match the 1D scattering energy eigenfunctions on the left with the correct potential profile (if any). Provide an explanation for your answer.

Here the telling characteristic is the wavelength of the wavefunction in the three regions. We expect the wavefunction to always be a traveling wave, because the energy is always larger than the potential. The wavelength in region I and III is the same ( $\lambda = \frac{2\pi\hbar}{\sqrt{2mE}}$ ), while in region II we expect a longer wavelength for the case A-2 (since  $E - V_{II} < E$



A-2



B-1

in that case) and a shorter wavelength for case B-1 (as there the potential is s.t.  $E - V_{II} > E - V_I$ , giving a larger kinetic energy, a larger wavenumber  $k$  and short wavelength  $\lambda = 2\pi/k$ ).

### Problem 4: Alpha Particle Tunneling

20 points

A beam of alpha particles is directed at a potential barrier as indicated in the figure below. The beam has a high flux of,  $\Gamma = 6 \times 10^{21}$  particles/sec. The energy of the alpha particles is  $E = 5$  MeV, and the potential barrier height is,  $V_B = 85$  MeV and its width is  $L = 10$  fm. You may assume the alpha rest mass to be  $m_\alpha c^2 = 4000$  MeV.

a) Make a drawing of the eigenfunction for this problem, assuming the alpha particles are shot in from the left.

The particle will be represented by a traveling wave from the left (here I plot the real part of the wavefunction) with an amplitude  $A$  for the incoming flux and  $R$  for the reflected one traveling from right to left. Inside the barrier the wavefunction has a complex wave-number  $\kappa = \sqrt{2m(V_B - E)}/\hbar$ , thus it is represented by a decaying exponential. The amplitude after the barrier is approximately  $|A|^2 P_{tun}$  so it will be much smaller. However the particle will still be represented by a traveling wave (only a component traveling from left to right) and it will have the same wavelength of the original wave,  $\lambda_1 = \lambda_3$ .

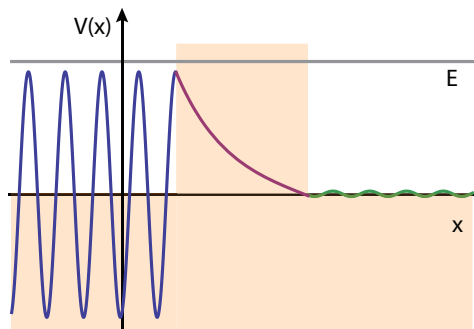


Figure 1: Scattering potential and wavefunction

b) Estimate the probability of tunneling,  $P_{tun}$ , through the barrier. Write the generic formula for tunneling and then estimate the numbers quantitatively.

The approximate tunneling formula is given by  $P_{tun} = 4e^{-2\kappa L}$  (as seen in class, or  $P_{tun} = e^{-2\kappa L}$  as some of you might have seen from 8.04. Since both these formulas are valid when  $P_{tun} \approx 0$  they are equivalent).

We have  $\kappa = \sqrt{2m(V_B - E)}/\hbar = \sqrt{2mc^2(V_B - E)}/(\hbar c) = \frac{\sqrt{2 \cdot 4000 \text{ MeV} \cdot (85 - 5) \text{ MeV}}}{200 \text{ MeV fm}} = \frac{\sqrt{8000 \cdot 80}}{200} \text{ fm}^{-1} = 4 \text{ fm}^{-1}$ . Thus, the probability is  $P_{tun} = 4e^{-2 \cdot 4 \text{ fm}^{-1} \cdot 10 \text{ fm}} = 4e^{-80} \approx 4 \times 10^{-35}$  (as you could have inferred from the graph).

c) If the incoming beam is described by the wavefunction,  $\psi(x) = Ae^{ikx}$  calculate  $A$  that gives the flux  $\Gamma = p(x)v$  (where  $p(x)dx$  is the probability of finding the particle at  $\{x, x + dx\}$  and  $v$  the particle velocity).

We can calculate  $p(x)$  from the wavefunction amplitude as usual  $p(x) = |\psi(x)|^2 = |A|^2$ . The particle velocity is given by  $v = \frac{p}{m} = \frac{\hbar k}{m}$ . Thus we have  $\Gamma = |A|^2 \frac{\hbar k}{m}$  from which we can calculate  $|A|^2 = \frac{\Gamma m}{\hbar k}$ . The wave-number  $k$  is calculated from  $\frac{\hbar^2 k^2}{2m} = E \rightarrow k = \sqrt{2mE}/\hbar = \sqrt{2mc^2 E}/(\hbar c) = \sqrt{2 \cdot 4000 \text{ MeV} \cdot 5 \text{ MeV}}/200 \text{ MeV fm} = 1 \text{ fm}^{-1}$ . Finally:  $|A| = \sqrt{\frac{\Gamma mc^2}{\hbar ck}} = \sqrt{\frac{6 \times 10^{21} \text{ s}^{-1} \cdot 4000 \text{ MeV}}{200 \text{ MeV fm} \cdot 1.3 \times 10^8 \text{ m s}^{-1} \cdot 10^{15} \text{ m fm}^{-1}}} = 2/\sqrt{10} \text{ fm}^{-1/2}$ . Alternatively, you could have calculated the velocity from the kinetic energy  $v = \sqrt{2E/m} = \sqrt{2 \times 5 \text{ MeV} c^2 / 4000 \text{ MeV}} = c/20$ . And use this to calculate  $|A|$ . Two remarks:

You can only calculate  $|A|$ , the full value of  $A$  could have a phase factor  $Ae^{i\varphi}$  (that however is not observable).  $|A|$  has dimensions of  $1/\sqrt{\text{length}}$ . If you don't specify its unit, any numerical number you give is meaningless.

d) Assuming the flux of alphas emerging from the barrier is  $\Gamma_{tun} = P_{tun}\Gamma$ , how long should you wait to see an alpha particle exit from the barrier? (you need to calculate the time  $\tau$  such that  $\Gamma_{tun}\tau = 1$ )

Using the result above, we obtain  $\tau = \Gamma_{tun}^{-1} = \frac{1}{4} \times 10^{35} \cdot \frac{1}{6} \times 10^{-21} \text{ s} = \frac{1}{24} \times 10^{14} \text{ s}$  or about 130 thousand years.

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