Green's Function

Solution for the following problem:

$$L\{y(x)\} = f(x) \qquad \textit{for} \qquad a \leq x \leq b$$

and boundary condition

$$B\{y(x)\} = y(a) = 0$$

Problem's Green function

$$L\left\{G(x,\xi)\right\} = \delta(x-\xi)$$

subject to

$$G(a,\xi) = 0 \quad \textit{or} \quad B\left\{G(x,\xi)\right\} = 0$$

Solution for $y(x)$ including $G(x,\xi)$ is

$$y(x) = \int_a^b G(x,\xi)\, f(\xi)\, d\xi$$

Prove:

$$L\{y(x)\} = L\left\{ \int_a^b G(x,\xi)\, f(\xi)\, d\xi \right\}$$

$$L\{y(x)\} = \left\{ \int_a^b L\{G(x,\xi)\}\, f(\xi)\, d\xi \right\}$$

$$L\{y(x)\} = \left\{ \int_a^b \delta(x-\xi)\, f(\xi)\, d\xi \right\}$$

$$L\{y(x)\} = f(x)$$

For the boundary condition:

$$B\{y(x)\} = \left\{ \int_a^b B\{G(x,\xi)\} f(\xi)\, d\xi \right\}$$

$$B\{y(x)\} = 0$$

Green's Function Solution for the Scattering Problem

$$\left(\nabla^2 + k^2\right)G(\vec{r},\vec{r}\,') = \delta(\vec{r} - \vec{r}\,')$$

$$Defining \;\; \vec{R} = \vec{r} - \vec{r}\,'$$

$$Fourier\ Transform\ of\ G(\vec{R})$$

$$G(\vec{R}) = \frac{1}{(2\pi)^{3/2}} \int_{\tau} g(\vec{K}) e^{-i\vec{K}.\vec{R}} d\vec{K}$$

Recall that the Dirac $\delta(\vec{R})$ can be represented as

$$\delta(\vec{R}) = \frac{1}{(2\pi)^{3}} \int_{\tau} e^{-i\vec{K}.\vec{R}} d\vec{K}$$

$g(\vec{K})$ is the Fourier transform of $G(\vec{R})$

$$\left(\nabla^2 + k^2\right)\frac{1}{(2\pi)^{3/2}}\int_\tau g(\vec{K})e^{-i\vec{K}.\vec{R}}d\vec{K} = \frac{1}{(2\pi)^3}\int_\tau e^{-i\vec{K}.\vec{R}}d\vec{K}$$

$$\left(-K^2 + k^2\right)g(\vec{K}) = \frac{1}{(2\pi)^{3/2}}$$

or

$$g(\vec{K}) = -\frac{1}{(2\pi)^{3/2}}\frac{1}{K^2 - k^2}$$

Then $G(\vec{R})$ becomes,

$$G(\vec{R}) = -\frac{1}{(2\pi)^3}\int_{\tau} \frac{e^{-i\vec{K}.\vec{R}}}{K^2 - k^2} d\vec{K}$$

The integral is performed in spherical coordinates in the volume τ where

$$d\vec{K} \rightarrow K^2 dK \; d\varphi \sin\theta \, d\theta$$

The range of integration is

$$0 \le \varphi \le 2\pi$$

$$0 \le \theta \le \pi$$

$$0 \le K \le \infty$$

The integral in $G(\vec{R})$ becomes

$$G(\vec{R}) = -\frac{1}{(2\pi)^3}\int_0^{\infty} K^2 dK \int_0^{2\pi} d\varphi \int_0^{\pi} \frac{e^{-iKR\cos\theta}}{K^2 - k^2} \sin\theta \, d\theta$$

$$G(\vec{R}) = -\frac{1}{(2\pi)^3}\int_0^\infty \frac{K^2 dK}{K^2 - k^2}(2\pi)\left(\frac{2\sin KR}{KR}\right)$$

The function

$$f(K) = \frac{\sin KR}{K}\frac{1}{K^2 - k^2}$$

is even on K, therefore

$$G(\vec{R}) = -\frac{1}{4\pi^2 R}\int_{-\infty}^{\infty} \frac{K\, dK}{K^2 - k^2}\left(\frac{\sin KR}{K}\right)$$

Recall that

$$\sin KR = \frac{1}{2i}\left(e^{iKR} - e^{-iKR}\right)$$

$$G(\vec{R}) = -\frac{1}{4\pi^2 R}\left\{\frac{1}{2i}\left[\int_{-\infty}^{\infty}\frac{Ke^{iKR}}{K^2-k^2} - \int_{-\infty}^{\infty}\frac{Ke^{-iKR}}{K^2-k^2}\right]\right.$$

Use of the Residue Theorem

If $f(z)$ is an analytic function then

$$\oint_c f(z)\,dz = 2\pi i\sum_{j=1}^{n}\mathrm{Res}\, f(a_j)$$

Where a_j (singularity) is in the circle c.

For a m[th] order singularity

$$\mathrm{Res}\, f(a_j) = \frac{1}{(m-1)!}\lim_{z\to a_j}\left\{\frac{d^{m-1}}{dz^{m-1}}[(z-a_j)^m f(z)]\right\}$$

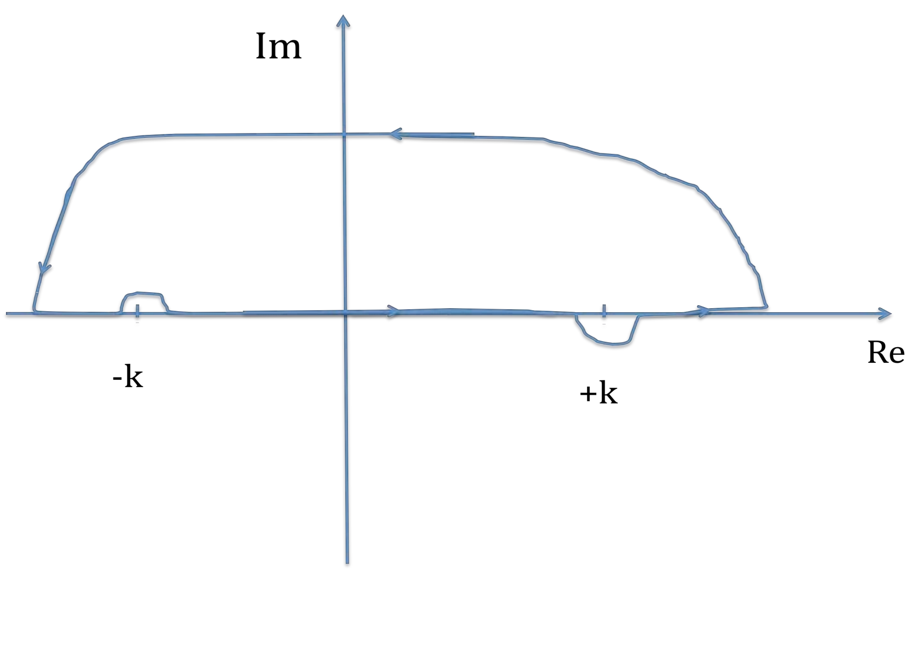
(a) First part of the integral

$$f(K) = \frac{Ke^{iKR}}{K^2 - k^2}$$

Singularities of order m=1 at

$$K = \pm k$$

Since $R > 0$



$$\operatorname{Re} s\, f(+k) = \lim_{K \to +k} (K-k) \frac{Ke^{iKR}}{(K-k)(K+k)}$$

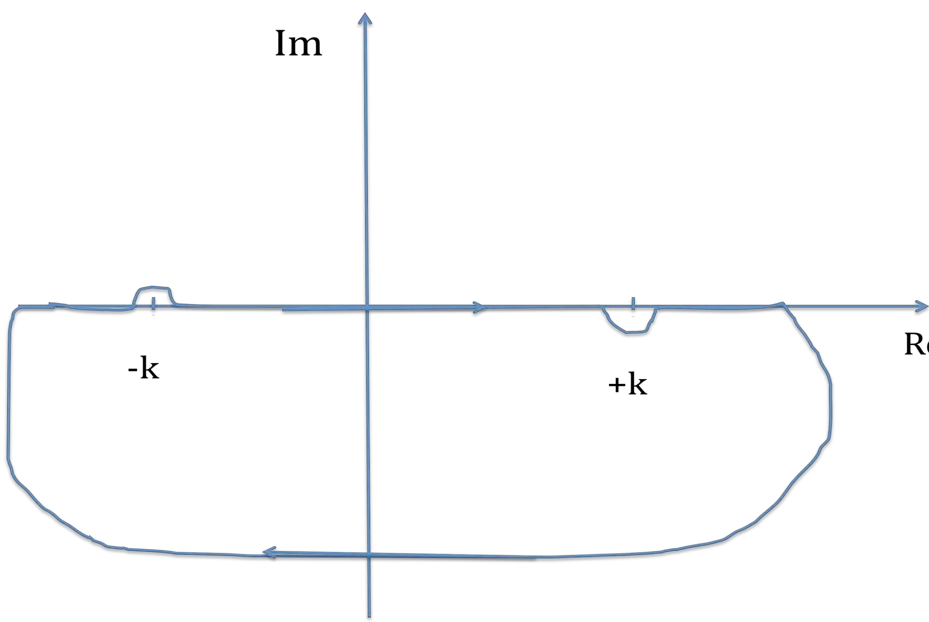
$$\operatorname{Re} s\, f(+k) = \frac{e^{ikR}}{2}$$

$$\oint_{c} f(z)\, dz = \pi i\, e^{ikR}$$

(b) Second part of the integral

$$f(K) = \frac{Ke^{-iKR}}{K^2 - k^2}$$

Since $R > 0$



$$\operatorname{Re} s\, f(+k) = \lim_{K \to -k} (K-k)\frac{Ke^{iKR}}{(K-k)(K+k)}$$

$$\operatorname{Res} f(+k) = (-k)\frac{e^{-i(-k)R}}{-2k}$$

$$\operatorname{Res} f(+k) = \frac{e^{ikR}}{2}$$

Note that the sense of integration is counter-clockwise.

$$\oint_c f(z)\,dz = -\pi i\, e^{ikR}$$

Hence $G(\vec{R})$ becomes

$$G(\vec{R}) = -\frac{1}{4\pi^2 R}\left\{\frac{1}{2i}\left[\pi i e^{ikR} - (-\pi i e^{ikR})\right]\right\}$$

$$G(\vec{R}) = -\frac{1}{4\pi R} e^{ikR}$$

Recall that

$$\vec{R} = \left|\vec{r} - \vec{r}\,'\right|$$

$$G(\vec{r},\vec{r}\,') = -\frac{1}{4\pi\left|\vec{r} - \vec{r}\,'\right|} e^{ik\left|\vec{r} - \vec{r}\,'\right|}$$