22.02 Intro to Applied Nuclear Physics

Final Exam

Tuesday May 17, 2011 Solution

Problem 1: Short Questions *20 points*

These short questions require only ***short answers*** *(but even for yes/no questions give a brief explanation)*

1) *True or False: if nuclide ${}^{A_1}X$ has a larger neutron scattering cross section than nuclide ${}^{A_2}X'$, ${}^{A_1}X$ is a better moderator than ${}^{A_2}X'$. (Explain)*

Solution:

This is true only if $A_1 \approx A_2$. Otherwise the lighter nuclide might be the best moderator, since the neutron can loose more energy in each single collision.

2) *A sheet of paper can stop a beam of alpha particle. What type of collisions are responsible for this? Indicate the type of interaction and target.*

Solution:

The collisions are due to the Coulomb interaction with the electrons in the material. (Interactions with nuclei are rare and will not contribute substantially to the stopping down).

3) *Why is the sky blue?*

Solution:

Light from the sun is scattered by the air via Rayleigh scattering. The probability of scattering (cross section) depends on the frequency as $\sigma \propto \omega^4$, thus blue light is scattered more away from the sun than other wavelengths and it can reach our eyes, when we look away from the sun (looking in a direction closer to the sun we see light that has not been scattered, thus it is closer to the original mix of wavelengths: white light).

4) *Berillium-8 is an even-even nuclide (N and Z even) in which all the nucleons are paired off, leading to a very energetically favorable configuration. Why then Be-8 is not found in nature?*
[Hint: what is the decay mode of Be-8?]

Solution:

Be-8 decays to two alpha particles: since alpha particles are so tightly bound, the decay from Be-8 to He-4 is so fast that Be-8 cannot be found in nature. As hinted in the question, beta decay to either ^{8}Li or ^{8}B is not favorable (Q¡0) because of the pairing term.

[Note: this was a tricky question, since we usually don't have alpha decay or f ssion for light isotopes, so I did not detract many points for it]

5) *The decay rates for gamma and beta decay can be obtain from the Fermi golden rule. What are the two factors that determine the decay rates?*

Solution:

Besides numerical factors, the Fermi golden rule is given by the product of the matrix element and of the density of states. The f rst one gives the probability of a transition from the initial state to a f nal state. The second one multiplies this probability by the number of states available, to f nally give the total transition rate.

6) *Fusion of two deuterium atoms is energetically favorable. Why then it does not happen unless we provide a lot of energy to them?*

Solution:

In order to reach the deep potential well of the bound He-4, the two deuteriums have to overcome the Coulomb potential barrier.

7) *Consider a free particle in quantum mechanics, initially in a state $\psi(x, t = 0)$ (for example a Gaussian wavepacket). What is the evolution of the expectation value of its position $\langle x\rangle$ and velocity $\langle v\rangle$?*
(write at least the equations of motion $\frac{d\langle x\rangle}{dt} = \dots$ and $\frac{d\langle v\rangle}{dt} = \dots$).

8) *Are the equations of motion you found in question 7 different than the classical equations of motion?*
How would you answer change if the particle was moving in a potential?

Solution:

From Ehrenfest theorem, we have $\frac{d\langle x\rangle}{dt} = \langle p\rangle = m\langle v\rangle$ and $\frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial V}{\partial x}\right\rangle = 0$, since a "free particle" means that is not subjected to any potential or force and $V = 0$. Thus $\langle v\rangle = v_0$ (its value at $t = 0$) and $\langle x(t)\rangle = v_0 t + x_0$ (where $x_0 = \langle x(0)\rangle$). The equation of motions thus give the classical result.

If we have a potential, we might have that $\left\langle \frac{\partial V}{\partial x}\right\rangle \neq \frac{\partial V(\langle x\rangle)}{\partial\langle x\rangle}$ (or $\langle F(x)\rangle \neq F(\langle x\rangle)$) it the wavefunction is spread out in space, thus the equation of motion for the quantum particle could differ from the classical case.

Problem 2: Radioactive Decay I *10 points*

Molybdenum-99 is produced in f ssion reactors, as one of the possible f ssion fragments. It decays to Technetium-99 with a half-life of 66 hours.

a) *What is the type of radioactive decay? Write the nuclear reaction describing this decay.*

Solution:

Since the mass number A is left unchanged, but the atomic number changes, the decay must be a beta decay. Specif - cally, it is Beta-minus β^- decay:

$$^{99}_{42}\mathrm{Mo} \rightarrow ^{99}_{43}\mathrm{Tc} + e^- + \bar{\nu}$$

where the underlying decay converts one neutron into one proton:

$$n \rightarrow p + e^- + \bar{\nu}$$

b) *What is the Q-value of this decay? Describe qualitatively how this energy is distributed among the reaction products.*

Solution:

Given the atomic masses of Tc and Mo we can calculate the Q value:

$$Q = [m_A(^{99}_{42}\mathrm{Mo}) - m_A(^{99}_{43}\mathrm{Tc})]c^2 = (98.90771 - 98.90625)\mathrm{amu} \times 931.46\mathrm{MeV}/c^2/\mathrm{amu} = 1.35993\mathrm{MeV}$$

Since we're using the atomic masses, the electron masses are already accounted for (notice this is different for β^+ decay).

The energy is distributed between the electron and anti-neutrino (Tc is very heavy so it does not acquire much kinetic energy due to conservation of momentum). The exact distribution of energy between electron and anti-neutrino can be calculated from the Fermi-Golden rule (in particular the density of state part) which gives the spectra of the beta decay (as calculated in class and plotted in the lecture notes, as well as you calculated in a Pset). Note that because the mass of neutrino is very small we expect it to get more kinetic energy, as we infer from a simple conservation of energy and momentum calculation.

c) ***Bonus question:*** *A metastable level of Technetium-99 (Technetium-99m) is used as a gamma emitter in medical imaging. Its half life is about 6 hours (and it decays to the ground state of Technetium-99 by emitting gamma radiation). Molybdenum-99 produced from fission is transported to hospitals where its radioactive decay product, Technetium-99m, is extracted. What is the optimal time at which the largest amount of Technetium-99m can be extracted?*

Solution:

Molybdenum decays as

$$\frac{dN_M}{dt} = -\lambda_M N_M(t)$$

while Technetium-99m dynamics is given by

$$\frac{dN_T}{dt} = \lambda_M N_M(t) - \lambda_T N_T(t)$$

with $\lambda_M = \log(2)/t_{\frac{1}{2}} = 2.9 \times 10^{-6} s^{-1}$ and $\lambda_T = \log(2)/t_{\frac{1}{2}} = 32 \times 10^{-6} s^{-1}$.
The optimum time is obtained when there is a maximum amount of Tc-99m, which can be calculated by putting the time derivative to zero. This impose the condition that the activity of the two isotopes are equal at the optimum time:

$$\frac{d\,N_T}{d\,t} = \lambda_M N_M(t_{opt}) - \lambda_T N_T(t_{opt}) = 0 \quad \rightarrow \quad A_M(t_{opt}) = A_T(t_{opt})$$

To f nd the optimal time we still need an explicit expression for the time evolution of the activities.
Solving the f rst equation (setting $N_M(t=0) = N_0$) we have: $N_M(t) = N_0 e^{-\lambda_M t}$. The second equation is then

$$\frac{d\,N_T}{d\,t} = \lambda_M N_0 e^{-\lambda_M t} - \lambda_T N_T(t)$$

with $N_T(t=0) = 0$. We can guess a solution of the form $N_T(t) = Ae^{-\lambda_M t} + Be^{-\lambda_T t}$, with A and B two parameters to be determined:

$$\frac{d\,N_T}{d\,t} = \lambda_M N_0 e^{-\lambda_M t} - \lambda_T N_T(t) \quad \rightarrow \quad -\lambda_M A e^{-\lambda_M t} - \lambda_T B e^{-\lambda_T t} = \lambda_M N_0 e^{-\lambda_M t} - \lambda_T (Ae^{-\lambda_M t} + Be^{-\lambda_T t})$$

The RHS equation is satisf ed for

$$A = N_0 \frac{\lambda_M}{\lambda_T - \lambda_M}$$

and

$$B = -N_0 \frac{\lambda_M}{\lambda_T - \lambda_M}$$

Thus giving:

$$N_T(t) = N_0 \frac{\lambda_M}{\lambda_T - \lambda_M}(e^{-\lambda_M t} - e^{-\lambda_T t})$$

The maximum quantity of Technetium-99m is found for $\frac{d\,N_T}{d\,t} = 0$ or $N_T = N_0 \frac{\lambda_M}{\lambda_T} e^{-\lambda_M t}$

$$\frac{\lambda_M}{\lambda_T - \lambda_M}(e^{-\lambda_M t} - e^{-\lambda_T t}) = \frac{\lambda_M}{\lambda_T} e^{-\lambda_M t}$$

$$\rightarrow \lambda_T (e^{-\lambda_M t} - e^{-\lambda_T t}) = (\lambda_T - \lambda_M) e^{-\lambda_M t} \quad \rightarrow \quad e^{-(\lambda_T - \lambda_M)t} = \frac{\lambda_M}{\lambda_T}$$

thus we f nd:

$$t_{\text{opt}} = \log\left(\frac{\lambda_M}{\lambda_T}\right)/(\lambda_T - \lambda_M) \approx 22 \text{ hours}$$

Problem 3: Shell Model — *20 points*

a) *Based on the shell model, what is the spin and parity of the ground states of Molybdenum-99 and of Technetium-99?*

Solution:

Molybdenum-99 has 57 neutrons. The most likely spin-parity assignment is $\frac{7}{2}^+$, or maybe $\frac{5}{2}^+$ if the orbit term is considered. [Note: it turns out to be $\frac{1}{2}^+$ but we can't obtain this result from a simple picture of the shell model].
Technetium-99 has 43 protons. In the valence nucleon model, the last unpaired proton determines the spin and parity of the nuclide. The proton occupies the $1g\frac{9}{2}$ level, thus Technetium-99 has spin $\frac{9}{2}$ and parity $+$, $\frac{9}{2}^+$

b) *Consider now Technetium-99. The first two excited states of Technetium-99 have very similar energy. Their spin-parity assignments are $\frac{7}{2}^+$ ($E_1 = 140.511$ keV) and $\frac{1}{2}^-$ ($E_2 = 142.63$keV). Explain the spin and parity of these two excited states. Draw a scheme of the energy levels involved (including the ground state) and which levels the nucleons occupy (you might want to indicate filled states with filled circles and empty states with empty circles).*

Solution:

The f rst excited state is explained by having the unpaired proton jump to the level $1g\frac{7}{2}$, which in the next shell (after the magic number 50).

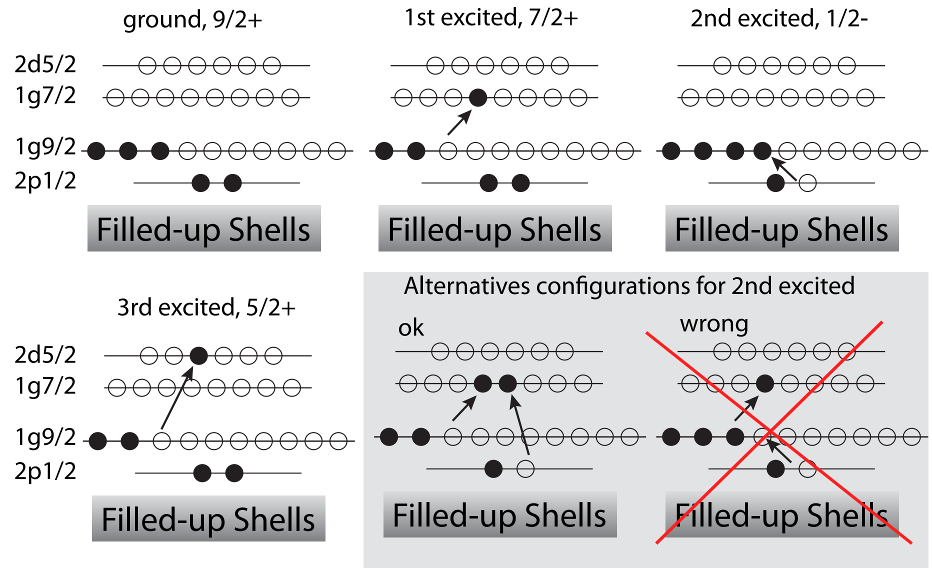

Figure 1: Shell-model configuration for Technetium. Note that students proposed different configurations for the second excited state (in the gray box). While the configuration on the left in the gray box is ok, it most probably would have higher energy than what found experimentally. The configuration on the right is not acceptable since it leaves three unpaired nucleons.

The second excited state is obtained by having one proton in the closed shell 2p$\frac{1}{2}$ being promoted to the ground state level 1g$\frac{9}{2}$. The unpaired proton is now in the 2p$\frac{1}{2}$ shell, yielding $\frac{1}{2}^-$ spin parity.

c) *The third excited state of Technetium-99 is the level 2d$\frac{5}{2}$, as we would have expected from a naive picture of the energy arrangement in the nuclear shell model. What characteristic of the nuclear force makes the energy of the level with spin/parity $\frac{1}{2}^-$ (the second excited state) smaller than the energy of the level 2d$\frac{5}{2}$?*

Solution:

We expect the level 2d$\frac{5}{2}$ to be higher in energy than the configuration 2p$\frac{1}{2}^-$ (with a closed shell1g$\frac{9}{2}$) because of a combination of the pairing term and spin-orbit coupling: the angular momentum dependence of the nuclear force makes leaving unpaired nucleons with higher angular momentum less favorable (more energetic).

d) *The spin-orbit potential $V_{so} = \frac{V_{so}^0}{\hbar^2}\hat{\vec{l}}\cdot\hat{\vec{s}}$ explains the energy difference between the ground state and the first excited state of Technetium-99.*

Given the energy difference between the first excited state and the ground state ($E_1 = 140.511$ keV), what is the value of V_{so}^0 for Technetium-99?

Solution:

The difference in energy is given by $\Delta E = V_{so}|_{1g\frac{7}{2}} - V_{so}|_{1g\frac{9}{2}}$. For a given value of the total and orbital angular momentum, the spin-orbit coupling is calculated as:

$$V_{so} = \frac{V_{so}^0}{\hbar^2}\hat{\vec{l}}\cdot\hat{\vec{s}} = V_{so} = \frac{V_{so}^0}{\hbar^2}\hbar(I(I+1) - l(l+1) - s(s+1))$$

since we have the vector operator relationship: $\hat{\vec{I}}^2 = (\hat{\vec{l}} + \hat{\vec{s}})^2 = \hat{\vec{l}}^2 + \hat{\vec{s}}^2 + 2\hat{\vec{l}}\cdot\hat{\vec{s}}$ and given the value of the operators $\hat{\vec{I}}^2$ when the system is in an eigenstate with eigenvalue $\hbar^2 I(I+1)$. Here $\hat{\vec{I}}$ is the total angular momentum, what we call the "spin" of the nucleus. $\hat{\vec{S}}$ is the last unpaired nucleon spin, which has always eigenvalue $\hbar^2 s(s+1)$ with $s = \frac{1}{2}$ and $\hat{\vec{L}}$ is the angular momentum (with eigenvalue $\hbar^2 l(l+1)$ identified by the shell spectroscopic notation: here $l = 4$ for a g-shell).

Since l (and s) is the same for the two states, we have

$$\Delta E = V_{so}^0[I_1(I_1+1) - I_0(I_0+1)] = V_{so}^0[\frac{7}{2}(\frac{7}{2}+1) - \frac{9}{2}(\frac{9}{2}+1)] = -V_{so}^0\frac{9}{2}$$

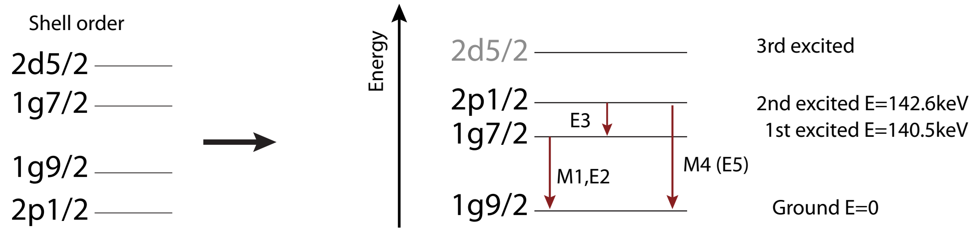

Figure 2: Gamma decays from the first **two** excited states of Tc-99. Note that although the shell order is as given in the left, the actual order of the energy levels (when ordered by energy) is as in the right side. **Three** transitions are possible, even when not considering the third excited level.

Then we have $V_{so}^0 = -140.511keV \times \frac{2}{9} = -31keV$.
Note that since $I = l \pm \frac{1}{2}$, we had also seen in lecture that we can also always write: $\Delta E = -V_{so}^0(l + \frac{1}{2}) = -V_{so}^0(4 + \frac{1}{2}) = -V_{so}^0 \frac{9}{2}$.

Problem 4: Radioactive Decay II *20 points*

Consider the gamma decay from the two first excited levels of Technetium-99.

a) *What are the allowed multipole transitions for each one of the three possible decays? Specify if there are possibly competing multipoles.*

Solution:

There are three possible decays:

$$\begin{array}{llll} 1g\frac{7}{2}^+ \to 1g\frac{9}{2}^+, & \Delta\Pi = \text{no}, & 1 \le \ell \le 8 & M1, E2, (M3, E4, M5, E6, M7, E8) \\ 2p\frac{1}{2}^- \to 1g\frac{9}{2}^+, & \Delta\Pi = \text{yes}, & 4 \le \ell \le 5 & M4, (E5) \\ 2p\frac{1}{2}^- \to 1g\frac{7}{2}^+, & \Delta\Pi = \text{yes}, & 3 \le \ell \le 4 & E3, (M4) \end{array}$$

The first decay allows odd magnetic multipoles and even electric multipoles. Thus we expect to have two competing decays, M1 and E2.
The second and third decays allow instead odd electric multipoles and even magnetic multipoles. Thus the second decay will be M4 (with maybe E5 competing) and the third E3. In this last case, we don't expect to have competing processes, because it would have much smaller decay rate.
Note that than we are considering "decay" processes, in which some energy is released (in the form of a gamma photon) the transition is always from the excited state (which has more energy) to the ground state (see figure).

b) *Estimate the decay rates for each transition (use the formulas given at the end of the exam sheet).*

Solution:

We can use the formulas in the appendix to estimate λ for the four multipoles found above, using A=99. Notice that the energy to be considered for the third decay is $\Delta E_3 = E_2 - E_1 = 142.63 - 140.511keV = 2.12keV$ We obtain:

$$\lambda(M1) = 1.55 \times 10^{11} s^{-1} \qquad \lambda(E2) = 1.83 \times 10^6 s^{-1}$$

$$\lambda(M4) = 1.08 \times 10^{-9} s^{-1}$$

$$\lambda(E3) = 6.39 \times 10^{-14} s^{-1}$$

c) *Based on your answer above and the 6 hours lifetime of Technetium-99m can you identify the excited energy level that corresponds to Technetium-99m? If not, what other radioactive process could be happening that could lead to the observed lifetime (by accelerating some of the decays)?*

Solution:

The decay from 1g$\frac{7}{2}$ is too fast, while the decay from 2p$\frac{1}{2}$ is too slow. Thus the level 2p$\frac{1}{2}$ (that we can identify with Tc-99m) must be decaying by **internal conversion** at a faster rate than gamma decay to explain the observed lifetime.

Internal conversion is a decay process that is always present and can have faster rates that gamma decay in some cases, as here.

d) *Although long-lived, Technetium-99 is not stable. By what radioactive process does its ground state decay? (write the nuclear reaction describing this decay).*

Solution:

We can f rst eliminate alpha decay, since the mass number is too low to allow it. Then we should decide if Tc-99 decays by β^- or β^+/electron capture processes. Note that the last two processes would create Molybdenum-99 again, but we know this is not possible, since the opposite decay happens.

Then, we f nd that Tc-99 decays by β^- decay to Ruthenium-99 ($^{99}_{44}$Ru):

$$^{99}_{43}Tc \rightarrow ^{99}_{44} Ru + e^- + \bar{\nu}$$

e) *The daughter nuclide of Technetium-99 can be also obtained from the decay of an isotope of Rhodium. What is the decay type? (write the reaction). Use the semi-empirical mass formula to explain this fact.*

Solution:

Ruthenium-99 can be obtained also from Rhodium-99 by changing one proton into a neutron. The two reactions that make this possible are β^+ decay and **electron capture**:

$$^{99}_{45}Rh \rightarrow ^{99}_{44} Ru + e^+ + \nu$$

$$^{99}_{45}Rh + e^- \rightarrow ^{99}_{44} Ru + \nu$$

It turns out that electron capture is the actual decay mode (although you could not have known, but you should at least listed the possibility).

Nuclides try to reach an optimal value of their Z/N ratio, as it can be explained by the mass parabola obtained from the semi-empirical mass formula plotted for a f xed A as a function of Z (see lecture notes for the plot of the mass parabola).

Problem 5: Neutron Scattering and Capture by Hydrogen *20 points*

a) *Consider the elastic scattering of neutrons of energy E_0 by Hydrogen initially at rest. What is the mean energy of the Hydrogen after one collision with a neutron? What is the variance of the Hydrogen energy?*

[Hint: start from what you know about the energy distribution of neutrons]

Solution:

In the elastic scattering of a neutron by a proton the two particles share the f nal energy. We know that the probability for the neutron to have f nal energy E_1 is $P(E_1) = \frac{(A+1)^2}{4AE_0} = \frac{1}{E_0}$, which is a uniform distribution between αE_0 and E_0 (here $\alpha = 0$). See the lecture notes on how we obtained this probability distribution function: qualitatively this is due to the variation in the angle at which the two particle are scattered.

Then the probability for the proton to have an energy E_p is $P(E_p)dE_p = P(E_1)dE_1 = \frac{1}{E_0}$ for $E_p \in [0, E_0]$. Then the mean energy of the proton is $\langle E_p \rangle = \frac{1}{2}E_0$ and the **variance** $\Delta E_p^2 = \langle E_p^2 \rangle - \langle E_p \rangle^2 = \frac{E_0^2}{12}$.

Note: the variance of a random variable is calculate by calculating the mean (or average) as:

$$\langle E \rangle = \int_{-\infty}^{\infty} P(E) E \, dE \qquad \rightarrow \qquad \text{here : } \int_0^{E_0} \frac{1}{E_0} E \, dE = \frac{E_0^2}{2E_0}$$

and the second moment:

$$\langle E^2 \rangle = \int_{-\infty}^{\infty} P(E) E^2 \, dE \qquad \rightarrow \qquad \text{here : } \int_0^{E_0} \frac{1}{E_0} E^2 \, dE = \frac{E_0^3}{3E_0}$$

and then taking the difference:

$$\langle E_p^2 \rangle - \langle E_p \rangle^2 = \frac{E_0^2}{3} - \frac{E_0^2}{4} = \frac{E_0^2}{12}.$$

You can review probability concepts on the notes (from Griff th) that I had posted some time ago.

b) *Now consider the synthesis of deuterium D from the collision of a neutron and a proton.*
The reaction is a neutron capture reaction, n+H→ γ+ D or H(n,γ)D. What is the energy of the gamma photon emitted?

Solution:

The gamma photon will carry out the Q energy released in the reaction. Thus we need to calculate Q:

$$Q = c^2[m(H) + m(n) - m(D)] = [(938.272 + 0.511 + 939.565) - 2.014 \times 931.46]MeV = 2.39MeV$$

Here I approximated the atomic mass of the hydrogen by the sum of the proton and electron mass (neglecting the binding energy).

c) *In many neutron reaction capture ^{A}X(n,γ)$^{A+1}$X some of the energy released in the reaction is given to the $^{A+1}$X nuclide, which is left in an excited state $^{A+1}X^{*}$.*
Can this happen in the neutron capture reaction producing deuterium, H(n,γ)D?
What does this imply about the relative values of the scattering cross section σ_{sc} and the capture cross section σ_{cp} of a neutron by Hydrogen? (which one is larger?)

Solution:

Deuterium does not have any bound excited state, thus the only possible reaction leading to the formation of deuterium is the one in which the gamma possess all the energy Q generated in the reaction. In practice, if some of this energy would to be given to the deuterium, the nucleus would again split apart in a neutron and a proton. This process would be then just a (possibly inelastic) collision. We can deduce that the probability of a collision is much higher than the creation of a deuterium: in other words, $\sigma_{sc} \gg \sigma_{cp}$.

d) *The neutron and proton that collide can have their spins aligned or anti-aligned. Does the probability of a collision leading to neutron capture instead of elastic scattering depend the spins of proton and neutron being aligned or anti-aligned? Explain.*
[Hint: What is the total spin of deuterium?]

Solution:

A proton and a neutron are different particles, so we do not have to worry about the Pauli exclusion principle. They both have spin-1/2, so their possible states are (in the coupled and uncoupled representations):

$$\begin{array}{l}
\left|s=\tfrac{1}{2}, m_s=+\tfrac{1}{2}\right\rangle_p \left|s=\tfrac{1}{2}, m_s=+\tfrac{1}{2}\right\rangle_n = |\uparrow\uparrow\rangle \\
\left|s=\tfrac{1}{2}, m_s=+\tfrac{1}{2}\right\rangle_p \left|s=\tfrac{1}{2}, m_s=-\tfrac{1}{2}\right\rangle_n = |\uparrow\downarrow\rangle \\
\left|s=\tfrac{1}{2}, m_s=-\tfrac{1}{2}\right\rangle_p \left|s=\tfrac{1}{2}, m_s=+\tfrac{1}{2}\right\rangle_n = |\downarrow\uparrow\rangle \\
\left|s=\tfrac{1}{2}, m_s=-\tfrac{1}{2}\right\rangle_p \left|s=\tfrac{1}{2}, m_s=-\tfrac{1}{2}\right\rangle_n = |\downarrow\downarrow\rangle
\end{array}$$

$$\begin{array}{ll}
|S=0, N_s=0\rangle = & \frac{|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle}{\sqrt{2}} \\
|S=1, M_s=-1\rangle = & |\downarrow,\downarrow\rangle \\
|S=1, M_s=0\rangle = & \frac{|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle}{\sqrt{2}} \\
|S=1, M_s=1\rangle = & |\uparrow,\uparrow\rangle
\end{array}$$

Note that the total spin can only be $S = 0, 1$ (no negative values!!) while for each value of S we have $M_s = -S, -S+1, \dots, 0, \dots, S$.
The only bound state of deuterium has total spin $S = 1$. This spin state is reached if the spins of proton and neutron are aligned but also if they are anti-aligned. In this last case however, the spins can either form the desired triplet (with $S = 1$) but also a singlet (i.e. the state with $S = 0$); if this is the case, then neutron capture and deuterium formation cannot happen. Thus we expect that the probability of neutron capture is higher if the spins are aligned (and conversely a lower scattering cross-section).

e) ***Bonus question:*** *Denote by $\sigma_{cp}^{\uparrow\uparrow}$ the cross-section for neutron capture if the neutron and proton spins are aligned and $\sigma_{cp}^{\uparrow\downarrow}$ the cross-section if the two nucleons are anti-aligned. Assuming that the collision and capture does not change the spins of neutron and proton, what is the ratio $\sigma_{cp}^{\uparrow\downarrow}/\sigma_{cp}^{\uparrow\uparrow}$? (you do not need to find an explicit expression for these cross-section to answer this question).*

Solution:

If we neglect spin effects, we would have a cross section σ_{cp}. If the two nucleons are aligned, the cross section is unchanged, since they can in all the case fusion to form deuterium, $\sigma_{cp}^{\uparrow\uparrow} = \sigma_{cp}$. However if the nucleons are antialigned, only in half of the cases they are going to be in the triplet state $S = 1$ that lead to a bound state, thus $\sigma_{cp}^{\uparrow\downarrow} = \frac{1}{2}\sigma_{cp}$ and the ratio of the two cross-sections is $\sigma_{cp}^{\uparrow\downarrow}/\sigma_{cp}^{\uparrow\uparrow} = \frac{1}{2}$.

Problem 6: Time Evolution *10 points*

Consider again a neutron and a proton (before the collision).
Assume that at time $t = 0$ the proton is in the state $|\psi(0)\rangle_p = \left|S = \frac{1}{2}, m_s = +\frac{1}{2}\right\rangle_p = |\uparrow\rangle_p$, while the neutron is the state $|\psi(0)\rangle_n = \frac{1}{2}\left|S = \frac{1}{2}, m_s = +\frac{1}{2}\right\rangle_n + \frac{\sqrt{3}}{2}\left|S = \frac{1}{2}, m_s = -\frac{1}{2}\right\rangle_n = \frac{1}{2}|\uparrow\rangle_n + \frac{\sqrt{3}}{2}|\downarrow\rangle_n$.
The states $\left|S = \frac{1}{2}, m_s = +\frac{1}{2}\right\rangle = |\uparrow\rangle$ and $\left|S = \frac{1}{2}, m_s = -\frac{1}{2}\right\rangle = |\downarrow\rangle$ are eigenstates of the spin Hamiltonian, with energies E_+ and E_- respectively.

a) *What is the state of the proton at a time $t > 0$? What is the state of the neutron at the same time?*

Solution:

Since it is in an energy eigenstate, the evolution of the proton is simply $|\psi(t)\rangle_p = e^{-iE_+t/\hbar}|\uparrow\rangle_p$. Notice that this is only an ”apparent” evolution, since it does not change any observable property of the proton.
The evolution of the neutron is instead given by:

$$|\psi(t)\rangle_n = \frac{1}{2}e^{-iE_+t/\hbar}|\uparrow\rangle_n + \frac{\sqrt{3}}{2}e^{-iE_-t/\hbar}|\downarrow\rangle_n$$

which is a different state than the initial one.

b) *What is the probability for the two spins to be aligned at a time t? (assume that the two particle evolve independently)*

Solution:

Since the state of the proton does not really change with time and it’s always $|\uparrow\rangle_p = \left|S = \frac{1}{2}, m_s = +\frac{1}{2}\right\rangle_p$, we have to find the probability for the neutron to be as well in the state $|\uparrow\rangle_n = \left|S = \frac{1}{2}, m_s = +\frac{1}{2}\right\rangle_n$. This is given by $|\langle\uparrow|\psi(t)\rangle_n|^2 = |\frac{1}{2}e^{-iE_+t/\hbar}|^2 = \frac{1}{4}$.

c) ***Bonus question:*** *What is the probability at time t for the proton and neutron to be in a (common) state with total spin $S = 1$?*

Solution:

The probability of having spin $S = 1$, P_1 is the probability of not having spin $S = 0$, $P_1 = 1 - P_0$. In turns,

$$P_0 = |\langle S = 0, m_s = 0|\psi_p(t)\psi_n(t)\rangle|^2$$

$$= |\left(\frac{\langle\uparrow\downarrow| - \langle\downarrow\uparrow|}{\sqrt{2}}\right)\left(\frac{1}{2}e^{-iE_+t/\hbar}|\uparrow\uparrow\rangle + \frac{\sqrt{3}}{2}e^{-iE_-t/\hbar}|\uparrow\downarrow\rangle\right)|^2 = |\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}e^{-iE_-t/\hbar}\langle\uparrow\downarrow|\uparrow\downarrow\rangle|^2 = \frac{3}{8}.$$

Thus, $P_1 = 1 - \frac{3}{8} = \frac{5}{8}$.

Appendix: Useful Formulas and Constants

Masses: Proton mass, $m_pc^2 = 938.272$ MeV; Neutron mass, $m_nc^2 = 939.565$ MeV; Deuteron mass = 2.014amu; Electron mass=0.511 MeV. Atomic mass of $^{99}_{42}$Mo: 98.90771 amu. Atomic mass of $^{99}_{43}$Tc: 98.90625 amu.

Constants: $\hbar c = 197$MeV fm; $\frac{e^2}{\hbar c} = \frac{1}{137}$; $c = 3 \times 10^8$m/s; $R_0 = 1.2$fm. 1 amu $\approx$ 931.46 MeV/c^2

Relativistic energy: $E = \sqrt{p^2c^2 + m^2c^4}$

SEMF coefficients: $a_v \approx 15.5$MeV, $a_s \approx 16$MeV, $a_c \approx 0.6$MeV, $a_{sym} \approx 24$MeV, $a_p \approx 34$MeV.

Spin-orbit interaction $V_{so} = \frac{V_{so}^0}{\hbar^2}\hat{\vec{l}} \cdot \hat{\vec{s}}$

Fermi's Golden Rule: $W = \frac{2\pi}{\hbar}|\langle\psi_f|V|\psi_i\rangle|^2\rho(E_f)$

Gamma Decay: Estimates for the rates of different electric multipoles (energies in MeV):

- $\lambda(E1) = 1.0 \times 10^{14} A^{2/3} E^3$
- $\lambda(E2) = 7.3 \times 10^7 A^{4/3} E^5$
- $\lambda(E3) = 34 A^2 E^7$
- $\lambda(E4) = 1.1 \times 10^{-5} A^{8/3} E^9$

- $\lambda(M1) = 5.6 \times 10^{13} E^3$
- $\lambda(M2) = 3.5 \times 10^7 A^{2/3} E^5$
- $\lambda(M3) = 16 A^{4/3} E^7$
- $\lambda(M4) = 4.5 \times 10^{-6} A^2 E^9$

Cross Sections

Differential cross-section: $\frac{d\sigma}{d\Omega} = \frac{r(\vartheta,\varphi)}{4\pi I_a n}$ Doubly differential cross section: $\frac{d^2\sigma}{d\Omega dE_b}$

Coulomb : $\sigma_c = 2\pi r_e^2 \frac{Z_\alpha^2}{\beta^4} \ln\Lambda$ (with $r_e = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_ec^2} \sim 2.8fm$ –or $4\pi r_e^2 \sim 1$barn– and $\ln\Lambda \sim 10$)

Stopping power: $-\frac{dE}{dx} = \sigma_c n_e \Delta E$. Rutherford's: $\frac{d\sigma}{d\Omega} = \left(\frac{zZe^2}{4\pi\epsilon_0}\right)^2 (4T_a)^{-2} \sin^{-4}\left(\frac{\vartheta}{2}\right)$

Rayleigh: $\sigma_R = \frac{8}{3}\pi r_e^2 \frac{\omega^4}{\omega_0^4}$ Thomson: $\sigma_T = \frac{8}{3}\pi r_e^2$

Compton: $\sigma_C = \sigma_T \frac{m_ec^2}{\hbar\omega}$, with a change in the photon's wavelength: $\Delta\lambda = \frac{2\pi\hbar}{m_ec}(1 - \cos\vartheta)$.

Number of collisions to reach neutron thermalization

^1H	^2H	^4He	^{12}C	^{238}U
18	25	43	110	2200

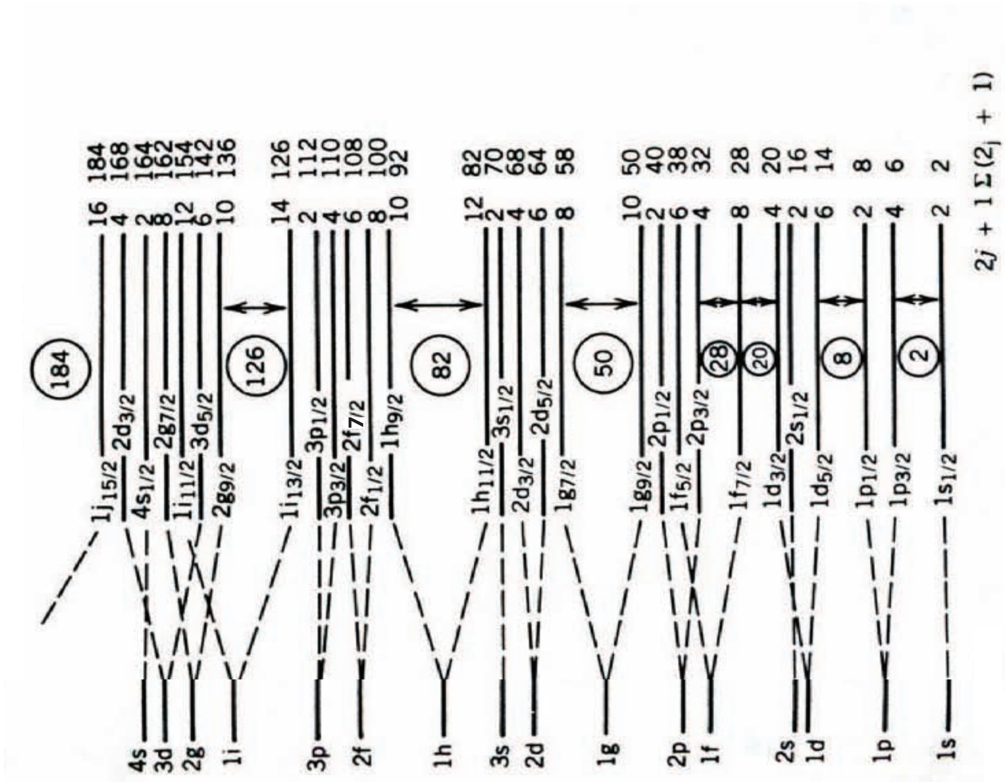

Figure 3: **Shell Model**

The Periodic Table of the Elements

1 **H** Hydrogen 1.00794																	2 **He** Helium 4.003
3 **Li** Lithium 6.941	4 **Be** Beryllium 9.012182											5 **B** Boron 10.811	6 **C** Carbon 12.0107	7 **N** Nitrogen 14.00674	8 **O** Oxygen 15.9994	9 **F** Fluorine 18.9984032	10 **Ne** Neon 20.1797
11 **Na** Sodium 22.989770	12 **Mg** Magnesium 24.3050											13 **Al** Aluminum 26.981538	14 **Si** Silicon 28.0855	15 **P** Phosphorus 30.973761	16 **S** Sulfur 32.066	17 **Cl** Chlorine 35.4527	18 **Ar** Argon 39.948
19 **K** Potassium 39.0983	20 **Ca** Calcium 40.078	21 **Sc** Scandium 44.955910	22 **Ti** Titanium 47.867	23 **V** Vanadium 50.9415	24 **Cr** Chromium 51.9961	25 **Mn** Manganese 54.938049	26 **Fe** Iron 55.845	27 **Co** Cobalt 58.933200	28 **Ni** Nickel 58.6934	29 **Cu** Copper 63.546	30 **Zn** Zinc 65.39	31 **Ga** Gallium 69.723	32 **Ge** Germanium 72.61	33 **As** Arsenic 74.92160	34 **Se** Selenium 78.96	35 **Br** Bromine 79.904	36 **Kr** Krypton 83.80
37 **Rb** Rubidium 85.4678	38 **Sr** Strontium 87.62	39 **Y** Yttrium 88.90585	40 **Zr** Zirconium 91.224	41 **Nb** Niobium 92.90638	42 **Mo** Molybdenum 95.94	43 **Tc** Technetium (98)	44 **Ru** Ruthenium 101.07	45 **Rh** Rhodium 102.90550	46 **Pd** Palladium 106.42	47 **Ag** Silver 107.8682	48 **Cd** Cadmium 112.411	49 **In** Indium 114.818	50 **Sn** Tin 118.710	51 **Sb** Antimony 121.760	52 **Te** Tellurium 127.60	53 **I** Iodine 126.90447	54 **Xe** Xenon 131.29
55 **Cs** Cesium 132.90545	56 **Ba** Barium 137.327	57 **La** Lanthanum 138.9055	72 **Hf** Hafnium 178.49	73 **Ta** Tantalum 180.9479	74 **W** Tungsten 183.84	75 **Re** Rhenium 186.207	76 **Os** Osmium 190.23	77 **Ir** Iridium 192.217	78 **Pt** Platinum 195.078	79 **Au** Gold 196.96655	80 **Hg** Mercury 200.59	81 **Tl** Thallium 204.3833	82 **Pb** Lead 207.2	83 **Bi** Bismuth 208.98038	84 **Po** Polonium (209)	85 **At** Astatine (210)	86 **Rn** Radon (222)
87 **Fr** Francium (223)	88 **Ra** Radium (226)	89 **Ac** Actinium (227)	104 **Rf** Rutherfordium (261)	105 **Db** Dubnium (262)	106 **Sg** Seaborgium (263)	107 **Bh** Bohrium (262)	108 **Hs** Hassium (265)	109 **Mt** Meitnerium (266)	110 (269)	111 (272)	112 (277)	113	114				

58 **Ce** Cerium 140.116	59 **Pr** Praseodymium 140.90765	60 **Nd** Neodymium 144.24	61 **Pm** Promethium (145)	62 **Sm** Samarium 150.36	63 **Eu** Europium 151.964	64 **Gd** Gadolinium 157.25	65 **Tb** Terbium 158.92534	66 **Dy** Dysprosium 162.50	67 **Ho** Holmium 164.93032	68 **Er** Erbium 167.26	69 **Tm** Thulium 168.93421	70 **Yb** Ytterbium 173.04	71 **Lu** Lutetium 174.967
90 **Th** Thorium 232.0381	91 **Pa** Protactinium 231.03588	92 **U** Uranium 238.0289	93 **Np** Neptunium (237)	94 **Pu** Plutonium (244)	95 **Am** Americium (243)	96 **Cm** Curium (247)	97 **Bk** Berkelium (247)	98 **Cf** Californium (251)	99 **Es** Einsteinium (252)	100 **Fm** Fermium (257)	101 **Md** Mendelevium (258)	102 **No** Nobelium (259)	103 **Lr** Lawrencium (262)

Figure 4: **Periodic Table of the elements**

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