

## 22.101 Applied Nuclear Physics (Fall 2006)

### Lecture 12 (10/25/06)

#### Empirical Binding Energy Formula and Mass Parabolas

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##### References:

W. E. Meyerhof, *Elements of Nuclear Physics* (McGraw-Hill, New York, 1967), Chap. 2.

P. Marmier and E. Sheldon, *Physics of Nuclei and Particles* (Academic Press, New York, 1969), Vol. I, Chap. 2.

R. D. Evans, *The Atomic Nucleus* (McGraw-Hill, New York, 1955).

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The binding energy curve we have discussed in the last lecture is an overall representation of how the stability of nuclides varies across the entire range of mass number  $A$ . The curve shown in Fig. 10.1 was based on experimental data on atomic masses. One way to analyze this curve is to decompose the binding energy into various contributions from the interactions among the nucleons. An empirical formula for the binding energy consisting of contributions representing volume, surface, Coulomb and other effects was first proposed by von Weizsäcker in 1935. Such a formula is useful because it not only allows one to calculate the mass of a nucleus, thereby eliminating the need for table of mass data, but also it leads to qualitative understanding of the essential features of nuclear binding. More detailed theories exist, for example, Bruecker et al., *Physical Review* **121**, 255 (1961), but they are beyond the scope of our study.

The empirical mass formula we consider here was derived on the basis of the liquid drop model of the nucleus. The essential assumptions are:

1. The nucleus is composed of incompressible matter, thus  $R \sim A^{1/3}$ .
2. The nuclear force is the same among neutrons and protons (excluding Coulomb interactions).
3. The nuclear force saturates (meaning it is very short ranged).

The empirical mass formula is usually given in terms of the binding energy,

$$B(A, Z) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(N-Z)^2}{A} + \delta \quad (11.1)$$

where the coefficients  $a$  are to be determined (by fitting the mass data), with subscripts v, s, c, and a referring to volume, surface, Coulomb, and asymmetry respectively. The last term in (11.1) represents the pairing effects,

$$\begin{aligned} \delta &= a_p / \sqrt{A} && \text{even-even nuclei} \\ &= 0 && \text{even-odd, odd-even nuclei} \\ &= -a_p / \sqrt{A} && \text{odd-odd nuclei} \end{aligned}$$

where coefficient  $a_p$  is also a fitting parameter. A set of values for the five coefficients in (11.1) is:

$$\begin{array}{cccccc} \underline{a_v} & \underline{a_s} & \underline{a_c} & \underline{a_a} & \underline{a_p} & \\ 16 & 18 & 0.72 & 23.5 & 11 & \text{Mev} \end{array}$$

Since the fitting to experimental data is not perfect one can find several slightly different sets of coefficients in the literature. The average accuracy of (11.1) is about 2 Mev except where strong shell effects are present. One can add a term,  $\sim 1$  to 2 Mev, to (11.1) to represent the shell effects, extra binding for nuclei with closed shells of neutrons or protons.

A simple way to interpret (11.1) is to regard the first term as a first approximation to the binding energy. That is to say, the binding energy is proportional to the volume of the nucleus or the mass number A. This assumes every nucleon is like every other nucleon. Of course this is an oversimplification, the remaining terms then can be regarded as corrections to this first approximation. That is why the terms representing surface, Coulomb and asymmetry come in with negative signs, each one subtracting from

the volume effect. It is quite understandable that the surface term should vary with  $A^{2/3}$ , or  $R^2$ . The Coulomb term is also quite self-evident considering that  $Z(Z-1)/2$  is the number of pairs that one can form from  $Z$  protons, and the  $1/A^{1/3}$  factor comes from the  $1/R$ . The asymmetry term in (11.1) is less obvious, so we digress to derive it.

What we would like to estimate is the energy difference between an actual nucleus, where  $N > Z$ , and an ideal nucleus, where  $N = Z = A/2$ . This is then the energy to transform a symmetric nucleus, in the sense of  $N = Z$ , to an asymmetric one,  $N > Z$ . For fixed  $N$  and  $Z$ , the number of protons that we need to transform into neutrons is  $\nu$ , with  $N = (A/2) + \nu$  and  $Z = (A/2) - \nu$ . Thus,  $\nu = (N - Z)/2$ . Now consider a set of energy levels for the neutrons and another set for the protons, each one filled to a certain level. To transform  $\nu$  protons into neutrons the protons in question have to go into unoccupied energy levels above the last neutron. This means the amount of energy involved is  $\nu$  (the number of nucleons that have to be transformed) times  $\nu\Delta$  (energy change for each nucleon to be transformed)  $= \nu^2\Delta = (N - Z)^2\Delta/4$ , where  $\Delta$  is the spacing between energy levels (assumed to be the same for all the levels). To estimate  $\Delta$ , we note that  $\Delta \sim E_F/A$ , where  $E_F$  is the Fermi energy (see Figs. 9.7 and 9.8) which is known to be independent of  $A$ . Thus  $\Delta \sim 1/A$ , and we have the expression for the asymmetry term in (11.1).

The magnitudes of the various contributions to the binding energy curve are depicted in Fig. 11.1. The initial rise of  $B/A$  with  $A$  is seen to be due to the decreasing importance of the surface contribution as  $A$  increases. The Coulomb repulsion effect grows in importance with  $A$ , causing a maximum in  $B/A$  at  $A \sim 60$ , and a subsequent decrease of  $B/A$  at larger  $A$ . Except for the extreme ends of the mass number range the semi-empirical mass formula generally can give binding energies accurate to within 1%

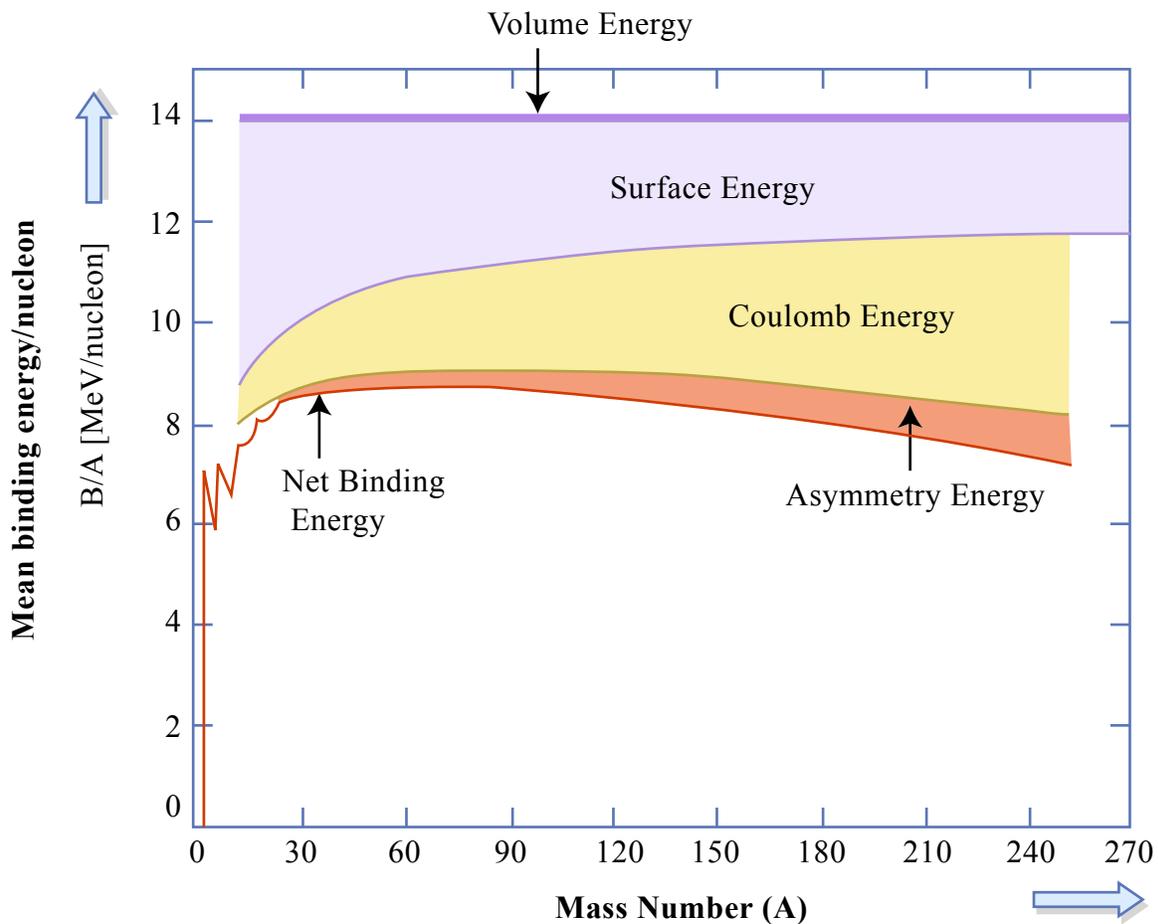


Figure by MIT OCW. Adapted from Evans.

**Fig. 11.1.** Relative contributions to the binding energy per nucleon showing the importance of the various terms in the semi-empirical Weizsäcker formula. [from Evans]

of the experimental values [Evans, p. 382]. This means that atomic masses can be calculated correctly to roughly 1 part in  $10^4$ . However, there are conspicuous discrepancies in the neighborhood of magic nuclei. Attempts have been made to take into account the nuclear shell effects by generalizing the mass formula. In addition to what we have already mentioned, one can consider another term representing nuclear deformation [see Marmier and Sheldon, pp. 39, for references].

One can use the mass formula to determine the constant  $r_0$  in the expression for the nuclear radius,  $R = r_0 A^{1/3}$ . The radius appears in the coefficients  $a_v$  and  $a_s$ . In this way one obtains  $r_0 = 1.24 \times 10^{-13}$  cm.

### ***Mass Parabolas and Stability Line***

The mass formula can be rearranged to give the mass  $M(A,Z)$ ,

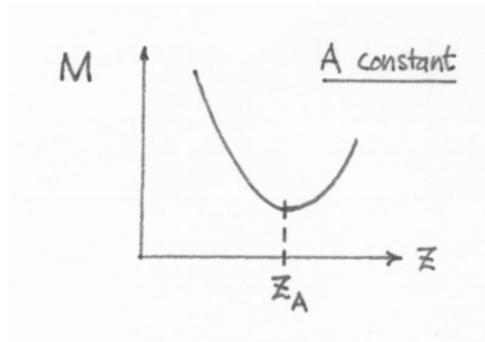
$$M(A,Z)c^2 \cong A[M_n c^2 - a_v + a_a + a_s / A^{1/3}] + xZ + yZ^2 - \delta \quad (11.2)$$

where

$$x = -4a_a - (M_n - M_H)c^2 \cong -4a_a \quad (11.3)$$

$$y = \frac{4a_a}{A} + \frac{a_c}{A^{1/3}} \quad (11.4)$$

Notice that (11.2) is not an exact rearrangement of (11.1), certain small terms having been neglected. What is important about (11.2) is that it shows that with A held constant the variation of M(A,Z) with Z is given by a parabola, as sketched below. The minimum



of this parabola occurs at the atomic number, which we label as  $Z_A$ , of the stable nucleus for the given A. This therefore represents a way of determining the stable nuclides.

We can analyze (11.2) further by considering  $\partial M / \partial Z|_{Z_A} = 0$ . This gives

$$Z_A = -x/2y \approx \frac{A/2}{1 + \frac{1}{4} \left( \frac{a_c}{a_a} \right) A^{2/3}} \quad (11.5)$$

Notice that if we had considered only the volume, surface and Coulomb terms in B(A,Z), then we would have found instead of (11.5) the expression

$$Z_A \approx \frac{(M_n - M_H)c^2 A^{1/3}}{2a_c} \sim 0.9A^{1/3} \quad (11.6)$$

This is a very different result because for a stable nucleus with  $Z_A = 20$  the corresponding mass number given by (11.6) would be  $\sim 9,000$ , which is clearly unrealistic. Fitting (11.5) to the experimental data gives  $a_c / 4a_a = 0.0078$ , or  $a_a \sim 20 - 23$  Mev. We see therefore the deviation of the stability line from  $N = Z = A/2$  is the result of Coulomb effects, which favor  $Z_A < A/2$ , becoming relatively more important than the asymmetry effects, which favor  $Z_A = A/2$ .

We can ask what happens when a nuclide is unstable because it is proton-rich. The answer is that a nucleus with too many protons for stability can emit a positron (positive electron  $e^+$  or  $\beta^+$ ) and thus convert a proton into a neutron. In this process a neutrino ( $\nu$ ) is also emitted. An example of a positron decay is



By the same token if a nucleus has too many neutrons, then it can emit an electron ( $e^-$  or  $\beta^-$ ) and an antineutrino  $\bar{\nu}$ , converting a neutron into a proton. An electron decay for the isobar  $A = 16$  is



A competing process with positron decay is electron capture (EC). In this process an inner shell atomic electron is captured by the nucleus so the nuclear charge is reduced from  $Z$  to  $Z - 1$ . (Note: Orbital electrons can spend a fraction of their time inside the nucleus.) The atom as a whole would remain neutral but it is left in an excited state because a vacancy has been created in one of its inner shells.

As far as atomic mass balance is concerned, the requirement for each process to be energetically allowed is:

$$M(A, Z+1) > M(A, Z) + 2 m_e \quad \beta^+ \text{ - decay} \quad (11.9)$$

$$M(A,Z) > M(A, Z+1) \quad \beta^- \text{ - decay} \quad (11.10)$$

$$M(A, Z+1) > M(A,Z) \quad \text{EC} \quad (11.11)$$

where  $M(A,Z)$  now denotes atomic mass. Notice that EC is a less stringent condition for the nucleus to decrease its atomic number. If the energy difference between initial and final states is less than twice the electron rest mass (1.02 Mev), the transition can take place via EC while it would be energetically forbidden via positron decay. The reason for the appearance of the electron rest mass in (11.9) may be explained by looking at an energy balance in terms of nuclear mass  $M'(A,Z)$ , which is related to the atomic mass by  $M(A,Z) = Zm_e + M'(A,Z)$ , if we ignore the binding energy of the electrons in the atom. For  $\beta^+$ -decay the energy balance is

$$M'(A, Z) = M'(A, Z - 1) + m_e + \nu \quad (11.12)$$

which we can rewrite as

$$Zm_e + M'(A, Z) = (Z - 1)m_e + M'(A, Z - 1) + 2m_e + \nu \quad (11.13)$$

The LHS is just  $M(A,Z)$  while the RHS is at least  $M(A,Z-1) + 2m_e$  with the neutrino having a variable energy. Thus one obtains (11.9). Another way to look at this condition is that in addition to the positron emitted, the daughter nuclide also has to eject an electron (from an outer shell) in order to preserve charge neutrality.

Having discussed how a nucleus can change its atomic number  $Z$  while preserving its mass number  $A$ , we can predict what transitions will occur as an unstable nuclide moves along the mass parabola toward the point of stability. Since the pairing term  $\delta$  vanishes for odd- $A$  isobars, one has a single mass parabola in this case in contrast to two mass parabolas for the even- $A$  isobars. One might then expect that when  $A$  is odd there can be only one stable isobar. This is generally true with two exceptions, at  $A = 113$  and  $123$ . In these two cases the discrepancies arise from small mass differences which cause

one of the isobars in each case to have exceptionally long half life. In the case of even A there can be stable even-even isobars (three is the largest number found). Since the odd-odd isobars lie on the upper mass parabola, one would expect there should be no stable odd-odd nuclides. Yet there are several exceptions,  $H^2$ ,  $Li^6$ ,  $B^{10}$  and  $N^{14}$ . One explanation is that there are rapid variations of the binding energy for the very light nuclides due to nuclear structure effects that are not taken into account in the semi-empirical mass formula. For certain odd-odd nuclides both conditions for  $\beta^+$  and  $\beta^-$  decays are satisfied, and indeed both decays do occur in the same nucleus. Examples of odd- and even-A mass parabolas are shown in Fig. 11.2.

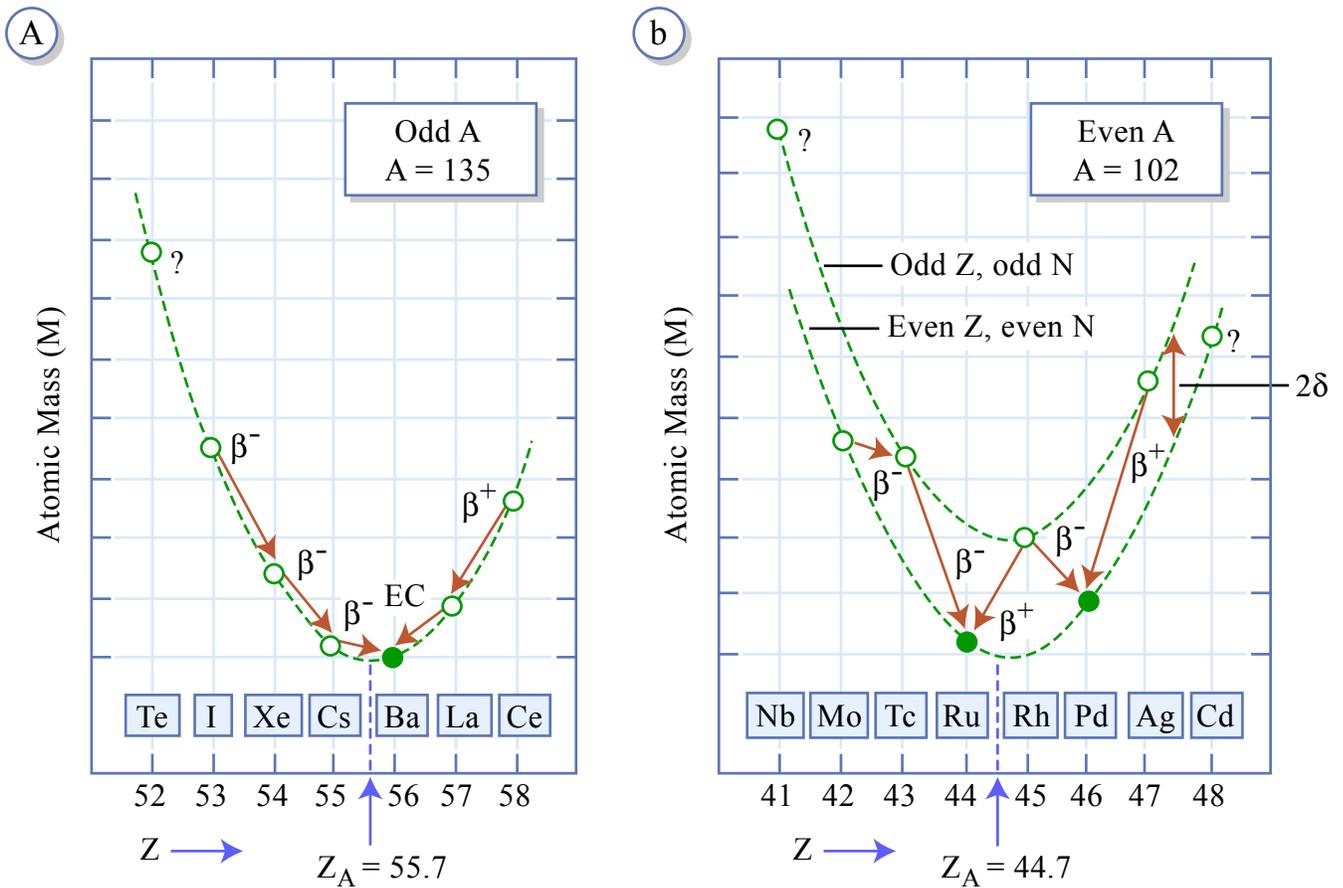


Figure by MIT OCW. Adapted from Meyerhof.

**Fig. 11.2.** Mass parabolas for odd and even isobars. Stable and radioactive nuclides are denoted by closed and open circles respectively. [from Meyerhof]