Chapter 4

Radiation By Moving Charges

4.1 Potentials and Fields of a moving point charge

The general solution

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{[\![\rho]\!]}{|\mathbf{x} - \mathbf{x}'|} d^3x' \tag{4.1}$$

Looks as if it will give the result for a point charge directly in the same way as the static solution. For a stationary point charge $\rho = q\delta(\mathbf{x} - \mathbf{r})$, where $\mathbf{r}$ is the charge position, $\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{x}-\mathbf{r}|}$. For brevity let's write $\mathbf{R} \equiv \mathbf{x} - \mathbf{r}$. One might think for a moving charge $\phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}[\![1/R]\!]$ but this is incorrect. We haven't taken care with derivatives etc. of

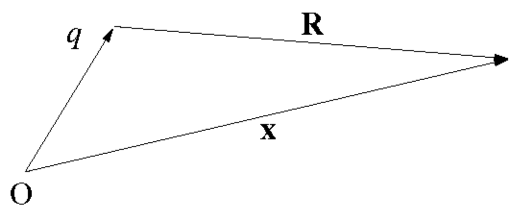

Figure 4.1: Vector coordinates of charge and field point.

retarded quantities. Let's go carefully! The charge density is $\rho(x, t) = q\delta(\mathbf{x} - \mathbf{r}(t))$ where $\mathbf{r}$ is now allowed to vary with time so we want

$$\begin{aligned} \phi(\mathbf{x}, t) &= \frac{q}{4\pi\epsilon_0} \int \frac{\delta(\mathbf{x}' - \mathbf{r}(t'))}{|\mathbf{x} - \mathbf{x}'|} d^3x' \quad , \quad t' = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{r}(t')|} \int \delta(\mathbf{x}' - \mathbf{r}(t'))\, d^3x' \end{aligned} \tag{4.2}$$

where we make use of the fact that the delta function is non-zero only where its argument is zero, so all the contribution to the integral comes from the place where $\mathbf{x}' = \mathbf{r}(t')$, which is where the particle is at retarded time i.e.

$$\mathbf{r}\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right) \tag{4.3}$$

[This requires self-referential notation which is one reason we write it $[\![\mathbf{r}]\!]$.] Now we have to do the integral $\int \delta\left(\mathbf{x}' - \mathbf{r}\left(t'\right)\right) d^3x'$. This is not unity because $\mathbf{x}'$ appears inside $\mathbf{r}(t')$ as well as in $\mathbf{x}'$. The delta function is defined such that

$$\int \delta\left(\mathbf{y}\right) d^3y = 1 \tag{4.4}$$

but now its argument is $\mathbf{y} = \mathbf{x}' - \mathbf{r}\left(t'\right)$. We need to relate d^3y to d^3x' for the integral we want. Consider the gradient of one component:

$$\begin{aligned}
\nabla' y_i &= \nabla'\left(x_i' - r_i\left(t'\right)\right) = \nabla'[\![x_i' - r_i]\!] \\
&= [\![\nabla'\left(x_i' - r_i\right)]\!] + \frac{\mathbf{x}-\mathbf{x}'}{c|\mathbf{x}-\mathbf{x}'|}\left[\!\!\left[\frac{\partial}{\partial t}\left(x_i' - r_i\right)\right]\!\!\right] \\
&= [\![\nabla' x_i']\!] + \frac{1}{c}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\left[\!\!\left[-\frac{\partial r_i}{\partial t}\right]\!\!\right]
\end{aligned} \tag{4.5}$$

[since r_i is a function of t but not $\mathbf{x}'$ directly.] Choose axes such that $\mathbf{x}' = (x_1', x_2', x_3')$ with component 1 in the $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ direction. Then the second term is present only for the $\mathbf{x_1}$ component not the other two (because they are $\perp$ to $\mathbf{x} - \mathbf{x}'$). Also $(\nabla' x_i') = \delta_{ij}$. [i.e. 1 iff $i = j$]. Thus

$$\begin{aligned}
\frac{\partial y_1}{\partial x_1'} &= 1 + \frac{1}{c}\left[\!\!\left[-\frac{\partial r_1}{\partial t}\right]\!\!\right] \\
\frac{\partial y_2}{\partial x_2'} &= 1 \qquad ; \qquad \frac{\partial y_3}{\partial x_3'} = 1 \quad .
\end{aligned} \tag{4.6}$$

Consequently

$$\begin{aligned}
d^3y &= dy_1 dy_2 dy_3 = \left(1 - \frac{1}{c}\left[\!\!\left[\frac{\partial r_1}{\partial t}\right]\!\!\right]\right) dx_1' dx_2' dx_3' \\
&= \left[\!\!\left[1 - \frac{1}{c}\frac{\mathbf{R}}{R}.\frac{\partial \mathbf{r}}{\partial t}\right]\!\!\right] d^3x'
\end{aligned} \tag{4.7}$$

Let's write

$$\kappa \equiv 1 - \frac{1}{c}\frac{\mathbf{R}}{R}.\frac{\partial \mathbf{r}}{\partial t} = 1 - \frac{1}{c}\hat{\mathbf{R}}.\mathbf{v} \tag{4.8}$$

Then

$$\int \delta\left(\mathbf{y}\right) d^3\mathbf{x}' = \int \delta\left(\mathbf{y}\right)\frac{d^3y}{[\![\kappa]\!]} = \frac{1}{[\![\kappa]\!]} \tag{4.9}$$

And finally

$$\phi\left(\mathbf{x}, t\right) = \frac{q}{4\pi\epsilon_0}\left[\!\!\left[\frac{1}{\kappa R}\right]\!\!\right] \tag{4.10}$$

By exactly the same process we can obtain the correct value for each component of $\mathbf{A}$ and in total

$$\mathbf{A}\left(\mathbf{x}, t\right) = \frac{\mu_0 q}{4\pi}\left[\!\!\left[\frac{\mathbf{v}}{\kappa R}\right]\!\!\right] \quad . \tag{4.11}$$

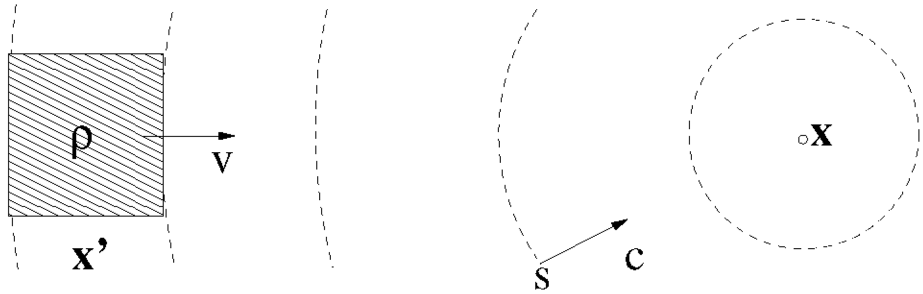

Figure 4.2: Integral of charge density over a square-shaped moving charge at retarded time.

$(\mathbf{v} = \partial \mathbf{r}/\partial t)$, $\mathbf{j} = q\mathbf{v}\delta$. These expressions are called the "Liénard-Wiechert" potentials of a moving point charge. Since the κ correction factor is so important and the scientific literature is strewn with papers that get it wrong, let's obtain the result graphically. The retarded integral $\int [\![\rho]\!]\, d^3x'$ can be viewed as composed of contributions from a spherical surface S which sweeps inward towards the observation (field) point $\mathbf{x}$, at the speed of light, arriving at time t. The charge that we integrate is the value of ρ when the surface S sweeps past. If we are dealing with a localized charge density, such as illustrated, the surface can be approximated as planar at the charge. If the charge region is moving rigidly at speed v towards $\mathbf{x}$, then its influence or contribution to the integral is increased because by the time the surface S has swept from front to back, the charge has moved. Consequently, the volume of the contribution (in x') is larger by the ratio $\frac{L'}{L}$ of the $\frac{\text{additional volume + charge volume}}{\text{charge volume}}$ How much is this? When does S reach 'front' of charge? At the moment S reaches the front,

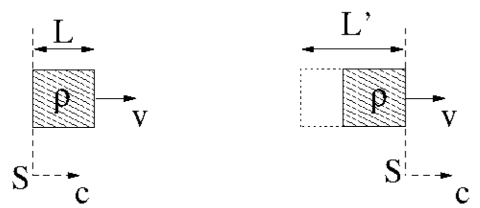

Figure 4.3: Snapshots as the integration surface, S, crosses the back and the front of the charge.

$$L' = c\Delta t = v\Delta t + L \tag{4.12}$$

So

$$\Delta t = (c - v)\, L \qquad \text{and} \qquad L' = \frac{c}{c - v} L = \frac{1}{1 - \frac{v}{c}} L \tag{4.13}$$

Thus

$$\frac{L'}{L} = \frac{1}{1 - \frac{v}{c}} = \frac{1}{\kappa} \tag{4.14}$$

as before. Notice that transverse velocity does nothing, and that approximations implicit in taking S to be planar become exact for a point charge, with spatial extent $\to 0$. The quantity κ can also be seen to relate intervals of time, dt, to the corresponding retarded time intervals, dt'.

$$t' = t - \frac{R'}{c} \qquad \text{or} \qquad t = t' + \frac{R'}{c} \tag{4.15}$$

So

$$\frac{dt}{dt'} = 1 + \frac{1}{c}\frac{dR'}{dt'} \quad . \tag{4.16}$$

But

$$\begin{aligned} \frac{dR'}{dt'} &= \frac{d}{dt'}|\mathbf{x} - \mathbf{x}'| = \frac{d}{dt'}\left\{(\mathbf{x} - \mathbf{x}') \, . \, (\mathbf{x} - \mathbf{x}')\right\}^{1/2} \\ &= \frac{-\mathbf{v} \, . \, (\mathbf{x} - \mathbf{x}')}{R'} = -\frac{\mathbf{v}.\mathbf{R}'}{R'} \end{aligned} \tag{4.17}$$

Hence

$$\frac{dt}{dt'} = 1 - \frac{\mathbf{v}.\mathbf{R}'}{cR'} = \kappa \tag{4.18}$$

Strictly speaking, it is the value of κ at retarded time, when the surface S passes the particle, that is required here if $\mathbf{v}$ is changing.

4.2 Potential of a Point Charge in Uniform Motion

An important special case is when $\mathbf{v} = \dot{\mathbf{r}} = \text{const.}$ From the retarded solution Lorentz derived his transformation, which is the basis of special relativity. Take axes such that $\mathbf{v} = v\hat{\mathbf{x}}$. We need to calculate the potential at $\mathbf{x} = (x, y, z)$ and we'll suppose that the particle is at the origin at time of interest, $(t = 0)$. Tricky part is just to calculate the retarded time t' and position $\mathbf{x}'$. By definition

$$c^2 (-t')^2 = R'^2 = (x - vt')^2 + y^2 + z^2 \quad . \tag{4.19}$$

Substitute $(-t')v = -x'$

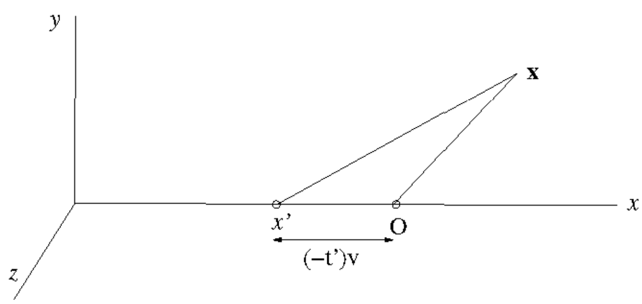

Figure 4.4: Coordinates of a uniformly moving charge at $\mathbf{x}(t)$.

$$\frac{c^2}{v^2}x'^2 = (x - x')^2 + y^2 + z^2 \tag{4.20}$$

[1][Feynman 21-6]

Gather terms

$$x'^2\left(\frac{c^2}{v^2}-1\right)+2xx'-\left(x^2+y^2+z^2\right)=0 \tag{4.21}$$

solution

$$\left(\frac{c^2}{v^2}-1\right)x'=-x\pm\sqrt{x^2+(x^2+y^2+z^2)\left(\frac{c^2}{v^2}-1\right)} \tag{4.22}$$

(where the - sign must be taken). And

$$R'=-\frac{c}{v}x'=\frac{c}{v}\frac{x+\sqrt{x^2+(x^2+y^2+z^2)\left(\frac{c^2}{v^2}-1\right)}}{\frac{c^2}{v^2}-1} \tag{4.23}$$

we also need the retarded value of κ i.e. $1-(\mathbf{R}'/R').(\mathbf{v}/c)$.

$$\kappa'=1-\frac{v}{c}.\frac{x-x'}{R'} \tag{4.24}$$

and

$$\kappa'R'=R'-\frac{v}{c}\left(x-x'\right)=R'-\frac{v}{c}\left(x+\frac{v}{c}R'\right)=\left(1-\frac{v^2}{c^2}\right)R'-\frac{v}{c}x \tag{4.25}$$

Substituting for R' we get

$$\begin{aligned}\kappa'R' &= \frac{v}{c}\left\{x+\sqrt{x^2+(x^2+y^2+z^2)\left(\frac{c^2}{v^2}-1\right)}-x\right\}\\ &= \frac{v}{c}\sqrt{x^2+(x^2+y^2+z^2)\left(\frac{c^2}{v^2}-1\right)}\\ &= \sqrt{\frac{x^2}{1-\frac{v^2}{c^2}}+y^2+z^2}\sqrt{1-\frac{v^2}{c^2}}\end{aligned} \tag{4.26}$$

This is the value at time $t=0$. At any other time t, the particle is at the position $x=vt$ instead of at the origin, $x=0$. Our formula was developed for the particle at the origin. So to use it we must move the origin to $x=vt$, which means we simply have to replace x in this formula with $x-vt$. So finally, substituting the general result for $\kappa'R'$ into the Liénard-Wiechert formula we get

$$\phi(\mathbf{x},t)=\frac{q}{4\pi\epsilon_0}\left[\!\left[\frac{1}{\kappa R}\right]\!\right]-\frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{\left(\frac{x-vt}{\sqrt{1-v^2/c^2}}\right)^2+y^2+z^2}} \tag{4.27}$$

See how we have the beginnings of relativity. We get electromagnetic potential dependence on spatial coordinates that can only be consistent with the formula in the frame of reference in which the particle is at rest:

$$\phi=\frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(x_1^2+y_1^2+z_1^2)}} \tag{4.28}$$

if coordinates transform as

$$x_1 = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , \quad y_1 = y \quad , \quad z_1 = z \quad . \tag{4.29}$$

This is the (spatial part of the) Lorentz transformation, incorporating the Fitzgerald contraction in the direction of motion. Now we also need to recognize there is a vector potential

$$\mathbf{A} = \frac{\mu_0 q}{4\pi}\left[\!\!\left[\frac{\mathbf{v}}{\kappa R}\right]\!\!\right] = \frac{\mu_0 q}{4\pi}\frac{\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}\sqrt{\left(\frac{x - vt}{\sqrt{1 - v^2/c^2}}\right)^2 + y^2 + z^2}} \tag{4.30}$$

So the electric field has both contributions:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad . \tag{4.31}$$

To evaluate these, denote by R'' the quantity in the denominator of ϕ and $\mathbf{A}$:

$$R'' \equiv \sqrt{\left(\frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}\right)^2 + y^2 + z^2} \quad . \tag{4.32}$$

[Note that this is <u>not</u> R', the retarded radius]. Its derivatives are

$$\frac{\partial R''}{\partial x} = \frac{x - vt}{1 - \frac{v^2}{c^2}}\frac{1}{R''} \quad ; \quad \frac{\partial R''}{\partial y} = \frac{y}{R''} \quad ; \quad \frac{\partial R''}{\partial z} = \frac{z}{R''} \quad ; \quad \frac{\partial R''}{\partial t} = \frac{-v\,(x - vt)}{\left(1 - \frac{v^2}{c^2}\right)R''} \quad . \tag{4.33}$$

Consequently

$$\nabla\frac{1}{R''} = \frac{-1}{R''^2}\nabla R'' = -\frac{1}{R''^3}\left(\frac{x - vt}{1 - v^2/c^2}\ , y, z\right) \tag{4.34}$$

giving

$$-\nabla\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\frac{1}{R''^3}\left(\frac{x - vt}{1 - v^2/c^2}\ , y, z\right) \tag{4.35}$$

and

$$-\frac{\partial \mathbf{A}}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{1 - v^2/c^2}}\frac{1}{R''^3}\left(\frac{(-v^2/c^2)\,(x - vt)}{(1 - v^2/c^2)}\ , 0, 0\right) \tag{4.36}$$

so

$$\mathbf{E} = -\nabla\phi - \mathbf{A} = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{1 - v^2/c^2}}\frac{1}{R''^3}\,(x - vt\ , y, z) \quad . \tag{4.37}$$

This is a remarkable result. It shows that despite the fact that contributions to E arise from the <u>retarded</u> position of the particle, the direction of $\mathbf{E}$ is actually radially outward from the instantaneous (i.e. <u>non</u> retarded) position. The $\mathbf{E}$ field at $t = 0$ is along the radius vector (x, y, z). The electric field is not just the same as for a stationary charge. The field is not

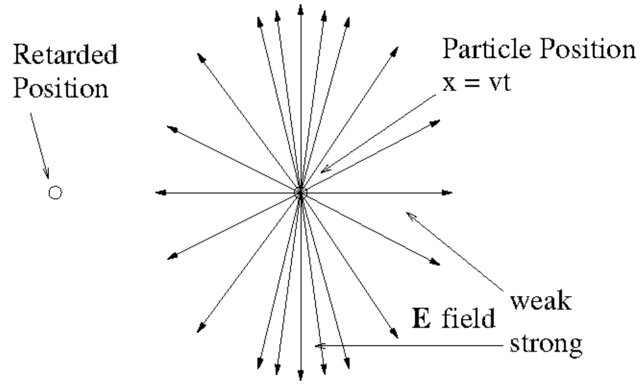

Figure 4.5: Electric field lines of a charge in uniform motion point outward from the instantaneous (not retarded) position but the field strength is not symmetric.

spherically symmetric, since it is proportional to

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}\left(\left(\frac{x-vt}{\sqrt{(1-v^2/c^2)}}\right)^2+y^2+z^2\right)^{3/2}} \tag{4.38}$$

which makes it stronger in the perpendicular direction and weaker in the parallel direction.

The magnetic field may be obtained from $\mathbf{B} = \nabla\wedge\mathbf{A}$ by recognizing $\nabla\wedge(f\mathbf{v}) = -\mathbf{v}\wedge\nabla f$, if $\mathbf{v}$ is constant. Hence, using $\mathbf{A} = \mathbf{v}\phi/c^2$,

$$\mathbf{B} = -\frac{\mathbf{v}}{c^2}\wedge\nabla\phi = \frac{\mathbf{v}}{c^2}\wedge\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right) = \frac{1}{c^2}\mathbf{v}\wedge\mathbf{E} \quad . \tag{4.39}$$

[The latter form uses the fact that ($\mathbf{A}$ and) $\frac{\partial\mathbf{A}}{\partial t}$ are parallel to $\mathbf{v}$ so $\mathbf{v}\wedge\frac{\partial\mathbf{A}}{\partial t} = 0$]. This expression for the magnetic field can also be rewritten, by noticing that $\mathbf{E}$ is in the direction of $\mathbf{R}$, $\mathbf{R}'\wedge\mathbf{R} = (t-t')\mathbf{v}\wedge\mathbf{R}$ and $t-t' = R'/c$; so $\mathbf{v}\wedge\mathbf{E} = (\mathbf{R}'c/R')\wedge\mathbf{E}$. To summarize:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{v}t}{\sqrt{1-\frac{v^2}{c^2}}\left(\left(\frac{x-vt}{\sqrt{(1-v^2/c^2)}}\right)^2+y^2+z^2\right)^{3/2}} \tag{4.40}$$

and

$$\mathbf{B} = \frac{1}{c^2}\mathbf{v}\wedge\mathbf{E} = \frac{1}{cR'}\mathbf{R}'\wedge\mathbf{E} \quad . \tag{4.41}$$

A helpful way to think of the result that the electric field is still radial but with a non-spherically-symmetric distribution, is to think about what happens to the field lines when viewed in the lab frame of reference [components (x, y, z)] compared with a frame of reference in which the particle is at rest [components (x_1, y_1, z_1)]. It turns out that the electric field we have calculated is exactly that which would be obtained by assuming that the spherically symmetric distribution of field-lines in the rest-frame is simply compressed together with the rest of space in the x-direction through the coordinate transform of eq 4.29. This contraction is illustrated in figure 4.6.

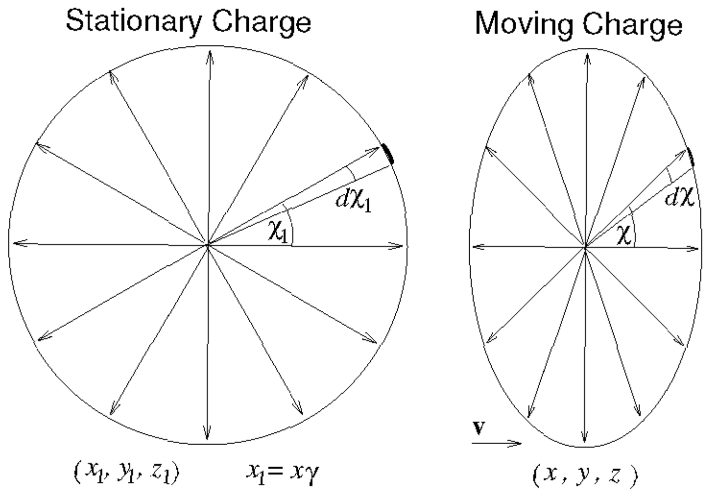

Figure 4.6: Contraction of space which gives the electric field-line distribution of a moving charge.

For a purely geometric compression in one dimension like this, the angles between the direction of $\mathbf{R}$ and $\mathbf{v}$ (for the two cases) are related by

$$\tan\chi_1 = y_1/x_1 = y/(\gamma x) = (1/\gamma)\tan\chi \quad , \tag{4.42}$$

where $\gamma = 1/\surd(1 - v^2/c^2)$. Consequently

$$\cos\chi_1 = \frac{1}{\sqrt{1+\tan^2\chi_1}} = \frac{1}{\sqrt{1+(1/\gamma)^2\tan^2\chi}} \quad . \tag{4.43}$$

Now the element of solid angle corresponding to an angle increment $d\chi$ is $\mathbf{d\Omega} = 2\pi\sin\chi\, d\chi$ and

$$\begin{aligned}\sin\chi_1\, d\chi_1 = -d(\cos\chi_1) &= -d\left(\frac{\gamma}{\sqrt{\gamma^2+\tan^2\chi}}\right) = \frac{\gamma\tan\chi\ \sec^2\chi}{(\gamma^2+\tan^2\chi)^{3/2}}\, d\chi \\ &= \frac{\gamma}{(\gamma^2\cos^2\chi+\sin^2\chi)^{3/2}}\,\sin\chi\, d\chi \quad .\end{aligned} \tag{4.44}$$

So the relationship between corresponding solid-angles is

$$\begin{aligned} d\Omega_1 &= \frac{\gamma}{(\gamma^2\cos^2\chi+\sin^2\chi)^{3/2}}\, d\Omega \\ &= \frac{\gamma R^3}{(\gamma^2x^2+y^2+z^2)^{3/2}}\, d\Omega \\ &= \frac{\gamma R^3}{R''^3}\, d\Omega \quad , \end{aligned} \tag{4.45}$$

where $R = x^2+y^2+z^2$. Therefore if the field-lines are compressed in this purely geometrical way, the number of field-lines per unit solid angle, which is proportional to the electric field intensity, in the lab-frame is equal to the value in the rest-frame times the factor $d\Omega_1/d\Omega = \gamma R^3/R''^3$. Thus the geometric compression would lead to an electric field:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{R}}{R^3}\,\frac{\gamma R^3}{R''^3} \quad . \tag{4.46}$$

This is precisely what we calculated directly from the equations of the fields. In other words, we can regard the non-symmetric electric field of eq 4.40 as arising from a compression of space corresponding to the Lorentz transformation (eq 4.29).

We are not here invoking the Lorentz transformation based on an understanding of special relativity. In fact the opposite is the historic situation. Lorentz's transform was part of the prior basis for the discovery of relativity. See Jackson 1998 pp. 514-518 for a discussion of electromagnetism as the historic foundation of relativity. Maxwell's equations are already fully relativistic. They don't need to be corrected for relativistic effects, the way Newton's laws require correction for example. Of course the point is stronger than that: Maxwell's equations can only be consistent when special relativity applies (i.e. Lorentz, not Galilean transformations). We don't have time to cover relativity but we don't have to make a special point of it since EM equations already are relativisitc.

4.3 Fields of a Generally-Moving Charge

The Lienard Wiechert potentials give the general potential solution. From them we can obtain the general $\mathbf{E}$ and $\mathbf{B}$ fields from a particle moving with arbitrary velocity: not just uniform $\mathbf{v}$. Since both potentials and fields depend only on the values at retarded time, our calculation will be almost the same as for the uniform motion with the exception that we must use the value of $\mathbf{v}$ at that retarded time *and* we must account for possible time-derivatives of $\mathbf{v}$. Our derivations of ϕ and $\mathbf{A}$ go through exactly as before except that the origin of coordinates is at a point $\mathbf{x}' + \mathbf{v}'t'$ along the projected path of the particle if it were to continue past the retarded time with constant speed $\mathbf{v}'$. [Here we are putting prime on $\mathbf{v}$ to remind that it is the retarded value we require.]

$$\phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(x - v't) + (y^2 + z^2)\left(1 - \frac{v'^2}{c^2}\right)}} \tag{4.47}$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}'}{\sqrt{(x - v't) + (y^2 + z^2)\left(1 - \frac{v'^2}{c^2}\right)}} \tag{4.48}$$

Now we need to get the fields by differentiation. We get exactly the same terms as before plus extra terms arising from the time derivative of $\mathbf{v}$. We could do this directly by taking into account all the contributions. Instead, let's do a vector calculation starting with the Lienard-Wiechert forms:

$$\phi = \frac{q}{4\pi\epsilon_0}\left[\!\left[\frac{1}{\kappa R}\right]\!\right] \quad ; \quad \mathbf{A} = \frac{q}{4\pi\epsilon_0 c^2}\left[\!\left[\frac{\mathbf{v}}{\kappa R}\right]\!\right] \quad . \tag{4.49}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = \frac{q}{4\pi\epsilon_0}\left\{-\nabla\left[\!\left[\frac{1}{\kappa R}\right]\!\right] - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\!\left[\frac{\mathbf{v}}{\kappa R}\right]\!\right]\right\} \tag{4.50}$$

Again, extreme care must be taken with the differentials. For any function $f(\mathbf{x}, t)$,

$$\nabla[\![f]\!] = \nabla f\left(\mathbf{x}, t - \tfrac{|\mathbf{x}-\mathbf{x}'|}{c}\right) \tag{4.51}$$

$$= [\![\nabla f]\!] - \left[\!\left[\frac{\partial f}{\partial t}\right]\!\right]\frac{1}{c}\nabla|\mathbf{x}-\mathbf{x}'| \tag{4.52}$$

This is not the same situation as we had before. There we had ∇' i.e. gradient with respect to retarded position, $\mathbf{x}'$, keeping $\mathbf{x}$ and t fixed. Here we are talking about gradient w.r.t. $\mathbf{x}$ keeping t fixed. Apply the above equation to the function $|\mathbf{x}-\mathbf{x}'|$ which is, strictly speaking, $[\![|\mathbf{x}-\mathbf{r}|]\!]$ or $[\![R]\!]$. We get

$$\nabla[\![|\mathbf{x}-\mathbf{r}|]\!] = [\![\nabla|\mathbf{x}-\mathbf{r}|]\!] - \left[\!\left[\frac{\partial}{\partial t}|\mathbf{x}-\mathbf{r}|\right]\!\right]\frac{1}{c}\nabla[\![\mathbf{x}-\mathbf{r}]\!] \tag{4.53}$$

i.e.

$$\left[\!\left[\left(1 - \frac{\mathbf{R}'}{R'}.\frac{\mathbf{v}}{c}\right)\right]\!\right]\nabla[\![x-r]\!] = [\![\nabla|\mathbf{x}-\mathbf{r}|]\!] \tag{4.54}$$

So

$$\nabla[\![R]\!] = \left[\!\left[\frac{1}{\kappa}\nabla R\right]\!\right] = \left[\!\left[\frac{\mathbf{R}}{\kappa R}\right]\!\right] \quad . \tag{4.55}$$

Then returning to the general identity:

$$\nabla[\![f]\!] = [\![\nabla f]\!] - \left[\!\left[\frac{\partial f}{\partial t}\right]\!\right]\frac{1}{c}\left[\!\left[\frac{\mathbf{R}}{\kappa R}\right]\!\right] \tag{4.56}$$

$$= \left[\!\left[\nabla f - \frac{\mathbf{R}}{c\kappa R}\frac{\partial f}{\partial t}\right]\!\right] \quad . \tag{4.57}$$

In the same way

$$\frac{\partial}{\partial t}[\![f]\!] = \frac{\partial}{\partial t}f\left(\mathbf{x}, t - \frac{[\![|\mathbf{x}-\mathbf{r}|]\!]}{c}\right) \tag{4.58}$$

$$= \left[\!\left[\frac{\partial f}{\partial t}\right]\!\right]\left(1 - \frac{1}{c}\frac{\partial}{\partial t}[\![|\mathbf{x}-\mathbf{r}|]\!]\right) \tag{4.59}$$

Substituting $R = |\mathbf{x}-\mathbf{r}|$ for f shows

$$\frac{\partial}{\partial t}[\![R]\!] = \left[\!\left[\frac{1}{\kappa}\frac{\partial R}{\partial t}\right]\!\right] \tag{4.60}$$

and, since

$$\frac{1}{c}\frac{\partial R}{\partial t} = \kappa - 1 \quad , \tag{4.61}$$

we have generally

$$\frac{\partial}{\partial t}[\![f]\!] = \left[\!\!\left[\frac{1}{\kappa}\frac{\partial f}{\partial t}\right]\!\!\right] \quad . \tag{4.62}$$

Ok, now we have the tools to evaluate $\mathbf{E}$:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\!\!\left[-\nabla\left(\frac{1}{\kappa R}\right) + \frac{\mathbf{R}}{c\kappa R}\frac{\partial}{\partial t}\left(\frac{1}{\kappa R}\right) - \frac{1}{c^2\kappa}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{\kappa R}\right)\right]\!\!\right] \quad . \tag{4.63}$$

With everything inside the retardation operator, it is safe to proceed with algebra as if $\mathbf{x}'$ is fixed. In particular,

$$\begin{aligned}
\kappa R &= R - \frac{\mathbf{R.v}}{c} && (4.64)\\
\nabla(\kappa R) &= \frac{\mathbf{R}}{|R|} - \frac{\mathbf{v}}{c} && (4.65)\\
\frac{\partial}{\partial t}(\kappa R) &= \frac{\partial R}{\partial t} - \frac{1}{c}\frac{\partial \mathbf{R}}{\partial t}.\mathbf{v} - \frac{\mathbf{R}}{c}.\frac{\partial \mathbf{v}}{\partial t}\\
&= \dot{R} + \frac{v^2}{c} - \frac{1}{c}\mathbf{R}.\dot{\mathbf{v}} && (4.66)
\end{aligned}$$

where dot denotes $\frac{\partial}{\partial t}$. The terms in $\mathbf{E}$ are then

$$\begin{aligned}
-\nabla\left(\frac{1}{\kappa R}\right) &= \frac{1}{\kappa^2R^2}\nabla(\kappa R) = \frac{1}{\kappa^2R^2}\left(\frac{\mathbf{R}}{R} - \frac{\mathbf{v}}{c}\right) && (4.67)\\
\frac{1}{c\kappa}\frac{\mathbf{R}}{R}\frac{\partial}{\partial t}\left(\frac{1}{\kappa R}\right) &= -\frac{1}{c\kappa^3R^3}\mathbf{R}\frac{\partial}{\partial t}(\kappa R) = -\frac{1}{c\kappa^3R^3}\mathbf{R}\left(\dot{R} + \frac{v^2}{c} - \frac{1}{c}\mathbf{R}.\dot{\mathbf{v}}\right) && (4.68)\\
-\frac{1}{c^2\kappa}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{\kappa R}\right) &= -\frac{1}{c\kappa^3R^2}\left(\dot{R} + \frac{v^2}{c} - \frac{1}{c}\mathbf{R}.\dot{\mathbf{v}}\right)\mathbf{v} - \frac{1}{c^2\kappa^2R}\dot{\mathbf{v}} && (4.69)
\end{aligned}$$

Gathering terms together, and denoting $\hat{\mathbf{R}} = \frac{\mathbf{R}}{R}$, we get

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\!\!\left[\frac{1}{\kappa^3R^2}\left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right)\left(1 - \frac{v^2}{c^2}\right) - \frac{1}{c\kappa^3R}\left\{\kappa\frac{\dot{\mathbf{v}}}{c} - \left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right)\left(\hat{\mathbf{R}}.\frac{\dot{\mathbf{v}}}{c}\right)\right\}\right]\!\!\right] \tag{4.70}$$

or alternatively, using vector triple product identities,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\!\!\left[\frac{1}{\kappa^3R^2}\left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \frac{1}{c\kappa^3R}\hat{\mathbf{R}}\wedge\left[\left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right)\wedge\frac{\dot{\mathbf{v}}}{c}\right]\right]\!\!\right] \tag{4.71}$$

The magnetic field is $\mathbf{B} = \nabla\wedge\mathbf{A}$ which is

$$\begin{aligned}
\mathbf{B} &= \frac{q}{4\pi\epsilon_0c^2}\left\{\nabla\wedge\left[\!\!\left[\frac{\mathbf{v}}{\kappa R}\right]\!\!\right]\right\}\\
&= \frac{q}{4\pi\epsilon_0c^2}\left[\!\!\left[\nabla\wedge\left(\frac{\mathbf{v}}{\kappa R}\right) - \frac{\mathbf{R}}{c\kappa R}\wedge\frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{\kappa R}\right)\right]\!\!\right] \quad , && (4.72)
\end{aligned}$$

by an identity directly analogous to the one we showed for gradient. Also

$$\begin{aligned}
\nabla \wedge \left(\frac{\mathbf{v}}{\kappa R}\right) &= -\mathbf{v} \wedge \nabla \left(\frac{1}{\kappa R}\right) \\
&= -\frac{1}{\kappa^2 R^2} \mathbf{v} \wedge \left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right) \\
&= -\frac{1}{\kappa^2 R^2} c\hat{\mathbf{R}} \wedge \left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right) = -c\hat{\mathbf{R}} \wedge \nabla \left(\frac{1}{\kappa r}\right) \quad . \qquad (4.73)
\end{aligned}$$

Hence

$$\mathbf{B} = \frac{q}{4\pi\epsilon_0 c^2} \left[\!\!\left[-c\hat{\mathbf{R}} \wedge \nabla \left(\frac{1}{\kappa R}\right) - \hat{\mathbf{R}} \wedge \frac{1}{c\kappa} \frac{\partial}{\partial t} \left(\frac{\mathbf{v}}{\kappa R}\right) \right]\!\!\right] = \frac{1}{c} [\![\hat{\mathbf{R}}]\!] \wedge \mathbf{E} \quad , \qquad (4.74)$$

by comparison with our expression (4.63) for $\mathbf{E}$.

Summarizing our results, the fields due to a point charge q moving with variable velocity $\mathbf{v}$ such that the radius vector from the charge to the field-point is $\mathbf{R}$ may be expressed using $\kappa \equiv 1 - \hat{\mathbf{R}}.\mathbf{v}/c$ as:

$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0} \left[\!\!\left[\frac{1}{\kappa^3 R^2} \left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right) \left(1 - \frac{v^2}{c^2}\right) + \frac{1}{c\kappa^3 R} \hat{\mathbf{R}} \wedge \left[\left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right) \wedge \frac{\dot{\mathbf{v}}}{c} \right] \right]\!\!\right] \qquad (4.75) \\
\mathbf{B} &= \frac{1}{c} [\![\hat{\mathbf{R}}]\!] \wedge \mathbf{E} \quad . \qquad (4.76)
\end{aligned}$$

There are several different forms of these expressions, useful to illustrate different aspects of the fields of a moving point charge. See Jackson and Feynman for discussion of some of these.

4.4 Radiation from Moving Charges

4.4.1 Near Field and Radiation Terms

The form for $\mathbf{E}$ that we obtained was exhibited in a way that had 2 separate terms. The first of those terms does not contain $\dot{\mathbf{v}}$ while the second is proportional to $\dot{\mathbf{v}}$. Therefore the first term is exactly what would be obtained for uniform motion $\dot{\mathbf{v}} = 0$ (although this is not obvious when comparing with our earlier formula expressed in coordinates). Also, everything inside the brackets is dimensionless ($\hat{R}, \frac{\mathbf{v}}{c}$) *except* $\frac{1}{\kappa^3 R^2}$ and $\frac{1}{c\kappa^3 R} \frac{\dot{\mathbf{v}}}{c}$. These factors decide the behaviour of their respective terms at large field-point distances, R. The 'static' (constant $\mathbf{v}$) term is $\propto \frac{1}{R^2}$ but the $\dot{\mathbf{v}}$ term is $\propto \frac{1}{R}$. Consequently, the Poynting vector is

$$\mathbf{E} \wedge \mathbf{H} = \frac{1}{\mu_0} \mathbf{E} \wedge \mathbf{B} = \frac{1}{\mu_0 c} \mathbf{E} \wedge \left([\![\hat{\mathbf{R}}]\!] \wedge \mathbf{E} \right) \propto \frac{1}{R^4} \quad \text{or} \quad \frac{1}{R^2} \qquad (4.77)$$

respectively. If we ask about the total EM power flux across a spherical surface far from the charge, that value scales like the surface area $4\pi R^2$ tims $\mathbf{E} \wedge \mathbf{H}$. Thus power flux $\propto \frac{1}{R^2}$ for the constant v term, and $\propto 1$ for the $\dot{\mathbf{v}}$ term. We see then, that the constant-v term gives rise to vanishingly small power flux far from the charge but the $\dot{\mathbf{v}}$ term gives rise to finite power flux even at infinity. This distinction requires us to regard these two terms as the "near field" term:

$$E \propto \frac{1}{R^2} v \tag{4.78}$$

and "radiation" term:

$$\mathbf{E} \propto \frac{1}{R} \dot{v} \quad . \tag{4.79}$$

A charged particle *radiates* only if it *accelerates*.

4.4.2 Radiation into a Specific Solid-angle

Having identified just the $1/R$ term as the radiation term, we will drop the other, near field, term from consideration. Imagine, then, a sphere of radius R surrounding the retarded position of the particle. The Poynting vector of the radiation term there is

$$\begin{aligned} \mathbf{E} \wedge \mathbf{B}/\mu_0 &= \mathbf{E} \wedge \left(\left[\!\left[\hat{\mathbf{R}} \right]\!\right] \wedge \mathbf{E} \right) / c\mu_0 = \left(E^2 / c\mu_0 \right) \left[\!\left[\hat{\mathbf{R}} \right]\!\right] - \left(\mathbf{E} . \left[\!\left[\hat{\mathbf{R}} \right]\!\right] \right) \mathbf{E} / c\mu_0 \\ &= \frac{1}{c\mu_0} E^2 \left[\!\left[\hat{\mathbf{R}} \right]\!\right] \quad , \end{aligned} \tag{4.80}$$

where the last form recognizes that the radiation term has $\mathbf{E}$ perpendicular to $\left[\!\left[\hat{\mathbf{R}} \right]\!\right]$. Radiated energy thus crosses the sphere, normal to its surface with a local intensity (energy/unit area/unit time) $E^2/c\mu_0$, with E given by the second term of eq 4.71. One is very often interested in the power radiated per unit solid angle, Ω_s, subtended by the area at the point of radiation. By definition of solid angle, a small area of the sphere, A, subtends a solid angle A/R^2. Consequently the power per unit solid angle is $R^2 E^2 / c\mu_0$. The extra term R^2 cancels the R^2 occurring in E^2, leaving an expression independent of the radius, R, of the sphere. By convention we can write the power per unit solid angle using the notation

$$\frac{dP}{d\Omega_s} = \frac{R^2 E^2}{c\mu_0} = \frac{q^2}{4\pi\epsilon_0} \frac{1}{4\pi c} \left| \left[\!\left[\frac{1}{\kappa^3} \hat{\mathbf{R}} \wedge \left\{ \left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c} \right) \wedge \frac{\dot{\mathbf{v}}}{c} \right\} \right]\!\right] \right|^2 \tag{4.81}$$

4.4.3 Radiation from Non-relativistic Particles: Dipole Approximation

Considerable algebraic simplifications occur when $v/c << 1$ and so we can approximate $(\hat{\mathbf{R}} - \mathbf{v}/c) \simeq \hat{\mathbf{R}}$, and $\kappa = 1$. Then

$$\frac{dP}{d\Omega_s} = \frac{q^2}{4\pi\epsilon_0} \frac{1}{4\pi c} |\hat{\mathbf{R}} \wedge \left(\hat{\mathbf{R}} \wedge \frac{\dot{\mathbf{v}}}{c} \right) |^2 = \frac{q^2}{4\pi\epsilon_0} \frac{1}{4\pi c} \left(\frac{\dot{\mathbf{v}}}{c} \right)^2 \sin^2 \alpha \tag{4.82}$$

where α is the angle between $\hat{\mathbf{R}}$, the direction of the solid angle (propagation), and $\dot{\mathbf{v}}$, the acceleration. An integration of the total radiated power over the entire sphere (all solid-angles) can readily be done. Taking the direction of $\dot{\mathbf{v}}$ to be the polar direction, the integral is such that

$$d\Omega_s = 2\pi \sin \alpha \; d\alpha \quad . \tag{4.83}$$

So, noting that

$$\int \sin^2\alpha \; 2\pi \sin\alpha \; d\alpha = 2\pi \int \left(1-\cos^2\alpha\right)\sin\alpha \; d\alpha = 2\pi \left[-\cos\alpha + \frac{1}{3}\cos^3\alpha\right]_0^\pi = \frac{8\pi}{3} \quad , \tag{4.84}$$

we get

$$P = \int \frac{dP}{d\Omega_s} d\Omega_s = \frac{q^2}{4\pi\epsilon_0}\frac{2}{3c}\left(\frac{\dot{v}}{c}\right)^2 \quad . \tag{4.85}$$

This expression for the total radiation from a non-relativistic accelerated charge is known as Larmor's formula. The non-relativistic expressions for P and $dP/d\Omega_s$ are often referred to as the "dipole approximation" because they are exactly what is obtained for the radiation from a stationary oscillating dipole electric distribution when the electric dipole moment $\mathbf{p}$, is such that

$$\ddot{\mathbf{p}} = q\dot{\mathbf{v}} \quad . \tag{4.86}$$

Thus this radiation pattern and intensity is what is obtained also from dipole antennas that are much smaller than the radiation wavelength.

4.5 Radiation from Relativistic Particles

The general expression for radiation by an accelerated particle, without invoking approximations requiring $v << c$, is given by eq (4.81). However an important distinction must be drawn in discussions of energy per unit time between expressions based on time-at-field-point, t, such as eq (4.81), and expressions referring to time-at-particle, retarded time t'. If we want to know how much energy a particle is radiating per unit time-at-particle, which is what we do want if, for example, we want to calculate how rapidly the particle is losing energy, or indeed if we want to calculate the total energy radiated per unit volume by adding up the energy radiated by all the particles in that volume, then we must multiply expressions for energy per time-at-field-point by the ratio $dt/dt' = \kappa$. This conversion lowers the power of κ in the denominator by one. We shall work henceforth with such expressions of energy per unit time-at-particle and will indicate this by a prime on the power: P'. Even so, we still have a factor κ^5 in the denominator of $dP'/d\Omega_s$. This factor is the most important effect. Since $\kappa = 1 - \hat{\mathbf{R}}.\mathbf{v}/c = 1 - \beta\cos\theta$, when we are dealing with particles moving near the speed of light, κ becomes extremely small when $\theta \simeq 0$, that is for radiation in the direction along the particle's velocity. As a result, the radiation is greatly enhanced in this forward direction, an effect that is sometimes called the relativistic "headlight" effect.

4.5.1 Acceleration Parallel to v

The simplest case algebraically is when $\mathbf{v}$ and $\dot{\mathbf{v}}$ are parallel. The radiation is then rotationally symmetric about this direction, having the κ factor as its only alteration, from the dipole formula:

$$\frac{dP'}{d\Omega_s} = \frac{q^2}{4\pi\epsilon_0}\frac{\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \quad . \tag{4.87}$$

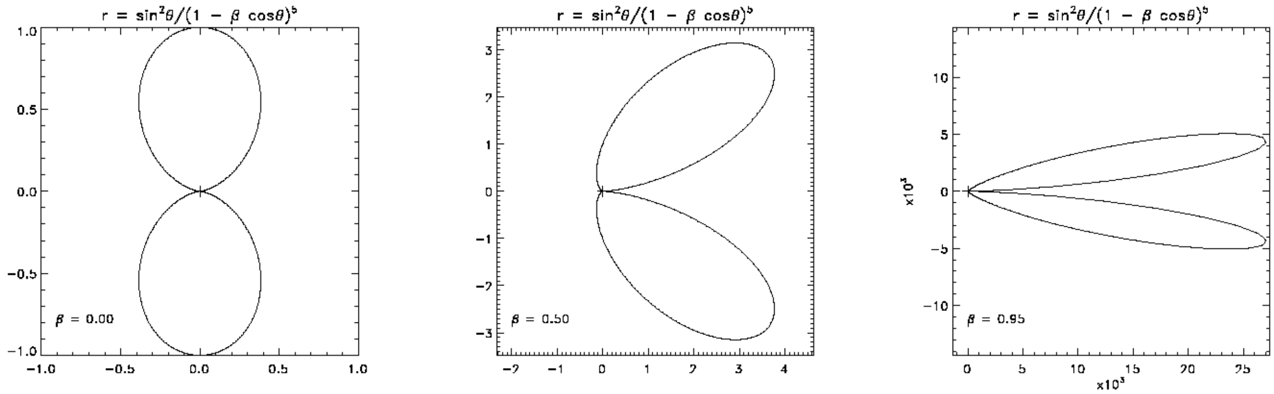

Figure 4.7: Polar plots of the radiation intensity as a function of direction, with acceleration parallel to $\mathbf{v}$, for different values of $\beta = v/c$. Velocity is in the x-direction.

The radiation in the exactly forward direction $\theta = 0$ is zero because of the $\sin^2\theta$ term. The maximum radiation is in the direction $\theta_m \simeq 1/(2\gamma)$ when $\beta \sim 1$. Here γ is the relativistic factor $(1 - v^2/c^2)^{-1/2}$. Moreover the intensity in this direction becomes extremely large as β gets close to one.

4.5.2 Acceleration Perpendicular to v

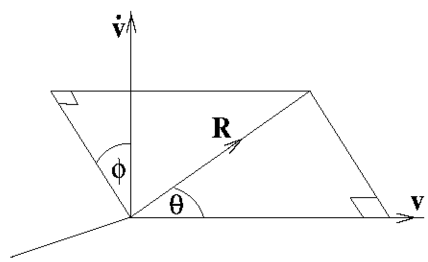

Figure 4.8: Definition of the angles for radiation when $\dot{\mathbf{v}}$ is perpendicular to $\mathbf{v}$.

An even more important case is when $\mathbf{v}$ and $\dot{\mathbf{v}}$ are perpendicular. One can then obtain

$$\frac{dP'}{d\Omega_s} = \frac{q^2}{4\pi\epsilon_0}\frac{\dot{v}^2}{4\pi c^3}\frac{1}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi\,(1-\beta^2)}{(1-\beta\cos\theta)^2}\right] \tag{4.88}$$

where θ is the angle of $\hat{\mathbf{R}}$ with respect to $\mathbf{v}$ and ϕ is the polar angle of $\hat{\mathbf{R}}$ about $\mathbf{v}$ measured with respect to $\dot{\mathbf{v}}$ as zero.

This distribution is likewise highly peaked in the forward direction for $\beta \sim 1$, having a typical half-angle extent of approximately $\frac{1}{\gamma}$.

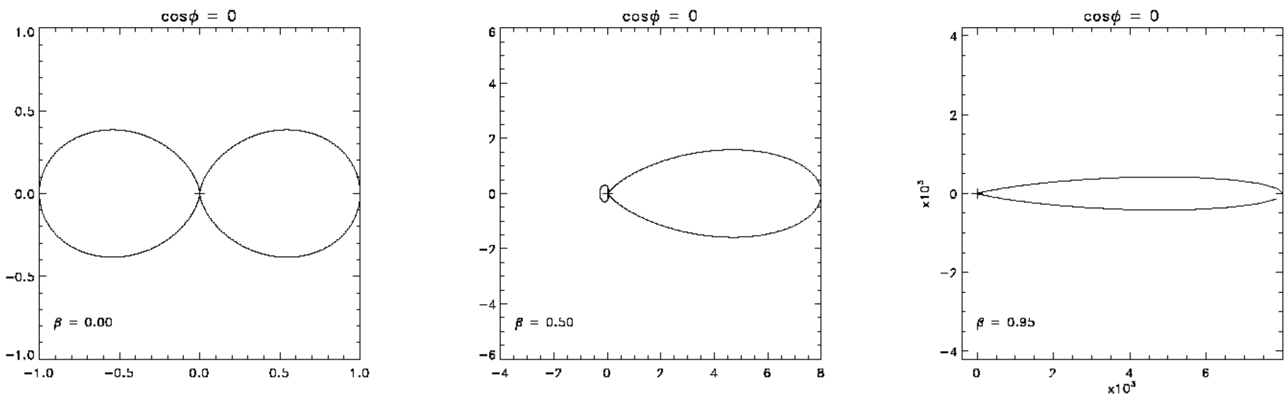

Figure 4.9: Polar plots of the radiation intensity as a function of direction, with acceleration perpendicular to $\mathbf{v}$, for the case where $\hat{\mathbf{R}}$ lies in the plane of $\mathbf{v}$ and $\dot{\mathbf{v}} \wedge \mathbf{v}$.

4.5.3 Total Radiated Power

The general expression (4.81) can be integrated over solid angles by elementary but tedious methods to obtain

$$\begin{aligned} P' &= \frac{q^2}{4\pi\epsilon_o}\frac{2}{3c}\gamma^6\left[\left(\frac{\dot{v}}{c}\right)^2 - \left(\frac{\dot{\mathbf{v}}}{c}\wedge\frac{\mathbf{v}}{c}\right)^2\right] \\ &= \frac{q^2}{4\pi\epsilon_0}\frac{2}{3c}\gamma^4\left[\left(\frac{\dot{v}}{c}\right)^2 - \left(\frac{\dot{\mathbf{v}}}{c}.\frac{\mathbf{v}}{c}\right)^2\right] \quad , \end{aligned} \tag{4.89}$$

the first form of which was obtained by Lienard (1898). These two alternate forms are convenient for obtaining the power when $\dot{\mathbf{v}}$ is parallel to $\mathbf{v}$:

$$P' = \frac{q^2}{4\pi\epsilon_0}\frac{2}{3c}\frac{\dot{v}^2}{c^2}\gamma^6 \tag{4.90}$$

and $\dot{\mathbf{v}}$ is perpendicular to $\mathbf{v}$:

$$P' = \frac{q^2}{4\pi\epsilon_0}\frac{2}{3c}\frac{\dot{v}^2}{c^2}\gamma^4 \quad . \tag{4.91}$$

These expressions give important quantitative information about the rate of energy loss by a charge undergoing acceleration. The first thing we can see is that a charge could never be accelerated through the velocity c, because $\gamma \to \infty$ at $\beta \to 1$ and so infinite amounts of radiation would be emitted. This remark is quite independent of Einstein's theory of relativity which shows that the mass becomes infinite as $\beta \to 1$. Thus in 1898 when Lienard obtained his expression he already could have deduced that a charge could not be accelerated past $v = c$. Second, let us compare the rate of radiative energy loss to the energy gain from an accelerating electrostatic force.

Write the field as equivalent to the field a distance r from a charge Zq so that the acceleration is

$$\dot{v} = \frac{Zq^2}{4\pi\epsilon_0}\frac{1}{r^2 m_0\gamma} \quad , \tag{4.92}$$

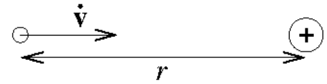

Figure 4.10: Comparing the radiative energy loss to the energy gained from force during acceleration due to a nearby charge.

accounting for the relativistic mass increase. Supposing $\dot{\mathbf{v}}$ to be parallel $\mathbf{v}$, the rate of radiative loss is

$$P' = \frac{q^2}{4\pi\epsilon_0}\frac{2}{3c^3}\frac{Z^2q^4}{(4\pi\epsilon_0)^2}\frac{1}{r^4}\frac{\gamma^4}{m_0^2} \quad . \tag{4.93}$$

This will equal the rate of gain of energy due to acceleration, namely

$$\frac{Zq^2}{4\pi\epsilon_0}\frac{v}{r^2} \quad ,$$

when

$$\left(\frac{Zq^2}{4\pi\epsilon_0 r}\right)^2 = Z\frac{3\beta\,(m_0c^2)^2}{2\gamma^4} \quad , \tag{4.94}$$

or

$$\frac{Zq^2}{4\pi\epsilon_0 r} = \left(\frac{3Z\beta}{\gamma^4}\right)^{1/2} m_0c^2 \quad . \tag{4.95}$$

The left-hand side, here, is the potential energy of the charge and the right-hand side is a square-root factor times the rest-mass of the charge (expressed as an energy). For modestly relativistic particles, when we can take the square-root factor to be of order unity, we therefore see that radiation would begin to have an important effect relative to the parallel acceleration only when an electron (for example) is in a potential well at a depth $\sim m_0c^2$ $= 511$ keV. Remembering that the binding energy of a hydrogen atom is only 13.6 eV this could happen only in the most exotic of situations (e.g. inner shells of heavy elements). Of course those situations would really have to be treated by quantum mechanics. Moreover these immensely strong electric fields ($\sim 10^{20}$ V/m) are never even approached in present accelerators. So radiation caused by acceleration parallel to v, such as in a linac, is never a serious consideration.

If $\mathbf{v}$ is perpendicular to $\dot{\mathbf{v}}$, however, the lowest order energy gain by the acceleration is zero. Compared with this the radiation may well be important. In the atomic force-field

$$P' = \frac{q^2}{4\pi\epsilon_0}\frac{2}{3c^3}\frac{Z^2q^4}{(4\pi\epsilon_0)^2}\frac{1}{r^4}\frac{\gamma^2}{m_0^2} \tag{4.96}$$

and the classical kinetic energy in a circular orbit at radius r is

$$\mathcal{E} = \frac{1}{2}\frac{Zq^2}{4\pi\epsilon_0 r} \quad . \tag{4.97}$$

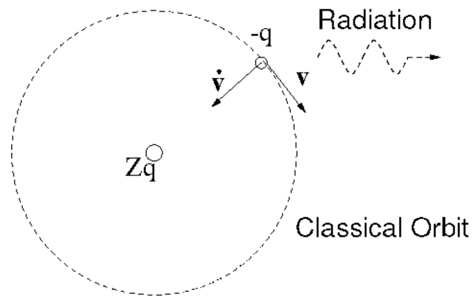

Figure 4.11: Radiation from a particle moving in a circular orbit arises from its perpendicular acceleration.

Hence the orbital energy is radiated with a characteristic time constant

$$\begin{aligned}\tau \sim \frac{\mathcal{E}}{P'} &= \frac{1}{2}\frac{Zq^2}{4\pi\epsilon_0 r}\left[\frac{q^2}{4\pi\epsilon_0}\frac{2c}{3}\left(\frac{Zq^2}{4\pi\epsilon_0 r}\right)^2\frac{1}{r^2}\frac{\gamma^2}{m_0^2c^4}\right]^{-1} \\ &= \frac{(m_0c^2)^2}{2\gamma^2}\frac{3r}{2c}Z\left(\frac{Zq^2}{4\pi\epsilon_0 r}\right)^{-2} = \frac{(m_0c^2)^2}{I^2}Z\frac{r}{c} \quad ,\end{aligned} \tag{4.98}$$

where I is the binding energy of the particle in this circular orbit. For a "classical" hydrogen atom circular orbit, $I = 13.6$eV, Z=1, and $r = a_0 = 5.29 \times 10^{-11}$ m (the Bohr radius) we get $\tau = 1.5 \times 10^{-9}s$. Thus the rate of loss of energy by an electron in a "classical" Bohr orbit is such that the electron would spiral into the nucleus in a few nanoseconds. This, of course, was one of the key problems with classical electrodynamics that physics faced in the early 1900s, which prompted the eventual discovery of quantum mechanics.

As an immediately practical matter, we can also ask how fast a particle radiates energy because of being accelerated by a magnetic field, in the circular orbit of a cyclotron, for example. In this case, the acceleration is $\dot{v} = v^2/d$, where d is the orbit radius. The power radiated is then, from eq (4.91),

$$P' = \frac{q^2}{4\pi\epsilon_0}\frac{2}{3c}\frac{v^4}{c^2r^2}\gamma^4 = \frac{q^2}{4\pi\epsilon_0}\frac{2c}{3}\frac{\beta^4\gamma^4}{r^2} \quad . \tag{4.99}$$

For a relativistic particle ($\beta \simeq 1$) the power therefore increases proportional to the fourth power of the energy (γ^4), and the energy loss per orbit for electrons moving with radius of curvature r can be written numerically in the form

$$\delta\mathcal{E}/\text{MeV} = 8.8 \times 10^{-2}\frac{(\mathcal{E}/\text{GeV})^4}{(r/\text{meters})} \quad . \tag{4.100}$$

This amounts to a major limitation for electron storage rings and accelerators above a few GeV energy. Jackson (p 668) cites the Cornell electron synchrotron with r = 100 meters having a loss of 8.8 MeV per turn at 10 GeV. The MIT Bates accelerator storage ring is designed for up to 1 GeV energy. With a bend radius of 9.1 m the loss is 9.8 keV per turn which is compensated by an accelerating stage within the ring.

4.6 Scattering of Electromagnetic Radiation

4.6.1 Thomson Scattering

We have seen that a non-relativistic accelerated charge radiates according to (eq 4.82),

$$\frac{dP}{d\Omega_s} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c}\left(\frac{\dot{\mathbf{v}}}{c}\right)^2 \sin^2\alpha \quad , \tag{4.101}$$

where α is the angle between the direction of radiation and the direction of the acceleration, $\dot{\mathbf{v}}$. If the acceleration arises from an electric field E_i, then

$$\dot{\mathbf{v}} = \frac{q}{m}\mathbf{E}_i \quad , \tag{4.102}$$

Therefore the power radiated per unit solid angle from a single electron can be written:

$$\frac{dP}{d\Omega_s} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{4\pi c}\left(\frac{e}{m_e c}E_i\right)^2 \sin^2\alpha = \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right)^2 c\epsilon_0 E_i^2 \sin^2\alpha \quad . \tag{4.103}$$

The combination of parameters arising in the last form of this equation,

$$r_e \equiv \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right) \tag{4.104}$$

has the dimensions of length, and is called the classical electron radius.

A steady electric field will not give rise to radiation that is particularly interesting, but if the electric field is oscillating, it will give rise to radiation that is at a corresponding frequency.

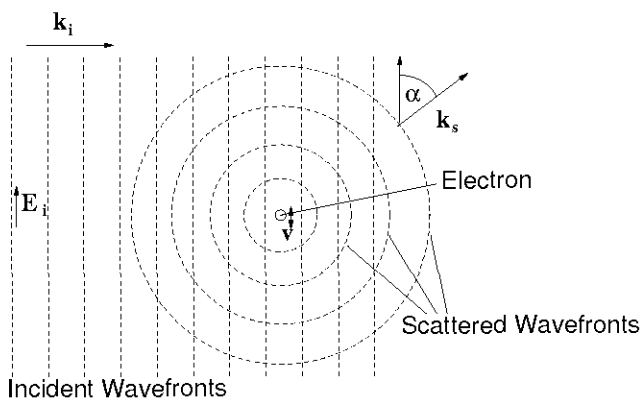

Figure 4.12: Schematic illustration of the process of Thomson Scattering.

The most elementary case one might consider is when the electric field varies sinusoidally with angular frequency ω. This is exactly the situation that arises if a charged particle such as an electron experiences the oscillating electric field of an incident electromagnetic wave at frequency ω. In this situation we speak of “scattering” of the incident wave by the electron.

This process of acceleration of a free electron by an incident wave and reradiation of a wave into other directions is known as Thomson scattering.

Now the instantaneous power per unit area of the incident wave is given by the Poynting vector whose magnitude is

$$s_i = |\mathbf{E} \wedge \mathbf{B}/\mu_0| = \frac{1}{c\mu_0} E_i^2 = c\epsilon_0 E_i^2 \quad , \tag{4.105}$$

and we evaluate it at retarded time t' (i.e. at the time necessary to give rise to radiation at the field point at later time t). Therefore the scattered power per unit solid angle from a single electron can be written:

$$\frac{dP}{d\Omega_s} = r_e^2 \sin^2\alpha \ s_i \quad . \tag{4.106}$$

The differential (energy) scattering cross-section is the ratio of $dP/d\Omega_s$ to the incident power density s_i. One can rapidly verify that this definition is in accord with the standard definition of a cross-section: that it should be such that the number of collisions per unit length is equal to the product of the cross-section and the density of targets. In this case the "projectiles" are represented by the incident energy of the wave. The projectiles can be considered to have a flux density proportional the wave power flux density, s_i. An alternative view of this cross-section is to regard it as the area across which the incident power flux density would have to flow in order to give rise to the power scattered. The cross-section is

$$\frac{d\sigma}{d\Omega_s} = r_e^2 \sin^2\alpha \quad , \tag{4.107}$$

where α is the angle between the scattering direction and the electric field (i.e. the polarization direction) of the incident wave.

Integrated over all scattering angles this expression yields the total Thomson scattering cross-section

$$\sigma = \frac{8\pi}{3} r_e^2 \quad . \tag{4.108}$$

If the electron is stationary apart from the oscillation that the wave imparts to it, then the scattered radiation will have exactly the same frequency (in this classical approximation) as the incident wave. However if the electron is moving prior to its perturbation by the incident wave, then there will be a Doppler shift of the scattered frequency both because the moving electron will experience the incident wave at a different frequency and because its radiation will be Doppler shifted at the observer. These two effects give a scattered frequency ω_s that is related to the incident frequency ω_i by

$$\omega_s = \omega_i + (\mathbf{k}_s - \mathbf{k}_i).\mathbf{v}_0 = \omega_i \frac{1 - \hat{\mathbf{k}}_i.\mathbf{v}_0/c}{1 - \hat{\mathbf{k}}_s.\mathbf{v}_0/c} \quad , \tag{4.109}$$

where $\mathbf{k}_i$ and $\mathbf{k}_s$ are the wave-vectors of the incident and scattered waves respectively, whose magnitudes are $k_i = \omega_i/c$ and $k_s = \omega_s/c$, and hats indicate unit vectors. The numerator and denominator of the fractional form for ω_s represent the two Doppler shifts just referred to. This one-to-one relationship between the scattered frequency and the component of the electron velocity along the direction $\mathbf{k}_s - \mathbf{k}_i$ is extremely helpful in plasma diagnostic applications. The velocity distribution of the electrons is directly revealed in the spectrum of Thomson scattered light.

4.6.2 Compton Scattering

One approximation implicit in our treatment of Thomson scattering is that all the incident wave does to the electron is to cause it to oscillate and that this oscillatory motion is added to an otherwise unperturbed prior motion. In other words, after the scattering has happened, the electron remains either stationary or moving at the same velocity as it had before. [In this section we will henceforward take the electron to be stationary prior to the scattering for simplicity.] But this cannot really be right, even on a classical picture, because we know that electromagnetic fields carry momentum. So if the wave is scattered, changing its momentum, then the electron's momentum must also be changed so as to conserve total momentum.

The classical effect can easily be calculated. By the symmetry of the $\sin^2\alpha$ angular distribution of scattering, the scattered radiation has zero momentum on average. Therefore the momentum imparted to the electron is just that of the incident radiation. We saw in section 3.2.3 that the momentum density of electromagnetic fields is equal to $1/c^2$ times the energy flux density. The force exerted by the incident radiation on the electron is equal to the total cross-section times the momentum flux density, which is c times the momentum density. So this force is

$$m_e\dot{v}_0 = \sigma s_i/c = \frac{8\pi}{3}r_e^2\epsilon_0 E_i^2\frac{1}{c} \quad . \tag{4.110}$$

In this classical picture there is a radiation pressure, applying over an area equal to the Thomson cross-section of the electron, which steadily pushes it in the direction of the incident radiation.

Quantum mechanics teaches us, however, that electromagnetic radiation is not smooth and infinitely divisible. Instead it takes the form of photons whose energy is $\hbar\omega$ when the angular frequency of the radiation is ω. If the size of the photon, the quantum of energy, is much less than the other energy scales in the problem, then the classical limit discussed above, can apply. If the photon energy is large, it cannot. Actually, the crucial question here is the *momentum* of the photon but this can be related to energy and compared with the rest energy of the electron ($m_0c^2 = 511$ keV) as we shall see. The quantum picture, then, is that each individual photon may, on encountering a free electron, bounce off in a scattering event. When it does so, the photon's momentum is changed, and the electron's momentum changes also so as satisfy conservation. As a consequence, for energetic (large momentum) photons, even an initially stationary electron recoils from a scattering event with substantial momentum. This recoil leads to a downshift in the energy (and hence frequency) of the scattered photon that will depend on the direction in which it is scattered.

The kinematics of the problem, momentum and energy conservation, are all that is needed to relate the energy shift to the angle of scattering. Scattering takes place in a scattering plane. We will suppose that the photon is scattered through an angle θ and the electron recoils in a direction at an angle ϕ to the initial direction of the photon. We have to do the problem relativistically and we appeal to the general relativistic relationship relating energy and momentum:

$$\mathcal{E} = \sqrt{p^2c^2 + (m_0c^2)^2} \tag{4.111}$$

We denote the final momentum of the electron by p, the photon energy by $\mathcal{E}$ before and $\mathcal{E}'$ after the scattering collision. Then the momentum of the photon is $\mathcal{E}/c$ (from the energy

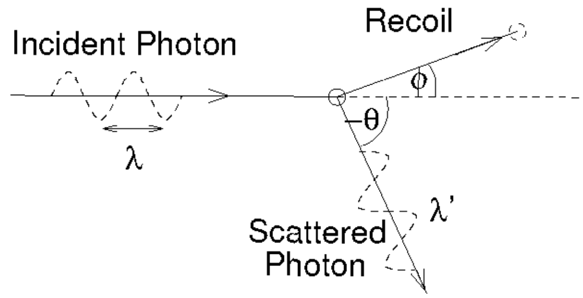

Figure 4.13: Compton scattering geometry in the scattering plane.

relationship above or from our knowledge about the relationship between energy flux and momentum of electromagnetic fields). Then we write down the two components of momentum conservation parallel

$$\frac{\mathcal{E}}{c} = \frac{\mathcal{E}'}{c}\cos\theta + p\cos\phi \tag{4.112}$$

and perpendicular

$$0 = \frac{\mathcal{E}'}{c}\sin\theta + p\sin\phi \tag{4.113}$$

to the incident photon, and the energy conservation:

$$\mathcal{E} + m_0c^2 = \mathcal{E}' + \sqrt{p^2c^2 + (m_0c^2)^2} \quad . \tag{4.114}$$

We eliminate ϕ by separating the ϕ terms in eqs 4.112 and 4.113 squaring and adding to get:

$$p^2 = \left(\frac{\mathcal{E}}{c}\right)^2 + \left(\frac{\mathcal{E}'}{c}\right)^2 - 2\frac{\mathcal{E}\mathcal{E}'}{c^2}\cos\theta \quad . \tag{4.115}$$

And then we eliminate the momentum p by squaring the square-root term of eq 4.114 to get

$$p^2c^2 = 2m_0c^2(\mathcal{E} - \mathcal{E}') + (\mathcal{E} - \mathcal{E}')^2 \tag{4.116}$$

and subtracting from c^2 times the previous equation to get

$$0 = \mathcal{E}\mathcal{E}'(1 - \cos\theta) - m_0c^2(\mathcal{E} - \mathcal{E}') \quad . \tag{4.117}$$

This is the equation that relates the photon energy downshift to the angle of photon scattering. It is most often written in a form governing the photon wavelength $\lambda = 2\pi c/\omega = hc/\mathcal{E}$ and using $1 - \cos\theta = 2\sin^2\theta/2$,

$$\lambda' - \lambda = \frac{h}{m_0c}2\sin^2\frac{\theta}{2} \quad , \tag{4.118}$$

which expresses the "Compton Shift" of wavelength in terms of the "Compton Wavelength", $\lambda_c \equiv h/m_0c = 2.426 \times 10^{-12}$ m, of the electron. A photon's wavelength equals the Compton wavelength when its energy is equal to the rest mass of the electron, $m_0c^2 = 511$ keV. Therefore the Compton shift is important only for very energetic x-rays and for γ-rays.

The energy of the scattered photon is

$$\mathcal{E}' = \frac{m_0 c^2}{1 - \cos\theta + m_0 c^2/\mathcal{E}} \tag{4.119}$$

and the energy lost by the photon, and hence gained as kinetic energy by the electron is

$$\mathcal{E}\frac{1 - \cos\theta}{1 - \cos\theta + m_0 c^2/\mathcal{E}} \tag{4.120}$$

Figure 4.14: Compton scattering cross-section angular variation. $[\alpha \equiv \mathcal{E}/m_e c^2]$.

The cross section for this scattering must reduce to the Thomson cross-section at low photon energy. It was first calculated using relativistic quantum mechanics (1928) by Klein and Nishina shortly after Dirac's formulation of the relativistic quantum equations for the electron, predicting spin and negative energy states. The agreement of the Klein-Nishina cross-section with experiments was one of the early triumphs of Dirac's theory. For unpolarized radiation, the differential cross-section for photon scattering (which is different from the energy scattering cross-section by virtue of the photon energy shift) per unit solid angle is:

$$\frac{d\sigma}{d\Omega_s} = \frac{r_e^2}{2}\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^2\left(\frac{\mathcal{E}}{\mathcal{E}'} + \frac{\mathcal{E}'}{\mathcal{E}} - \sin^2\theta\right) \quad . \tag{4.121}$$

In this form the reduction to the Thomson cross-section at low photon energy, so that $\mathcal{E}'/\mathcal{E} \to 1$, can be verified by integration of the Thomson formula over all possible incident radiation polarization directions. At high photon energy, $\mathcal{E} > m_e c^2$, forward or small angle scattering tends to dominate the cross-section, because the $(\mathcal{E}'/\mathcal{E})^2$ term becomes small at larger angles; although for those photons that are back-scattered $\theta \approx 180^o$, they lose practically all their energy to the electrons and retain only $\mathcal{E}' \to m_e c^2/2$. Figure 4.14 shows polar plots of the cross-section at different energies.

The Compton scattering process is a dominant attenuation mechanism in the 1 to 4 MeV photon energy range. It is sometimes helpful to distinguish between the cross-section for scattering of a photon, given above, and the cross-section for removal of energy from a photon beam, which is equal to the product of the scattering cross-section and the ratio of energy loss to initial photon energy. This later is sometimes called the Compton "absorption" cross-section since it represents the rate at which energy is transfered from photons to Compton scattered electrons. In either case, the attenuation of a photon stream of intensity I is governed by a differential equation:

$$\frac{dI}{d\ell} = -n_e \sigma I = -Z n_i \sigma I \quad , \tag{4.122}$$

where σ is the cross-section per electron, and the fact that the Z electrons are bound to each atom is ignored since the photon energy is so much higher than the electron binding energy. The solutions to this equation are exponential ($\propto \exp(-n_e \sigma \ell)$) with inverse decay length $n_e \sigma$, which is called the "attenuation coefficient".

Figure 4.15: Photon attenuation coefficients for lead. [From Evans]

Since the ratio of mass to charge of most nuclei is very similar, between 2 and 2.8, the greatest attenuation arises from the greatest electron density, which corresponds to the greatest mass density. Hence lead, for example, has one of the largest attenuation coefficients. Figure 4.15 shows the total (angle-integrated) Compton attenuation coefficients together with the photoelectric absorption and pair production coefficients. These latter processes will be discussed later. The Compton attenuation, since it is simply the product $n_e \sigma$ can be scaled to any other material by multiplying by the ratio of electron densities, that is (for elements) by the quantity

$$\frac{A_{\rm lead}}{\rho_{\rm lead} Z_{\rm lead}} \, \frac{\rho_{\rm other} Z_{\rm other}}{A_{\rm other}} \quad , \tag{4.123}$$

where ρ is mass density, A is atomic weight, and Z is atomic number.