2.1. Shells of Revolution

2.1.1. GEOMETRY OF SHELLS OF REVOLUTION

Most pressure vessels are shells of revolution, i.e. spheres, cylinders, cones, etc. Such a surface is obtained by rotating a plane curve about an axis lying in the plane of the curve. Figure 2.1 shows the curve, called the *meridian*. At a point P the radius of curvature of the

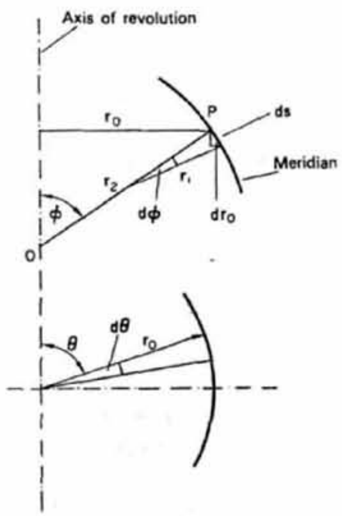


Fig. 2.1

meridian is r_1, and the circle of radius r_0 in the plan view of Fig. 2.1 is called a *parallel circle*. Thus an element of shell ($r_0\, d\theta \times r_1\, d\varphi$) is cut by two meridians and two parallel circles. OP is the normal to this element of shell and obviously r_1 is one radius of principal curvature. The second radius of principal curvature r_2 equals OP since by considering two adjacent points on the parallel circle at P, the

normals from these points will intersect the axis of rotation at O. From Fig 2.1 we obtain

$$r_0 = r_2 \sin \varphi \tag{2.1a}$$

$$dr_0 = ds \cos \varphi \tag{2.1b}$$

and

$$ds = r_1\, d\varphi \tag{2.1c}$$

from which

$$\frac{dr_0}{d\varphi} = r_1 \cos \varphi \tag{2.2}$$

2.1.2. EQUATIONS OF EQUILIBRIUM OF SHELLS OF REVOLUTION

Consider an element $r_0\, d\theta \times r_1\, d\varphi$ of the shell of revolution. Figure 2.2 shows the direct and shear stress resultants in the plane of the surface of the shell and also the transverse shear stress resultants. Figure

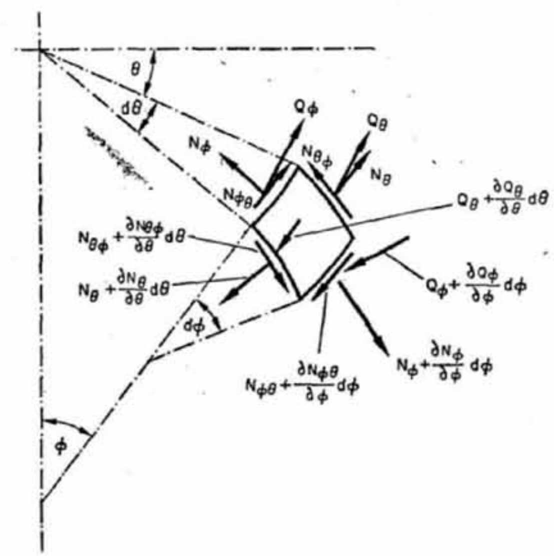


Fig. 2.2

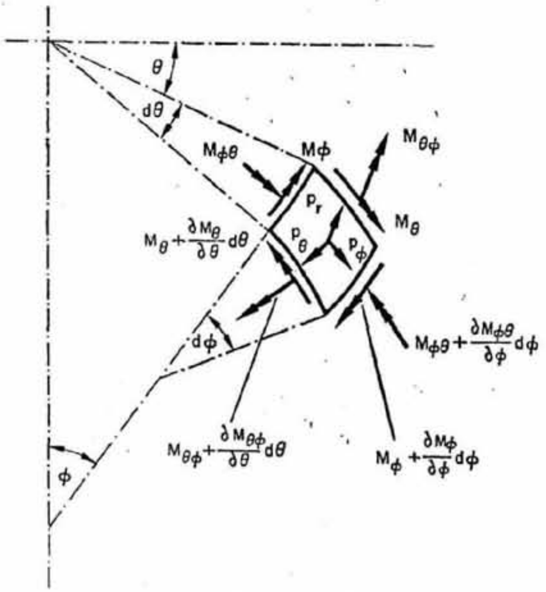


FIG. 2.3

2.3 shows the bending and twisting moments on the element. All stress resultants are shown in the positive directions, the right-hand screw rule being used for the moments in Fig. 2.3. The external forces on the shell element are p_r, p_θ, p_φ per unit area of shell and are shown in Fig. 2.3.

As is usual in thin shell analysis we ignore all stresses normal to the shell surface as being negligible since they are small, and the equations of equilibrium ignore all changes in shape of the shell due to the loads. The detailed steps for the derivation of the equations of equilibrium are given in various textbooks [2.1–4].

By resolving and taking moments for the element, the six equations are:

$$\frac{\partial}{\partial\varphi}(r_0N_\varphi)+r_1\frac{\partial N_{\theta\varphi}}{\partial\theta}-r_1N_\theta\cos\varphi-r_0Q_\varphi+r_0r_1p_\varphi = 0 \tag{2.3a}$$

$$\frac{\partial}{\partial\varphi}(r_0N_{\varphi\theta})+r_1\frac{\partial N_\theta}{\partial\theta}+r_1N_{\theta\varphi}\cos\varphi-r_1Q_\theta\sin\varphi+r_0r_1p_\theta = 0 \tag{2.3b}$$

$$r_1N_\theta\sin\varphi+r_0N_\varphi+r_1\frac{\partial Q_\theta}{\partial\theta}+\frac{\partial}{\partial\varphi}(r_0Q_\varphi)-r_0r_1p_r = 0 \tag{2.3c}$$

$$\frac{\partial}{\partial\varphi}(r_0M_\varphi)+r_1\frac{\partial M_{\theta\varphi}}{\partial\theta}-r_1M_\theta\cos\varphi-r_0r_1Q_\varphi = 0 \tag{2.3d}$$

$$\frac{\partial}{\partial\varphi}(r_0M_{\varphi\theta})+r_1\frac{\partial M_\theta}{\partial\theta}+r_1M_{\theta\varphi}\cos\varphi-r_0r_1Q_\theta = 0 \tag{2.3e}$$

$$r_0r_1N_{\theta\varphi}-r_0r_1N_{\varphi\theta}-r_1M_{\theta\varphi}\sin\varphi+r_0M_{\varphi\theta} = 0 \tag{2.3f}$$

If the loading is rotationally symmetric; i.e. if $p_\theta = 0$ then clearly $Q_\theta = N_{\theta\varphi} = N_{\varphi\theta} = M_{\theta\varphi} = M_{\varphi\theta} = 0$ and N_θ and M_θ are independent of θ. The equations of equilibrium then take the form:

$$\frac{d}{d\varphi}(r_0N_\varphi)-r_1N_\theta\cos\varphi-r_0Q_\varphi+r_0r_1p_\varphi = 0 \tag{2.4a}$$

$$\frac{d}{d\varphi}(r_0Q_\varphi)+r_1N_\theta\sin\varphi+r_0N_\varphi-r_0r_1p_r = 0 \tag{2.4b}$$

$$\frac{d}{d\varphi}(r_0M_\varphi)-r_1M_\theta\cos\varphi-r_0r_1Q_\varphi = 0 \tag{2.4c}$$

Eliminating N_θ from (2.4a) and (2.4b) we obtain

$$\frac{d}{d\varphi}(r_0N_\varphi)\sin\varphi-r_0Q_\varphi\sin\varphi+r_0r_1p_\varphi\sin\varphi$$
$$+r_0N_\varphi\cos\varphi+\frac{d}{d\varphi}(r_0Q_\varphi)\cos\varphi-r_0r_1p_r\cos\varphi = 0$$

i.e. $$\frac{d}{d\varphi}(r_0N_\varphi\sin\varphi)+\frac{d}{d\varphi}(r_0Q_\varphi\cos\varphi)+r_0r_1(p_\varphi\sin\varphi-p_r\cos\varphi) = 0$$

Therefore

$$2\pi[r_0N_\varphi\sin\varphi+r_0Q_\varphi\cos\varphi] = 2\pi\int r_0r_1(p_r\cos\varphi-p_\varphi\sin\varphi)\,d\varphi$$
$$= P\text{, the total load on the shell above the parallel circle.} \tag{2.4d}$$

This is a very useful but not independent equation of equilibrium and it follows obviously from Fig. 2.4.

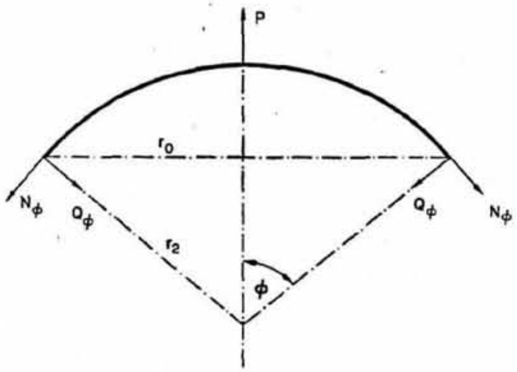


FIG. 2.4

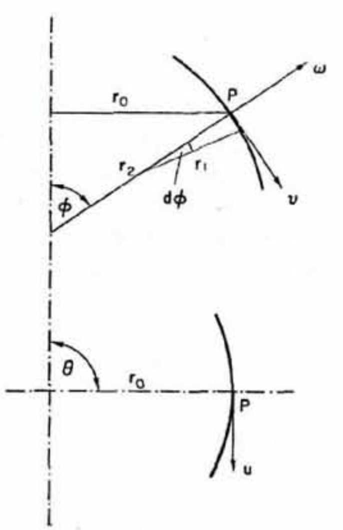


FIG. 2.5

2.1.3. STRAINS AND DISPLACEMENTS IN SHELLS OF REVOLUTION

Now consider the deformations of the shell of revolution and the consequent direct and shear strains, changes of curvature and twist. We make the assumption that during deformation the normals to the middle surface of the shell remain straight and normal to the deformed middle surface and undergo no change of length.

A point P in the middle surface of the shell with coordinates θ, φ has displacements u, v, w (assumed small) as shown in the positive directions in Fig. 2.5.

The strains of the middle surface in the circumferential and meridional directions are ε_θ and ε_φ and there will be a shear strain $\gamma_{\theta\varphi}$ in the plane of the shell. It is shown for example in [2.1] that

$$\varepsilon_\theta = \frac{1}{r_0}\frac{\partial u}{\partial \theta}+\frac{v}{r_1 r_0}\frac{\partial r_0}{\partial \varphi}+\frac{w}{r_2} = \frac{1}{r_0}\left[\frac{\partial u}{\partial \theta}+v\cos\varphi+w\sin\varphi\right] \quad (2.5a)$$

$$\varepsilon_\varphi = \frac{1}{r_1}\frac{\partial v}{\partial \varphi}+\frac{w}{r_1} \quad (2.5b)$$

$$\gamma_{\theta\varphi} = \frac{1}{r_1}\frac{\partial u}{\partial \varphi}-\frac{u\cos\varphi}{r_0}+\frac{1}{r_0}\frac{\partial v}{\partial \theta} \quad (2.5c)$$

If we proceeded with strict rigour at this point and followed an accurate analysis, see [2.1], we would proceed to find the values of $\bar{\varepsilon}_\theta$, $\bar{\varepsilon}_\varphi$, $\bar{\gamma}_{\theta\varphi}$ at a distance z from the middle surface (i.e. in the direction of the shell thickness) and the further analysis of the shell problem would use these strains and Hooke's law to derive the stresses at a distance z from the middle surface. These stresses would be integrated over the cross-section of the shell element to derive the stress resultants N_θ, N_φ, $N_{\theta\varphi}$, $N_{\varphi\theta}$, M_θ, M_φ, $M_{\theta\varphi}$, $M_{\varphi\theta}$. Because the shell element has a cross-section which is trapezoidal we would find that although the shear stress $\tau_{\theta\varphi} = \tau_{\varphi\theta} = \dfrac{E}{2(1+\nu)}\bar{\gamma}_{\theta\varphi}$ at a distance z from the middle surface, the integrations would give $N_{\theta\varphi} \neq N_{\varphi\theta}$ and $M_{\theta\varphi} \neq M_{\varphi\theta}$.

There would appear to be very few, if any, pressure vessel problems for which this rigorous analysis is required and we now take an engineering viewpoint and give the changes of curvature $\varkappa_\theta$ and $\varkappa_\varphi$ and the twist $\varkappa_{\theta\varphi}$ in terms of u, v, w.

$$\varkappa_\theta = \frac{1}{r_0}\left[\frac{\partial}{\partial\theta}\left\{\frac{1}{r_0}\left(\frac{\partial w}{\partial\theta}-u\sin\varphi\right)\right\}+\frac{\cos\varphi}{r_1}\left(\frac{\partial w}{\partial\varphi}-v\right)\right]$$

$$= \frac{1}{r_0^2}\frac{\partial^2 w}{\partial\theta^2}-\frac{1}{r_0 r_2}\frac{\partial u}{\partial\theta}+\frac{\cot\varphi}{r_2}\left[\frac{1}{r_1}\frac{\partial w}{\partial\varphi}-\frac{v}{r_1}\right] \tag{2.5d}$$

$$\varkappa_\varphi = \frac{1}{r_1}\frac{\partial}{\partial\varphi}\left[\frac{1}{r_1}\frac{\partial w}{\partial\varphi}-\frac{v}{r_1}\right] \tag{2.5e}$$

$$2\varkappa_{\theta\varphi} = \frac{r_0}{r_1}\frac{\partial}{\partial\varphi}\left[\frac{1}{r_0^2}\left(\frac{\partial w}{\partial\theta}-u\sin\varphi\right)\right]+\frac{1}{r_0}\frac{\partial}{\partial\theta}\left[\frac{1}{r_1}\frac{\partial w}{\partial\varphi}-\frac{v}{r_1}\right]$$

$$= \frac{2}{r_0 r_1}\frac{\partial^2 w}{\partial\theta\,\partial\varphi}-\frac{2\cos\varphi}{r_0^2}\frac{\partial w}{\partial\theta}-\frac{1}{r_0 r_1}\frac{\partial v}{\partial\theta}-\frac{\sin\varphi}{r_1 r_0}\frac{\partial u}{\partial\varphi}$$

$$-\frac{u\cot\varphi}{r_1 r_2^2}[r_2-2r_1] \tag{2.5f}$$

In these equations Flügge [2.1] suggests a simple form omitting the terms in u and v. The forms given here are the same as those given by Kraus [2.4].

Other forms of the expression for $\varkappa_{\theta\varphi}$ are given in the literature and there has been considerable discussion of their accuracy. Koiter [2.5] discusses this problem.

We also define V the angular rotation of a tangent to the meridian which is of great use in solving boundary conditions in pressure vessel problems:

$$V = \frac{1}{r_1}\left(\frac{\partial w}{\partial\varphi}-v\right) \tag{2.6}$$

This is a rotation in an anticlockwise direction in Fig. 2.5.

2.1.4. ELASTIC ANALYSIS OF SHELLS OF REVOLUTION

If we now consider the shell of revolution for an elastic material which obeys Hooke's law, we can derive relationships between the stress resultants and the strains and changes of curvature.

Using equations (2.5) the direct and shear strains $\bar\varepsilon_\theta$, $\bar\varepsilon_\varphi$, $\bar\gamma_{\theta\varphi}$ at a distance z from the middle surface are given by:

$$\bar\varepsilon_\theta = \varepsilon_\theta - z\varkappa_\theta \tag{2.7a}$$

$$\bar\varepsilon_\varphi = \varepsilon_\varphi - z\varkappa_\varphi \tag{2.7b}$$

$$\bar\gamma_{\theta\varphi} = \gamma_{\theta\varphi} - 2z\varkappa_{\theta\varphi} \tag{2.7c}$$

where z is positive in the direction of the outward normal.

Using Hooke's law and including the effect of temperature, T, we have:

$$\bar\varepsilon_\theta = \frac{\sigma_\theta}{E}-\frac{\nu\sigma_\varphi}{E}+\alpha T \tag{2.8a}$$

$$\bar\varepsilon_\varphi = \frac{\sigma_\varphi}{E}-\frac{\nu\sigma_\theta}{E}+\alpha T \tag{2.8b}$$

$$\bar\gamma_{\theta\varphi} = \frac{\tau_{\theta\varphi}}{G} \tag{2.8c}$$

where α is the coefficient of linear expansion. Solving for the stresses:

$$\sigma_\theta = \frac{E}{1-\nu^2}[(\bar\varepsilon_\theta+\nu\bar\varepsilon_\varphi)-(1+\nu)\alpha T] \tag{2.9a}$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}[(\bar\varepsilon_\varphi+\nu\bar\varepsilon_\theta)-(1+\nu)\alpha T] \tag{2.9b}$$

$$\tau_{\theta\varphi} = \frac{E}{2(1+\nu)}\bar\gamma_{\theta\varphi} \tag{2.9c}$$

and the stress resultants are:

$$N_\theta = \int_{-h/2}^{h/2}\sigma_\theta\,dz = K[(\varepsilon_\theta+\nu\varepsilon_\varphi)-(1+\nu)\alpha T_0] \tag{2.10a}$$

$$N_\varphi = \int_{-h/2}^{h/2} \sigma_\varphi\, dz = K[(\varepsilon_\varphi+\nu\varepsilon_\theta)-(1+\nu)\alpha T_0] \qquad (2.10b)$$

$$N_{\theta\varphi} = N_{\varphi\theta} = \int_{-h/2}^{h/2} \tau_{\theta\varphi}\, dz = \frac{K(1-\nu)}{2}\gamma_{\theta\varphi} \qquad (2.10c)$$

$$M_\theta = -\int_{-h/2}^{h/2} \sigma_\theta z\, dz = D[\varkappa_\theta+\nu\varkappa_\varphi-(1+\nu)\alpha T_1] \qquad (2.10d)$$

$$M_\varphi = -\int_{-h/2}^{h/2} \sigma_\varphi z\, dz = D[\varkappa_\varphi+\nu\varkappa_\theta-(1+\nu)\alpha T_1] \qquad (2.10e)$$

$$M_{\theta\varphi} = M_{\varphi\theta} = -\int_{-h/2}^{h/2} \tau_{\theta\varphi} z\, dz = D(1-\nu)\varkappa_{\theta\varphi} \qquad (2.10f)$$

where h is the shell thickness.

$$K = \frac{Eh}{1-\nu^2}, \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

$$T_0 = \int_{-h/2}^{h/2} \frac{T\, dz}{h}, \quad T_1 = \frac{-12}{h^3}\int_{-h/2}^{h/2} Tz\, dz$$

The minus signs in the expressions for bending moment and torsion arise from the sign conventions of these quantities used in the equilibrium equations.

As referred to previously, in deriving these expressions we have ignored the fact that the shell element has sides which are trapezoidal and have not calculated precisely the strains at a distance z from the middle surface. It is for these reasons that we obtain $N_{\theta\varphi} = N_{\varphi\theta}$ and $M_{\theta\varphi} = M_{\varphi\theta}$. Strictly, therefore, these stress resultants can never satisfy the equilibrium equation (2.3f) unless $r_1 = r_2$, but this is an acceptable situation in an engineering problem since only a very slight change of the values of the shear force and twist is needed [2.1] to satisfy the equation. Had we carried out an exact analysis we would have found that equation (2.3f) would be identically satisfied.

If we substitute for ε_θ, ε_φ, $\gamma_{\theta\varphi}$, $\varkappa_\theta$, $\varkappa_\varphi$, $\varkappa_{\theta\varphi}$ equations (2.5) into equations (2.10) we have six equations relating the stress resultants and the displacements. We also have six equations of equilibrium (2.3), but ignoring equation (2.3f) as discussed above, we have finally a total of 11 equations for 11 unknowns N_θ, N_φ, $N_{\theta\varphi}$, M_θ, M_φ, $M_{\theta\varphi}$, Q_θ, Q_φ, u, v, w, and hence the general problem of the elastic shell of revolution may in principle be solved.

2.1.5. MEMBRANE SOLUTIONS FOR SHELLS OF REVOLUTION

Membrane solutions to shell problems are of great technological importance. Apart from the obvious fact that to carry loads by direct stress with no bending stresses is an economic use of material, it will be seen that the combination of local edge bending solutions with membrane solutions is one of the important techniques in the analysis of pressure vessel problems.

For the membrane theory we put $M_\theta = M_\varphi = M_{\theta\varphi} = M_{\varphi\theta} = 0$ in the equations of equilibrium (2.3) and hence from (2.3d) and (2.3e) we see that $Q_\varphi = Q_\theta = 0$. The equations of the membrane theory are therefore

$$\frac{\partial}{\partial\varphi}(r_0 N_\varphi)+r_1\frac{\partial N_{\theta\varphi}}{\partial\theta}-r_1 N_\theta\cos\varphi+p_\varphi r_0 r_1 = 0 \qquad (2.11a)$$

$$\frac{\partial}{\partial\varphi}(r_0 N_{\varphi\theta})+r_1\frac{\partial N_\theta}{\partial\theta}+r_1 N_{\theta\varphi}\cos\varphi+p_\theta r_0 r_1 = 0 \qquad (2.11b)$$

$$r_1 N_\theta \sin\varphi+r_0 N_\varphi-p_r r_0 r_1 = 0 \qquad (2.11c)$$

$$N_{\theta\varphi}-N_{\varphi\theta} = 0 \qquad (2.11d)$$

Using the geometric relation (2.1) equation (2.11c) is

$$\frac{N_\theta}{r_2}+\frac{N_\varphi}{r_1} = p_r \qquad (2.12)$$

For rotationally symmetric loading, $p_\theta = N_{\theta\varphi} = N_{\varphi\theta} = 0$ and equation (2.11a) is not the most useful form. An alternative form follows from

(2.4d) with $Q_\varphi = 0$, i.e.

$$2\pi r_0 N_\varphi \sin\varphi = P \tag{2.13}$$

where P is the sum of all the external forces applied to the shell above the parallel circle.

To find the membrane solutions for a shell of revolution we see that the four equations (2.11) have four unknowns. Flügge [2.1] and Kraus [2.4] set up general equations for the case where p_φ, p_θ, p_r can be expressed in Fourier series form and give examples for the case of a spherical shell. They also consider the homogeneous solution for the sphere, i.e. $p_\varphi = p_\theta = p_r = 0$ corresponding to edge loads and point loads only. These homogeneous solutions are given in Chapter 3 as part of a general treatment of a spherical shell. For rotationally symmetric loading such as pressure, self-weight and fluid weight Flügge [2.1] and Kraus [2.4] give typical solutions.

In applying the membrane theory of shells it is important to note the conditions under which the membrane solution is possible and the conditions under which bending stresses are an essential part of the stress system. When we refer to a "boundary" of the shell in the following discussion we are referring to a parallel circle which in practice is often the junction of two parts of a vessel, e.g. cylinder and end closure. It can also refer to the case for example of a spherical vessel on a skirt support, the support line being the boundary for both the shell above and below the support line.

The important conditions to be fulfilled for the membrane solution are:

(1) At any boundary the reactions must be in the meridional tangent plane. Otherwise shear forces and bending moments are necessary for equilibrium.

(2) If we substitute the values of N_θ, N_φ, $N_{\theta\varphi}$ for the membrane solution into equations (2.10a, b, c) and use the values of ε_θ, ε_φ, $\gamma_{\theta\varphi}$ to derive the displacements u, v, w from equations (2.5a, b, c) these displacements will have particular values at the boundary and associated with them will be rotations and twists at the boundary. The boundary conditions on displacements must not restrain these displacements and rotations so as to require shear forces normal to the shell and bending moments.

(3) The membrane solution is not an acceptable solution for concentrated forces normal to the suface of the shell since shear forces and bending moments are a necessary part of the complete solution.

(4) We see that if we evaluate u, v, w as explained in (2) above and substitute these in (2.5d, e, f) there will in general be values of $\varkappa_\theta, \varkappa_\varphi, \varkappa_{\theta\varphi}$. When these are substituted in (2.10d, e, f) we obtain values of M_θ, M_φ, $M_{\theta\varphi}$ which were considered to be zero when we set up the equation of the membrane theory. If the shell is thin and hence has a small bending stiffness or when the changes of curvature are small, the values of the moments are small and may be neglected in an engineering theory and the solution of many practical problems. However, there are certain cases, of which the toroidal shell of certain proportions is a well-known example, where a membrane solution is completely unacceptable because of this effect and it is found that bending moments and shear forces are essential parts of the stress system.

2.1.6. BENDING AND EDGE BENDING SOLUTIONS FOR SHELLS OF REVOLUTION

At the end of section 2.1.4 it was stated that the general problem of the shell of revolution subjected to non-rotationally symmetric loading could be solved from 11 equations for 11 unknowns, but clearly a solution would be of great mathematical complexity and in general a solution would only be possible by numerical computer analysis.

However, for the case of the spherical shell the problem can be solved analytically and the solution is given at the beginning of Chapter 3. The solution presented there is valid for all values of φ and for any non-rotationally symmetric loading. Flügge [2.1] considers the spherical shell case in a manner similar to Havers [2.6]. He also considers the edge bending solutions under non-rotationally symmetric edge loads for conical shells. Leckie [2.7] obtained solutions for spheres for axisymmetric loading and also for the first harmonic loading. Leckie and Penny [2.8] solved the problem for higher harmonics and compared these solutions with various approximate solutions.

For the general shell of revolution, we now restrict our attention to rotationally symmetric loading and use equations (2.4), (2.5), and (2.10).

The solution reduces to two equations:

$$\frac{A\,d^2U}{d\varphi^2}+\frac{B\,dU}{d\varphi}+CU+r_1EhV = Fr_1 \tag{2.14a}$$

$$\frac{A\,d^2V}{d\varphi^2}+\frac{H\,dV}{d\varphi}+IV-\frac{r_1}{D}U = G \tag{2.14b}$$

where $U = r_2Q_\varphi$

$$V = \frac{1}{r_1}\left(\frac{dw}{d\varphi}-v\right) \quad \text{from equation (2.6)}$$

$$A = \frac{r_2}{r_1}$$

$$B = \frac{r_2}{r_1}\cot\varphi+\frac{d}{d\varphi}\left(\frac{r_2}{r_1}\right)-\frac{r_2}{r_1}\left[\frac{\dot h}{h}+\frac{\dot E}{E}\right]$$

$$C = -\frac{r_1}{r_2}\cot^2\varphi+\nu+\nu\cot\varphi\left[\frac{\dot h}{h}+\frac{\dot E}{E}\right]$$

$$F = \frac{R}{r_2\sin^2\varphi}\left[\frac{r_2}{r_1}\frac{d}{d\varphi}\left(\frac{r_2}{r_1}\right)-\frac{r_2}{r_1}\left(\frac{r_2}{r_1}+\nu\right)\left(\frac{\dot h}{h}+\frac{\dot E}{E}\right)\right.$$
$$\left.-\cot\varphi\left[\left(\frac{r_2}{r_1}\right)^2-1\right]\right]+\frac{Eh}{r_1}\left[r_2\frac{d}{d\varphi}(\alpha T_0)+\alpha T_0\frac{dr_2}{d\varphi}\right]$$
$$+Eh\alpha T_0\cot\varphi\left(\frac{r_2}{r_1}-1\right)+\left(\frac{r_2}{r_1}+\nu\right)(r_2p_\varphi)+\frac{1}{r_1}\frac{d}{d\varphi}(p_r r_2^2)$$
$$-p_r\frac{r_2^2}{r_1}\left[\frac{\dot h}{h}+\frac{\dot E}{E}\right]$$

and $\quad R = \int r_1r_2\sin\varphi[p_\varphi\sin\varphi-p_r\cos\varphi]\,d\varphi+\text{constant}$

the constant depending on the external loading at the shell edges.

$$H = \frac{r_2}{r_1}\cot\varphi+\frac{d}{d\varphi}\left(\frac{r_2}{r_1}\right)+\frac{r_2}{r_1}\left[\frac{\dot E}{E}+\frac{3\dot h}{h}\right]$$

$$I = \nu\cot\varphi\left[\frac{\dot E}{E}+\frac{3\dot h}{h}\right]-\nu-\frac{r_1}{r_2}\cot^2\varphi$$

$$G = r_2(1+\nu)\alpha\left[\frac{dT_1}{d\varphi}+T_1\left[\frac{\dot E}{E}+\frac{3\dot h}{h}\right]\right]$$

$$\dot h = \frac{dh}{d\varphi}, \quad \dot E = \frac{dE}{d\varphi}$$

From the values of U and V from these equations we obtain the following expressions for the stress resultants:

$$N_\varphi = -\frac{U}{r_2}\cot\varphi-\frac{R}{r_0}\operatorname{cosec}\varphi \tag{2.15a}$$

from equation (2.4d)

$$N_\theta = -\frac{1}{r_1}\frac{dU}{d\varphi}+\frac{R}{r_1}\operatorname{cosec}^2\varphi+p_r r_0\operatorname{cosec}\varphi \tag{2.15b}$$

from equation (2.4b)

$$M_\varphi = D\left[\frac{1}{r_1}\frac{dV}{d\varphi}+\frac{\nu V\cot\varphi}{r_2}-(1+\nu)\alpha T_1\right] \tag{2.15c}$$

$$M_\theta = D\left[\frac{V\cot\varphi}{r_2}+\frac{\nu}{r_1}\frac{dV}{d\varphi}-(1+\nu)\alpha T_1\right] \tag{2.15d}$$

from (2.10d, e), (2.5d, e) and (2.6).

In establishing compatibility conditions at the junction of the two shells we require V, the rotation of the tangent to the meridian and also δ, the deflection at right angles to the axis of symmetry, i.e. the increase in radius of a parallel circle

$$\delta = r_0\varepsilon_\theta = \frac{r_0}{Eh}(N_\theta-\nu N_\varphi)+r_0\alpha T_0 \tag{2.16}$$

In [2.9] a version of equations (2.14) is given which applies not only for the elastic case but also when plasticity and creep are present. In these cases, the terms F and G contain terms dependent on the current plastic strains.

Rigorous solutions to equations (2.14) can rarely be obtained and they involve complex mathematical analysis. A large amount of work on the numerical solution of these equations in which all the terms are included has been done and the work is reviewed in Chapter 8. The solution of the equations for creep is discussed in Chapter 10.

Equations (2.14) include the terms F and G on the right-hand side due to the applied loads and temperature. If we put $p_\varphi = p_r = 0$ and consider a uniform temperature, then $F = G = 0$ and the solutions to equations (2.14) are the edge bending solutions. They are the solutions for a shell of revolution subject to forces and moments uniformly distributed along a parallel circle.

These edge bending solutions are of great importance in pressure vessel problems. We have pointed out in section 2.1.5 that membrane solutions cannot always satisfy the boundary conditions. By applying self-equilibrating forces and bending moments at the boundary of a shell together with the membrane solutions we can satisfy the boundary conditions of equilibrium and compatibility of displacement and rotation. We mention again here that by a "boundary" in this context we often mean the junction of a cylindrical vessel and the end closure or the junction of the top and bottom parts of say a spherical vessel at a skirt support.

In order to illustrate the nature of edge bending solutions and their use in satisfying boundary conditions we will consider the simplest form of solution applied to a very simple problem.

Putting $F = G = 0$ in equations (2.14) with a uniform temperature for a shell of constant thickness and constant Young's modulus, the equations may be written:

$$L(U)+\frac{\nu}{r_1}U = -EhV \tag{2.17a}$$

$$L(V)-\frac{\nu}{r_1}V = \frac{U}{D} \tag{2.17b}$$

where $$L = \frac{r_2}{r_1^2}\frac{d^2(\ldots)}{d\varphi^2}+\frac{1}{r_1}\left[\frac{r_2}{r_1}\cot\varphi+\frac{d}{d\varphi}\left(\frac{r_2}{r_1}\right)\right]\frac{d(\ldots)}{d\varphi}-\frac{\cot^2\varphi}{r_2}(\ldots)$$

Now for the case of a spherical shell for which $r_1 = r_2 = R$, say, equations (2.17) may be written (using T for the thickness of the sphere)

$$\frac{d^2Q_\varphi}{d\varphi^2}+\cot\varphi\frac{dQ_\varphi}{d\varphi}-(\cot^2\varphi-\nu)Q_\varphi = -ETV \tag{2.18a}$$

$$\frac{d^2V}{d\varphi^2}+\cot\varphi\frac{dV}{d\varphi}-(\cot^2\varphi+\nu)V = \frac{R^2Q_\varphi}{D} \tag{2.18b}$$

Edge bending solutions are of a damped oscillatory character and if the shell is thin, the damping is very rapid. Hence we may assume that the second derivatives of Q_φ and V are large compared with the first derivatives and Q_φ and V themselves and we replace equation (2.18) by

$$\frac{d^2Q_\varphi}{d\varphi^2} = -ETV \tag{2.19a}$$

$$\frac{d^2V}{d\varphi^2} = \frac{R^2Q_\varphi}{D} \tag{2.19b}$$

We note, however, that in equations (2.18) Q_ϕ and V on the left-hand side are multiplied by $\cot^2\varphi$ and the first derivatives are multiplied by $\cot\varphi$. It follows, therefore, that this approximation will not be valid if φ is small and hence $\cot\varphi$ large. For the approximation to be valid we should have $\varphi > 30°$.

Equation (2.19) gives

$$\frac{d^4Q_\varphi}{d\varphi^4}+4\varkappa^4Q_\varphi = 0$$

where $$\varkappa^4 = \frac{ETR^2}{4D} = 3(1-\nu^2)\left(\frac{R}{T}\right)^2$$

for which the general solution is

$$Q_\varphi = A_1e^{\varkappa\varphi}\cos\varkappa\varphi+A_2e^{\varkappa\varphi}\sin\varkappa\varphi+A_3e^{-\varkappa\varphi}\cos\varkappa\varphi+A_4e^{-\varkappa\varphi}\sin\varkappa\varphi \tag{2.20}$$

where four arbitrary constants are available to satisfy the boundary conditions.

Having obtained Q_φ we can proceed to find all the other stress resultants, strains, displacements from our previous equations as for equations (2.15).

From (2.4d)

$$N_\varphi = -Q_\varphi \cot\varphi$$

The equilibrium equation (2.4b) for a sphere where

$$r_0 = r_1 \sin\varphi = R\sin\varphi$$

gives
$$N_\theta = -N_\varphi - \frac{dQ_\varphi}{d\varphi} - Q_\varphi \cot\varphi = -\frac{dQ_\varphi}{d\varphi}$$

From (2.19),

$$V = -\frac{1}{ET}\frac{d^2Q_\varphi}{d\varphi^2}$$

and from (2.15c and d)

$$M_\theta = \frac{\nu D}{R}\frac{dV}{d\varphi}$$

$$M_\varphi = \frac{D}{R}\frac{dV}{d\varphi}$$

where again we neglect V compared with $dV/d\varphi$ which is satisfactory if $\cot\varphi$ is small (i.e. $\varphi > 30°$).

Finally, we often require the radial displacement of the edge of a shell which is the radial displacement of a parallel circle.

We obtain (from (2.16))

$$\delta = \frac{(N_\theta - \nu N_\varphi)}{ET} R\sin\varphi$$

$$= -\frac{R\sin\varphi}{ET}\frac{dQ_\varphi}{d\varphi}$$

again neglecting Q_φ in comparison with $dQ_\varphi/d\varphi$.

We must emphasize that these simple solutions are only valid for thin shells for $\varphi > 30°$. A full treatment of edge bending solutions is given in Chapter 3 where it may be noted that the Hetényi solutions, equations (3.44) which are valid for the rotationally symmetric case and for the first harmonic loading (i.e. $n = 0$ and $n = 1$) are similar in form to equation (2.20) divided by $\sqrt{(\sin\varphi)}$. They are quite accurate when $\sqrt{(2)}\varkappa\varphi > 6$.

In section 2.2 we consider the shallow shell equations giving solutions when φ is small. All these different solutions are used where appropriate in solving pressure vessel problems, but it should be noted that the solutions given at the beginning of Chapter 3 are valid for all values of φ.

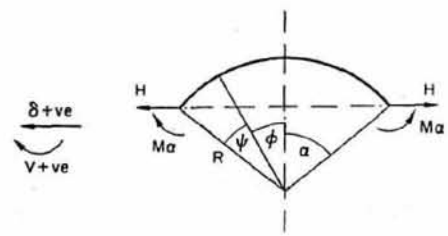


FIG. 2.6

Returning to the general solution (2.20) consider the example shown in Fig. 2.6 for a spherical cap of angle α with a moment M_α and outward force H applied to the bottom edge.

For this case A_3 and A_4 must be zero since as φ decreases away from the edge, the edge effects are damped out and hence terms involving $e^{-\varkappa\varphi}$ are omitted.

Changing the variables and using $\psi = (\alpha - \varphi)$ equation (2.20) can be written $Q_\varphi = Ae^{-\varkappa\psi}\sin(\varkappa\psi + \gamma)$. The boundary conditions at $\varphi = \alpha$, i.e. $\psi = 0$, are:

$$M_\varphi = M_\alpha \quad N_\varphi = H\cos\alpha \quad Q_\varphi = -H\sin\alpha$$

The complete solution is given in Table 2.1. The table shows the two cases where H and M_α are applied to the bottom edge as in Fig. 2.6 or

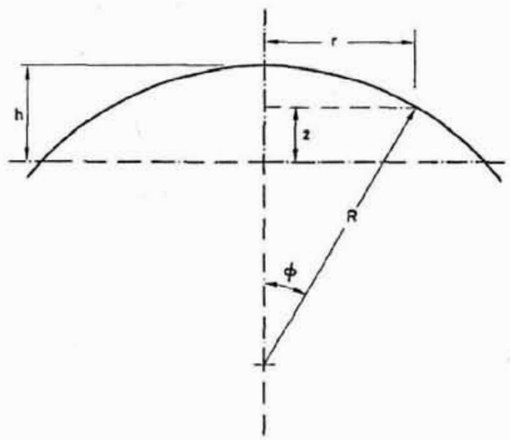


FIG. 2.8

Using N_r, M_r, $M_{\theta r}$, Q_r, p_r instead of N_φ, M_φ, $M_{\theta\varphi}$, Q_φ, p_φ and noting in equation (2.3) for a sphere that $r_1 = r_2 = R$, the equilibrium equations are:

$$\frac{\partial}{\partial r}(rN_r)+\frac{\partial N_{r\theta}}{\partial\theta}-N_\theta-\frac{r}{R}Q_r+rp_r = 0 \tag{2.21a}$$

$$\frac{\partial}{\partial r}(rN_{r\theta})+\frac{\partial N_\theta}{\partial\theta}+N_{r\theta}-\frac{r}{R}Q_\theta+rp_\theta = 0 \tag{2.21b}$$

$$\frac{r}{R}N_\theta+\frac{\partial Q_\theta}{\partial\theta}+\frac{r}{R}N_r+\frac{\partial}{\partial r}(rQ_r)-rp = 0 \tag{2.21c}$$

$$\frac{\partial}{\partial r}(rM_r)+\frac{\partial M_{\theta r}}{\partial\theta}-M_\theta-rQ_r = 0 \tag{2.21d}$$

$$\frac{\partial}{\partial r}(rM_{r\theta})+\frac{\partial M_\theta}{\partial\theta}+M_{\theta r}-rQ_\theta = 0 \tag{2.21e}$$

where p is now used for the radial, i.e. internal, pressure.

The strain–displacement equations (2.5) become

$$\varepsilon_r = \frac{\partial v}{\partial r}+\frac{w}{R} \tag{2.22a}$$

$$\varepsilon_\theta = \frac{1}{r}\frac{\partial u}{\partial\theta}+\frac{v}{r}+\frac{w}{R} \tag{2.22b}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{\partial u}{\partial r}-\frac{u}{r} = \frac{1}{r}\frac{\partial v}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{u}{r}\right) \tag{2.22c}$$

$$\varkappa_r = \frac{\partial^2 w}{\partial r^2} \tag{2.22d}$$

$$\varkappa_\theta = \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}+\frac{1}{r}\frac{\partial w}{\partial r} \tag{2.22e}$$

$$\varkappa_{\theta r} = -\frac{1}{r^2}\frac{\partial w}{\partial\theta}+\frac{1}{r}\frac{\partial^2 w}{\partial\theta\,\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right) \tag{2.22f}$$

With the appropriate change of notation (i.e. φ to r) equations (2.10a–f) are still applicable.

For rotationally symmetric edge loading and constant temperature we put $p_r = p = p_\theta = 0$ and we can introduce a stress function F such that $N_\theta = \frac{\partial^2 F}{\partial r^2}$, $N_r = \frac{1}{r}\frac{\partial F}{\partial r}$ which satisfies (2.21a) if we ignore the term $\frac{r}{R}Q_r$ which is negligible for shallow shells.

We obtain two equations in w and F

$$\nabla^2\nabla^2 F-\frac{ET}{R}\nabla^2 w = 0$$

$$D\nabla^2\nabla^2 w+\frac{1}{R}\nabla^2 F = 0$$

where
$$\nabla^2 = \frac{d^2(-)}{dr^2}+\frac{1}{r}\frac{d(-)}{dr}$$

which are the equations of the problem of a shallow spherical shell of constant thickness subject to forces and moments uniformly distributed along a parallel circle.

The solution is:

$$w = A_1 \operatorname{ber}\frac{r}{l} + A_2 \operatorname{bei}\frac{r}{l} + A_3 \ker\frac{r}{l} + A_4 \operatorname{kei}\frac{r}{l} + A_5 \tag{2.23}$$

$$F = \frac{ET^2}{\sqrt{[12(1-\nu^2)]}}\left[A_1 \operatorname{bei}\frac{r}{l} - A_2 \operatorname{ber}\frac{r}{l} + A_3 \operatorname{kei}\frac{r}{l} - A_4 \ker\frac{r}{l} + A_6 \log\frac{r}{l}\right]$$

where $l = \dfrac{\sqrt{(RT)}}{\sqrt[4]{[12(1-\nu^2)]}}$

The functions ber, bei, ker, kei are Kelvin functions [2.11] of argument

$$\frac{r}{l} = \frac{r}{\sqrt{(RT)}}\sqrt[4]{[12(1-\nu^2)]} = \varrho\cdot\sqrt[4]{[12(1-\nu^2)]}$$

where $\varrho = \dfrac{r}{R}\sqrt{\left(\dfrac{R}{T}\right)}$

N_θ and N_r are derived from F as given above and we also have:

$$M_r = D\left[\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right]$$

$$M_\theta = D\left[\frac{1}{r}\frac{dw}{dr} + \frac{\nu\, d^2w}{dr^2}\right]$$

$$Q_r = D\frac{d}{dr}(\nabla w^2)$$

Since all quantities are functions of ϱ it follows [2.12] that the stress concentration factor due to internal pressure at an unreinforced hole of radius r_0 in a spherical vessel of radius R and thickness T is a function of

$$\varrho_0 = \frac{r_0}{R}\sqrt{\left(\frac{R}{T}\right)}.$$

This is extremely useful, since in this problem the geometrical parameters of the structure are given by r_0/R and R/T and the fact that the dimensionless group ϱ_0 combines these two ratios is of great value in plotting results. Reference [2.12] uses this group in plotting stress concentration factors for nozzles in spheres. See also Chapter 3.

2.3. Cylindrical Shells

The circular cylinder is obviously a surface of revolution but because of its importance in pressure vessel work the relevant equations and principles of analysis are now stated independently.

2.3.1. EQUATIONS OF EQUILIBRIUM FOR CYLINDRICAL SHELLS

For the general case of asymmetric loading the shell element of length dx and width $r\cdot d\theta$ has the stress resultants shown in Figs. 2.9 and 2.10 the directions shown being taken as positive.

The detailed steps are given in [2.1], [2.2] and lead to the equilibrium equations:

$$\frac{r\,\partial N_x}{\partial x} + \frac{\partial N_{\theta x}}{\partial \theta} + rp_x = 0 \tag{2.24a}$$

$$\frac{\partial N_\theta}{\partial \theta} + \frac{r\,\partial N_{x\theta}}{\partial x} - Q_\theta + rp_\theta = 0 \tag{2.24b}$$

$$\frac{\partial Q_\theta}{\partial \theta} + \frac{r\,\partial Q_x}{\partial x} + N_\theta - rp_r = 0 \tag{2.24c}$$

$$\frac{\partial M_\theta}{\partial \theta} + \frac{r\,\partial M_{x\theta}}{\partial x} - rQ_\theta = 0 \tag{2.24d}$$

$$\frac{r\,\partial M_x}{\partial x} + \frac{\partial M_{\theta x}}{\partial \theta} - rQ_x = 0 \tag{2.24e}$$

$$rN_{x\theta} - rN_{\theta x} + M_{\theta x} = 0 \tag{2.24f}$$

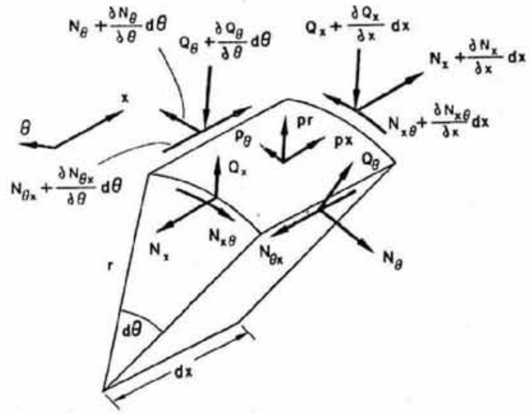


FIG. 2.9

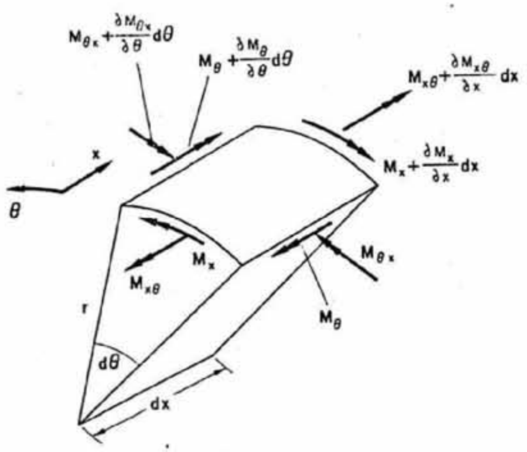


FIG. 2.10

For the membrane solution only we have $M_\theta = M_x = M_{x\theta} = M_{\theta x} = 0$ and hence $Q_x = Q_\theta = 0$.

If the loading is rotationally symmetric then $p_\theta = 0$ and $N_{x\theta} = N_{\theta x} = M_{x\theta} = M_{\theta x} = Q_\theta = 0$ and M_θ and N_θ are independent of θ and

the equations are:

$$\frac{dN_x}{dx} + p_x = 0 \tag{2.25a}$$

$$r\frac{dQ_x}{dx} + N_\theta - rp_r = 0 \tag{2.25b}$$

$$\frac{dM_x}{dx} = Q_x \tag{2.25c}$$

and the membrane solution for rotationally symmetric loading with $M_x = Q_x = 0$ is trivially obtained.

2.3.2. STRAINS AND DISPLACEMENTS IN CYLINDRICAL SHELLS

If the displacements of the mid-surface of the shell are u, v, w, defined as positive in the directions shown in Fig. 2.11, then the strains, curvatures and twists are given by [2.4]:

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{2.26a}$$

$$\varepsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r} \tag{2.26b}$$

$$\gamma_{\theta x} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} \tag{2.26c}$$

$$\varkappa_x = \frac{\partial^2 w}{\partial x^2} \tag{2.26d}$$

$$\varkappa_\theta = \frac{1}{r^2}\left[\frac{\partial^2 w}{\partial \theta^2} - \frac{dv}{\partial \theta}\right] \tag{2.26e}$$

$$2\varkappa_{\theta x} = \frac{1}{r}\left[\frac{2\,\partial^2 w}{\partial x\,\partial \theta} - \frac{\partial v}{\partial x}\right] \tag{2.26f}$$

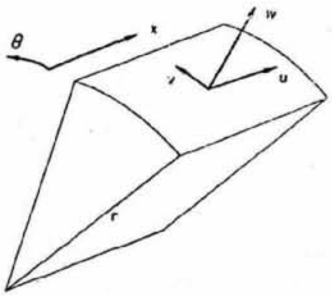


FIG. 2.11

2.3.3. ELASTIC ANALYSIS OF CYLINDRICAL SHELLS

Consider a cylindrical shell made of an elastic material which obeys Hooke's law.

If the strains at a distance z (+ve in direction of outward normal) from the middle surface are $\bar{\varepsilon}_x$, $\bar{\varepsilon}_\theta$, $\bar{\gamma}_{x\theta}$, then

$$\bar{\varepsilon}_x = \varepsilon_x - z\varkappa_x$$

$$\bar{\varepsilon}_\theta = \varepsilon_\theta - z\varkappa_\theta$$

$$\bar{\gamma}_{\theta x} = \gamma_{\theta x} - 2z\varkappa_{\theta x}$$

Using Hooke's law as for the shell of revolution we obtain equations similar to (2.8), (2.9), (2.10) replacing φ by x in those equations.

Omitting the temperature terms, we obtain:

$$N_\theta = K(\varepsilon_\theta + \nu\varepsilon_x) \tag{2.27a}$$

$$N_x = K(\varepsilon_x + \nu\varepsilon_\theta) \tag{2.27b}$$

$$N_{\theta x} = K\left(\frac{1-\nu}{2}\right)\gamma_{\theta x} \tag{2.27c}$$

$$M_\theta = D(\varkappa_\theta + \nu\varkappa_x) \tag{2.27d}$$

$$M_x = D(\varkappa_x + \nu\varkappa_\theta) \tag{2.27e}$$

$$M_{\theta x} = D(1-\nu)\varkappa_{\theta x} \tag{2.27f}$$

Now, following the arguments given by Kraus [2.4] we can drop the terms in v in the expressions for $\varkappa_\theta$ and $\varkappa_{\theta x}$, ignore the effect of Q_θ in equation (2.24b) and ignore equation (2.24f) for the reasons previously discussed in connection with general shells of revolution.

We can substitute the expressions for the stress resultants in terms of u, v, w in the equations of equilibrium and obtain (for the case of uniform temperature):

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1+\nu}{2r}\frac{\partial^2 v}{\partial x\,\partial \theta} + \frac{\nu}{r}\frac{\partial w}{\partial x} + \frac{p_x}{K} = 0 \tag{2.28a}$$

$$\frac{1+\nu}{2r}\frac{\partial^2 u}{\partial x\,\partial \theta} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial w}{\partial \theta} + \frac{p_\theta}{K} = 0 \tag{2.28b}$$

$$\frac{\nu\,\partial u}{\partial x} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r} + \frac{t^2}{12}\left[\frac{r\,\partial^4 w}{\partial x^4} + \frac{2}{r}\frac{\partial^4 w}{\partial x^2\,\partial \theta^2} + \frac{1}{r^3}\frac{\partial^4 w}{\partial \theta^4}\right] - \frac{rp_r}{K} = 0 \tag{2.28c}$$

These equations give the complete solution to the cylindrical shell problem under loading p_x, p_θ, p_r with the appropriate boundary conditions.

Equations (2.28) may also be written, and are often given in various papers, in the form (for $p_r = p_\theta = p_x = 0$)

$$\nabla^8 w + \frac{12(1-\nu^2)}{r^2 t^2}\frac{\partial^4 w}{\partial x^4} = 0 \tag{2.29a}$$

$$\nabla^4 v = -\frac{(2+\nu)}{r^2}\frac{\partial^3 w}{\partial x^2\,\partial \theta} - \frac{1}{r^4}\frac{\partial^3 w}{\partial \theta^3} \tag{2.29b}$$

$$\nabla^4 u = -\frac{\nu}{r}\frac{\partial^3 w}{\partial x^3} + \frac{1}{r^3}\frac{\partial^3 w}{\partial x\,\partial \theta^2} \tag{2.29c}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

These are known as the Donnell equations [2.13]. The corresponding equations for the case with surface loads and temperature are given by Kraus [2.4] who also discusses the accuracy of the equations.

In [2.14] Morley gives an alternative set of equations. Although the only difference from equations (2.29) is in the first term of the first equation, they are considerably more reliable [2.4].

In Chapter 5 the Donnell equations are used and the corresponding expressions for the stress resultants in terms of u, v, w, are given.

2.3.4. EDGE SOLUTIONS FOR CYLINDRICAL SHELLS

If we apply stress resultants at a plane x = constant of a circular cylinder with $p_x = p_\theta = p_r = 0$, the procedure given by Hoff [2.15] may be used.

Hoff puts

$$w = Ae^{\frac{px}{r}} \cos n\theta$$

$$u = Be^{\frac{px}{r}} \cos n\theta$$

$$v = Ce^{\frac{px}{r}} \sin n\theta$$

in equations (2.28) and obtains:

$$\begin{aligned} \frac{w}{\cos n\theta} &= e^{-\frac{\alpha_1 x}{r}}\left(A_1 \cos\frac{\beta_1 x}{r} + A_2 \sin\frac{\beta_1 x}{r}\right) \\ &+ e^{-\frac{\alpha_2 x}{r}}\left(A_3 \cos\frac{\beta_2 x}{r} + A_4 \sin\frac{\beta_2 x}{r}\right) \\ &+ e^{\frac{\alpha_1 x}{r}}\left(A_5 \cos\frac{\beta_1 x}{r} + A_6 \sin\frac{\beta_1 x}{r}\right) \\ &+ e^{\frac{\alpha_2 x}{r}}\left(A_7 \cos\frac{\beta_2 x}{r} + A_8 \sin\frac{\beta_2 x}{r}\right) \end{aligned} \tag{2.30}$$

where

$$\alpha_1 = \frac{1}{2}\left[\Omega+\lambda+\frac{n^2}{\Omega}\right] \quad \beta_1 = \frac{1}{2}\left[\Omega-\lambda-\frac{n^2}{\Omega}\right]$$

$$\alpha_2 = \frac{1}{2}\left[\Omega-\lambda+\frac{n^2}{\Omega}\right] \quad \beta_2 = \frac{1}{2}\left[\Omega+\lambda-\frac{n^2}{\Omega}\right]$$

$$\Omega = +\left[-\left(\frac{\lambda^2}{2}\right)+\left(\left(\frac{\lambda^2}{2}\right)^2+n^4\right)^{1/2}\right]^{1/2}$$

and
$$\lambda^4 = 3(1-\nu^2)\left(\frac{r}{t}\right)^2$$

Corresponding expressions for u, v and all the stress resultants are given by Hoff. Graphs of $\alpha_1, \beta_1, \alpha_2, \beta_2$ are given as functions of r/t for values of n from 1 to 10.

There is, however, one important point about the stress resultants at the boundary x = constant which should be noted:

Fig. 2.12(a) shows a part of the cylinder cross-section at x = constant (a boundary). The stress resultants on this section are N_x, M_x, Q_x, $N_{x\theta}$,

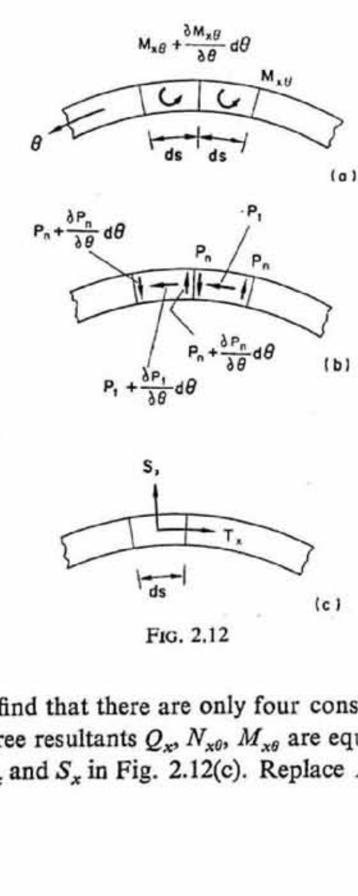


FIG. 2.12

$M_{x\theta}$, whereas we find that there are only four constants of integration available. The three resultants Q_x, $N_{x\theta}$, $M_{x\theta}$ are equivalent to the combined action of T_x and S_x in Fig. 2.12(c). Replace $M_{x\theta}$ by forces P_n as shown where $P_n\,ds = M_{x\theta}\,ds$ (Fig. 2.12(b)). These forces have a tangential resultant $P_n\,d\theta = P_t$ say, so that we see that the forces as shown in Fig. 2.12(b) are statically equivalent to $M_{x\theta}$. We are saying here that P_n and P_t are statically equivalent to the shear stresses which produce $M_{x\theta}$ but this is by ignoring local stress problems within the shell thickness which are not considered in thin shell theory.

By considering two adjacent elements and adding $N_{x\theta}$ and Q_x we see that the final resultants of Fig. 2.12(c) are

$$T_x\,ds = N_{x\theta}\,ds - P_t$$

$$= N_{x\theta}\,ds - M_{x\theta}\,d\theta$$

$$T_x = N_{x\theta} - \frac{M_{x\theta}}{r} \tag{2.31}$$

since $$\frac{d\theta}{ds} = \frac{1}{r}$$

Similarly

$$S_x\,ds = Q_x\,ds + \frac{\partial P_n}{\partial \theta}\cdot d\theta = Q_x\,ds + \frac{\partial M_{x\theta}}{\partial \theta}\cdot d\theta$$

$$S_x = Q_x + \frac{1}{r}\,\frac{\partial M_{x\theta}}{\partial \theta} \tag{2.32}$$

Hoff gives expressions for these quantities [2.15].

Returning to equation (2.30) for the particular case when $n = 1$, it is noted by Bailey and Hicks [2.16] that since λ is large (>4 say in many practical cases) then $\Omega \doteqdot 1/\lambda$ and $\alpha_1 \doteqdot -\beta_1 \doteqdot \lambda$, and $\alpha_2 \doteqdot \beta_2 \doteqdot 1/2\lambda$. Thus α_1 is much greater than α_2 so that any change in the radial deflections or stresses with x is due more or less entirely to terms containing α_1 and hence for $n = 1$ we may write

$$\frac{w}{\cos\theta} = e^{-\frac{\lambda x}{r}}\left[A_1\cos\frac{\lambda x}{r} - A_2\sin\frac{\lambda x}{r}\right] + e^{+\frac{\lambda x}{r}}\left[A_5\cos\frac{\lambda x}{r} - A_6\sin\frac{\lambda x}{r}\right] \tag{2.33}$$

Now consider the much simpler case of a circular cylinder subjected to rotationally symmetric edge loads.

It is easiest to start from first principles.

We have

$$\varepsilon_\theta = \frac{w}{r}$$

For no external loads, i.e. $p_r = 0$, $N_x = 0$,

we have $$N_\theta = \frac{Etw}{r}$$

$$M_x = D\frac{d^2w}{dx^2}$$

and substituting in the equilibrium equation

$$\frac{d^2M_x}{dx^2} = \frac{dQ_x}{dx} = -\frac{N_\theta}{r}$$

we obtain

$$\frac{D\,d^4w}{dx^4} + \frac{Etw}{r^2} = 0$$

i.e. $$\frac{d^4w}{dx^4} + 4\beta^4 w = 0$$

where $$\beta^4 = \frac{Et}{4Dr^2} = \frac{3(1-\nu^2)}{r^2t^2}$$

This is the differential equation for a circular cylinder subjected to rotationally symmetric edge loading. The solution is:

$$w = e^{\beta x}(k_1\cos\beta x + k_2\sin\beta x) + e^{-\beta x}(k_3\cos\beta x + k_4\sin\beta x) \tag{2.34}$$

Comparing (2.33) and (2.34) we note that $\lambda/r = \beta$ and hence the equations for $w/\cos\theta$ when $n = 1$ and w when $n = 0$ are the same. This is similar to the situation for spheres where the Hetényi solutions (see Chapter 3) are valid for $n = 0$ and $n = 1$. Consider an infinitely long cylinder (Fig. 2.13) subjected to rotationally symmetric edge shear Q_0 and edge moment M_0 as shown in the figure, giving a radial deflection w_0 and a slope θ_0. All quantities shown in the figure are positive. Note that due to M_0 and Q_0 it is obvious that θ_0 will be negative.

Because the cylinder is long, the coefficients k_1 and k_2 of terms containing $e^{\beta x}$ must be zero and equation (2.34) becomes

$$w = e^{-\beta x}(k_3 \cos \beta x + k_4 \sin \beta x) \tag{2.35}$$

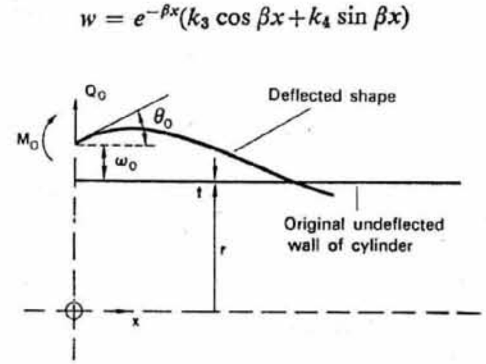


FIG. 2.13

and the constants k_3 and k_4 may be evaluated to suit the boundary conditions at the free edge. Now we will define four useful functions:

$$A_{\beta x} = e^{-\beta x}(\cos \beta x + \sin \beta x) \tag{2.36a}$$

$$B_{\beta x} = e^{-\beta x} \sin \beta x \tag{2.36b}$$

$$C_{\beta x} = e^{-\beta x}(\cos \beta x - \sin \beta x) \tag{2.36c}$$

$$D_{\beta x} = e^{-\beta x} \cos \beta x \tag{2.36d}$$

Graphs of these functions are shown in Fig. 2.14.

From the figure it is evident that for $\beta x > \pi$ the values of these quantities are negligible.

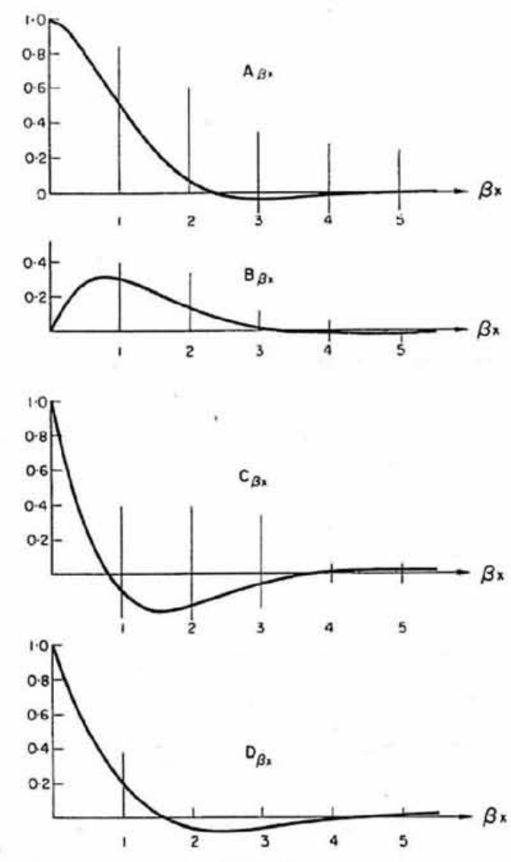


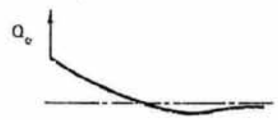
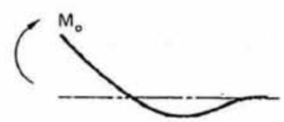
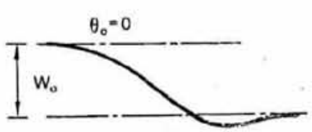
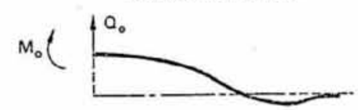
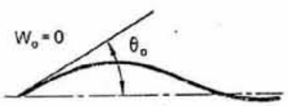
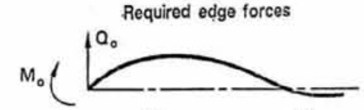
FIG. 2.14

It follows, therefore, that the effect of edge disturbances on a cylindrical pressure vessel are negligible when

$$x > 2{\cdot}45\sqrt{(rt)}$$

e.g. for a 5-foot diameter vessel $\frac{1}{2}$ inch thick $x > 9{\cdot}5$ inches, and this

TABLE 2.2. EDGE SOLUTIONS FOR LONG CYLINDRICAL SHELL

End condition	Deflected shape	Slope	Longitudinal b.m./unit width	Longitudinal s.f./unit width	Remarks
	$w = \frac{2Q_0\beta}{k} D_{\beta x}$	$-\frac{2Q_0\beta^2}{k} A_{\beta x}$	$\frac{Q_0}{\beta} B_{\beta x}$	$Q_0 C_{\beta x}$	w_0 and θ_0 found by putting $x = 0$ in expressions for deflection and slope.
	$w = \frac{2M_0\beta^2}{k} C_{\beta x}$	$-\frac{4M_0\beta^3}{k} D_{\beta x}$	$M_0 A_{\beta x}$	$-2\beta M_0 B_{\beta x}$	w_0 and θ_0 found by putting $x = 0$ in expressions for deflection and slope.
	$w = w_0 A_{\beta x}$	$-2\beta w_0 B_{\beta x}$	$-\frac{w_0 k}{2\beta^2} C_{\beta x}$	$\frac{k w_0}{\beta} D_{\beta x}$	Required edge forces  $Q_o = +\frac{kw_o}{\beta}$ $M_o = -\frac{kw_o}{2\beta^2}$
	$w = \frac{\theta_0}{\beta} B_{\beta x}$	$\theta_0 C_{\beta x}$	$-\frac{\theta_0 k}{2\beta^3} D_{\beta x}$	$\frac{\theta_0 k}{2\beta^2} A_{\beta x}$	Required edge forces  $Q_o = +\frac{k\theta_o}{2\beta^2}$ $M_o = -\frac{k\theta_o}{2\beta^3}$

$$k = \frac{Et}{r^2} \qquad \beta^4 = \frac{k}{4D} = \frac{3(1-\nu^2)}{r^2t^2}$$

$A_{\beta x}$, $B_{\beta x}$, $C_{\beta x}$, $D_{\beta x}$ as defined in equation (2.36).

is our justification for neglecting the coefficients of $e^{\beta x}$ in equation (2.34) to obtain equation (2.35).

For a cylinder whose length l gives $\beta l < \pi$ the edge disturbances at the two ends will interfere with one another and a complete solution involves using equation (2.34) and evaluating all four constants of integration for the edge conditions. These are considered in great detail by Hetényi [2.17].

Table 2.2 shows solutions of equation (2.35) for the various possible boundary conditions for a semi-infinite cylinder.

The membrane stresses due to internal pressure, i.e. pr/t and $pr/2t$, For the sphere from Table 2.1 putting $\alpha = \pi/2$

$$\delta_{2e} = \frac{2\varkappa^2 M}{Et} - \frac{2\varkappa r Q}{Et}$$

$$\theta_{2e} = \frac{4\varkappa^3 M}{Ert} - \frac{2\varkappa^2 Q}{Et}$$

Note that

$$\varkappa = \sqrt[4]{[3(1-\nu^2)]}\sqrt{\left(\frac{r}{t}\right)} \quad \text{and} \quad \beta = \frac{\sqrt[4]{[3(1-\nu^2)]}}{\sqrt{(rt)}}$$

For equality of slope:

$$\theta_{1e} = \theta_{2e}$$

$$-\frac{2Q\beta^2}{k} - \frac{4M\beta^3}{k} = \frac{4\varkappa^3 M}{Ert} - \frac{2\varkappa^2 Q}{Et}$$

Noting that $k = Et/r^2$ and putting $n = \sqrt[4]{[3(1-\nu^2)]}$

$$-2n^2 \frac{r}{t} Q - \frac{4Mn^3 r^2}{(rt)^{3/2}} = \frac{4n^3 M r^2}{(rt)^{3/2}} - 2n^2 \frac{r}{t} Q.$$

Also for equality of deflections:

$$\delta_1 + \delta_{1e} = \delta_2 + \delta_{2e}$$

Therefore $$\frac{pr^2}{2tE}(2-\nu) + \frac{2Q\beta}{k} + \frac{2M\beta^2}{k} = \frac{pr^2}{2tE}(1-\nu) + \frac{2\varkappa^2 M}{Et} - \frac{2r\varkappa Q}{Et}$$

Therefore $$\frac{pr^2}{2tE} = 2M\left[\frac{\varkappa^2}{Et} - \frac{2\beta^2}{k}\right] - 2Q\left[\frac{r\varkappa}{Et} + \frac{\beta}{k}\right]$$

We thus have two equations for M and Q and solving we obtain

$$M = 0$$

$$\frac{pr^2}{2tE} = -\frac{2Q}{Et}[nr\sqrt{(r/t)} + nr\sqrt{(r/t)}]$$

$$Q = -\frac{pr}{8n}\sqrt{\left(\frac{t}{r}\right)} = -\frac{p\sqrt{(rt)}}{8n} = -\frac{p}{8\beta}$$

It is interesting and important to note that the damping coefficients β, $\varkappa$ for the cylinder and sphere are effectively the same.

This now gives us the complete solution for this junction provided of course that the cylinder is sufficiently long so that there is no interference from any discontinuity stresses at the other end of the cylinder.

Finally the stresses are the algebraic sum of the membrane stresses and the stresses due to the discontinuity force Q. Expressions for the stresses due to Q are found from Tables 2.1 and 2.2.

For the cylinder

$$N_\theta = \frac{Ewt}{r} = \frac{Et}{r}\cdot\frac{2Q\beta}{k} D_{\beta x} = 2rQ\beta D_{\beta x} = -\frac{pr}{4} D_{\beta x}$$

$$M_x = \frac{Q}{\beta} B_{\beta x} = -\frac{p}{8\beta^2} B_{\beta x}$$

$$M_\theta = \nu M_x$$

$$N_x = 0$$

$$\sigma_x = \mp \frac{6M_x}{t^2} = \pm\frac{3p}{4\beta^2 t^2} B_{\beta x} = \pm\frac{3}{4}\frac{pr}{tn^2} B_{\beta x}$$

$$\sigma_\theta = -\frac{pr}{t}\left[\frac{D_{\beta x}}{4} \mp \frac{3\nu}{4n^2} B_{\beta x}\right]$$

The top sign applies to the outer surface.

For the sphere (from Table 2.1)

$$N_\theta = +\varkappa\sqrt{2}\,(\sqrt{2}\cdot Q)e^{-\varkappa\psi}\sin\left(\varkappa\psi - \frac{\pi}{2}\right)$$

$$= -2\varkappa Q e^{-\varkappa\psi}\cos\varkappa\psi = \frac{p\varkappa}{4\beta} e^{-\varkappa\psi}\cos\varkappa\psi$$

$$= \frac{pr}{4} e^{-\varkappa\psi}\cos\varkappa\psi$$

$$M_\varphi = \frac{r}{\varkappa\sqrt{2}}(\sqrt{2}\cdot Q)e^{-\varkappa\psi}\sin\varkappa\psi$$

$$= +\frac{prt}{8n^2} e^{-\varkappa\psi}\sin\varkappa\psi$$

$$M_\theta = \nu M_\varphi$$

$$\sigma_\varphi = \mp\frac{6M_\varphi}{t^2} = \mp\frac{3}{4}\frac{pr}{tn^2} e^{-\varkappa\psi}\sin\varkappa\psi$$

$$\sigma_\theta = +\frac{pr}{t}\left[\frac{e^{-\varkappa\psi}\cos\varkappa\psi}{4} \mp \frac{3\nu}{4n^2} e^{-\varkappa\psi}\sin\varkappa\psi\right]$$

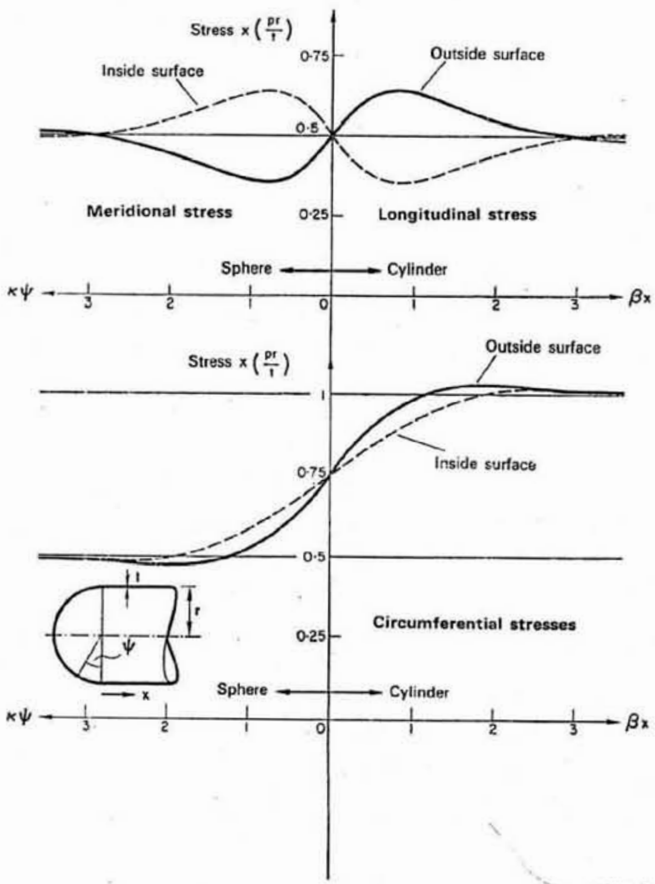


Fig. 2.16

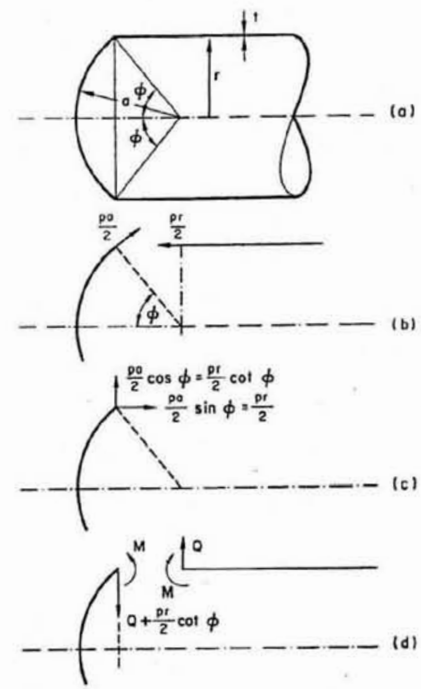


Fig. 2.17

To these stresses we add the membrane stresses and the final results are shown in Fig. 2.16.

One final point about the junction of two shells of different geometry is most easily explained by considering the junction of a cylinder and part of a sphere as shown in Fig. 2.17a. This would clearly be an extremely bad design for a pressure vessel, but it illustrates an important point which arises in the analysis of branches in spherical vessels.

If the vessel is under internal pressure then the membrane solutions give

$$\delta_1 = \frac{pr^2}{2tE}(2-\nu) \quad \text{for the cylinder}$$

For the sphere
$$N_\theta = N_\varphi = \frac{pa}{2} = \frac{pr}{2\sin\varphi}$$

$$\delta_2 = \frac{pr^2}{2Et\sin\varphi}(1-\nu)$$

When we consider the edge moments M and shear force Q to satisfy continuity of slope and deflection we cannot have equal and opposite Q as in Fig. 2.15 in the previous case.

The membrane stress and membrane deflections above require the edge forces as shown in Fig. 2.17b. The required force on the edge of the sphere has components as shown in Fig 2.17c and hence the edge forces and moments to produce compatibility are as shown in Fig. 2.17d.

2.5. Limit Analysis of Shells

The limit analysis of shells is a relatively new subject, but Hodge [2.18] has written a monograph which gives an excellent survey of the basic work up to the present time. Very little work has been carried out on non-rotationally symmetric problems (there are two papers in [2.19]) but this review of the principles is restricted to rotationally symmetric problems.

If we consider a structure made of material with an elastic-ideally plastic stress–strain curve as shown in Fig. 2.18a we can discuss what happens as we increase the load on the structure. Initially we have an elastic distribution of stresses, but as the load increases, yield will occur somewhere, but the small volume of yielded material is constrained by surrounding elastic material and the strains and deflections are small. As the load is further increased, yield develops until a region or a number of regions have yielded sufficiently so that the structure can flow under a constant load. This assumes no strain hardening and ignores any change in the shape of the structure under load.

The load at which flow occurs is called the limit load and the method of analysis is referred to as limit analysis. Because we do not consider any change of shape of the structure, the limit load calculated is strictly for a rigid-ideally plastic material with a stress–strain curve as shown in Fig. 2.18b.

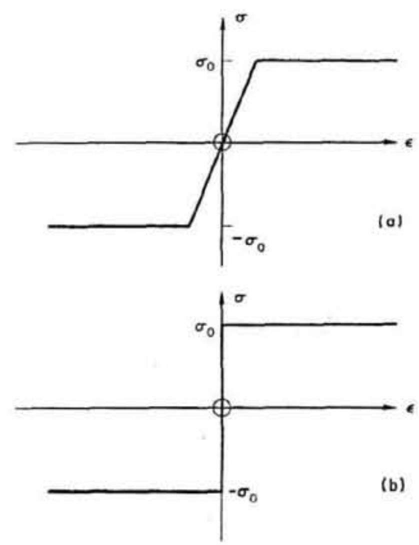


FIG. 2.18

The limit load of a structure has a value which is independent of any residual stresses in the structure at zero load. It is solely a function of the geometry of the structure and the yield stress of the material from which it is made. At the limit load, four conditions must be satisfied:

(1) A pattern of stresses will exist in the structure which are in equilibrium with the limit load and satisfy the stress boundary conditions.