Course 22.314 "Structural Mechanics in Nuclear Power Technology"

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APPENDIX: SOME INTRODUCTORY NOTES TO THE THEORY OF PLASTICITY

1. Distortion Energy Theory of Yield (von Mises Yield Criterion)

The distortion energy theory of yield, -known also as the von Mises criterion considers the strain energy in a stresses body to consist of two parts, one owing to distortional deformation and one owing to volume dilatation, and it postulates that only the distortional deformation has influence on plastic yielding, (which ordinarily is valid in the inelastic behavior or metals without voids).

The total strain energy U produced in an element of a stressed body consists of strain energy effecting distortion U_D and of strain energy producing a volume change U_V, thus

$$U = U_D + U_V. \tag{1}$$

In terms of principal stresses and strains the total strain energy can be expressed as follows:

$$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3. \tag{2}$$

Introducing the linear elastic stress-strain relations for isotropic materials

$$\begin{aligned}\varepsilon_1 &= \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]\\ \varepsilon_2 &= \frac{1}{E}\left[\sigma_2 - \nu(\sigma_1 + \sigma_3)\right]\\ \varepsilon_3 &= \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right]\end{aligned} \tag{3}$$

the following relation for the total strain energy in terms of principal stresses alone is obtained:

$$U = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{\nu}{E}\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right). \tag{4}$$

In accordance with Eq. (1), each of the principal stress components in the strain energy expression is considered to be composed of two parts, namely

$$(\sigma)_{1,2,3} = (s)_{1,2,3} + (\sigma_V)_{1,2,3} \tag{5}$$

where the stresses s_i are chosen so that the volume change produced by them is zero.

To achieve this, the stress tensor

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

is split up by subtraction of a hydrostatic pressure p_o from the diagonal terms of the tensor and compensating by adding it separately:

$$\sigma_{ij} = \begin{bmatrix} (\sigma_{11} - p_o) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - p_o) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - p_o) \end{bmatrix} + \begin{bmatrix} p_o & 0 & 0 \\ 0 & p_o & 0 \\ 0 & 0 & p_o \end{bmatrix} \tag{6}$$

thus

$$\sigma_{ij} = s_{ij} + \delta_{ij} p_o \tag{6a}$$

where $\delta_{ij} = \delta_{ji}$ is the Kronecker delta, which has diagonal terms of unity and off-diagonal zeros

$$\delta_{ij} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Associated with the hydrostatic pressure loading condition $\delta_{ij} p_0$ there is no distortion but only volume change.

As a special value for the hydrostatic pressure the mean principal stress is chosen:

$$p_o^{*} = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) = \frac{1}{3}\sigma_{kk} \tag{7}$$

which results in setting the mean normal stress in the modified stress tensor s_{ij} equal to zero:

$$\left(\sigma_1 - p_o^{*}\right) + \left(\sigma_2 - p_o^{*}\right) + \left(\sigma_3 - p_o^{*}\right) = 0. \tag{8}$$

The reference state of stress is termed the <u>stress deviator</u> or the stress deviation tensor:

$$s_{ij} = \sigma_{ij} - \delta_{ij} p_o^{*} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}. \tag{9}$$

The principal stress components of the stress deviator are:

$$s_1 = \sigma_1 - p_o^{*} = \frac{1}{3}\left(2\sigma_1 - \sigma_2 - \sigma_3\right) = \frac{1}{3}\left[\left(\sigma_1 - \sigma_2\right) - \left(\sigma_3 - \sigma_1\right)\right]$$

$$s_2 = \sigma_2 - p_o^{*} = \frac{1}{3}\left(2\sigma_2 - \sigma_1 - \sigma_3\right)$$

$$s_3 = \sigma_3 - p_o^{*} = \frac{1}{3}\left(2\sigma_3 - \sigma_1 - \sigma_2\right) \tag{10}$$

These stress deviator components satisfy the condition Eq. (8):

$$s_1 + s_2 + s_3 = 0. \tag{11}$$

Now, the volume dilatation part of the strain energy per unit volume can be expressed by utilizing the volumetric stress strain relation which follows from Eqs. (3):

$$\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1 - 2\nu}{E}\left(\sigma_1 + \sigma_2 + \sigma_3\right), \tag{12}$$

And it takes the following form:

$$U_V = \frac{1 - 2\nu}{6E}\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2. \tag{13}$$

The distortion energy

$$U_D = U - U_V$$

thus can be determined in terms of the principal stresses by utilizing Eqs. (4) and (13):

$$\mathrm{U_D} = \frac{1+\nu}{3E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right] \quad . \tag{14}$$

For simple tension $\sigma_2 = \sigma_3 = 0$, and

$$(\mathrm{U_D})_{\text{simple tension}} = \frac{1+\nu}{3E}\sigma_1^2 \tag{15}$$

According to the distortion energy theory, plastic yielding occurs when U_D becomes equal in magnitude to the distortion energy in the case of simple tension at yielding

$$\sigma_1 = \sigma_o = Y. \tag{16}$$

Equating Eqs. (14) and (15), utilizing Eq. (16) gives the following expression for the distortion energy at incipient yield:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 = \sigma_o^2 \tag{17}$$

or by simple arithmetic,

$$\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2 = 2\sigma_o^2 \tag{17a}$$

or

$$3(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - (\sigma_1 + \sigma_2 + \sigma_3)^2 = 2\sigma_o^2, \tag{17b}$$

that means that yielding under combined stresses occurs whenever

$$\frac{1}{\sqrt{2}}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right]^{\frac{1}{2}} = \sigma_o, \tag{18}$$

Where σ_o is the yield stress in simple tension and σ_1, σ_2, and σ_3 are the true principal stresses in combined loading.

The principal utility of the theory given by this equation is that σ_o, representing the yield point of a material under simple tensile loading, is equated to a system of combined stresses.

It is convenient to define an <u>equivalent</u> or <u>effective stress</u> in correspondence to Eq. (18) as

$$\sigma^* = \frac{1}{\sqrt{2}}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right]^{\frac{1}{2}}; \tag{19}$$

Thus, just as yielding begins, the effective stress reaches the value

$$\sigma^* = \sigma_o. \tag{20}$$

Because the hydrostatic component of the stresses makes no contribution to the incipient plastic flow, the von Mises yield criterion can also be expressed in terms of the stress deviators as defined by Eqs. (10). With these relations, Eqs. (17a,b) can be expressed as

$$\left(s_1 - s_2\right)^2 + \left(s_2 - s_3\right)^2 + \left(s_3 - s_1\right)^2 = 2\sigma_o^2 \tag{21a}$$

or

$$3(s_1^2 + s_2^2 + s_3^2) - (s_1 + s_2 + s_3)^2 = 2\sigma_o^2, \tag{21b}$$

and due to the condition Eq. (11) the von Mises yield criterion becomes

$$s_1^2 + s_2^2 + s_3^2 = \frac{2}{3}\sigma_o^2. \tag{22}$$

2. Invariants of the Stress Tensor and von Mises Yield Criterion

The principal total stresses σ_1, σ_2, σ_3 in an element of a body in three-dimensional state of stress can be found as the roots of the cubic equation

$$\sigma_p^3 - I_1\sigma_p^2 - I_2\sigma_p - I_3 = 0 . \tag{23}$$

This equation is obtained by expansion of the determinant of the coefficients of the stress tensor which is set equal to zero.

Since the principal stresses are physical quantities, they obviously do not depend on the coordinate axes chosen. Hence, no matter what coordinate axes have been chosen for relating the stress components, the numbers I_1, I_2 and I_3 appearing in Eq. (23) must remain the same, in order to give the same values for σ_1, σ_2, and σ_3. I_1, I_2 and I_3 are therefore called the first, second, and third invariants of the stress tensor.

If the principal directions are chosen as the directions of the coordinate axes, then the stress invariants take the following simple form:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \tag{24a}$$

$$I_2 = -\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \tag{24b}$$

$$I_3 = \sigma_1\sigma_2\sigma_3 . \tag{24c}$$

The invariants I_1, I_2, and I_3 are three independent scalar quantities which specify a state of stress just as well as σ_1, σ_2, and σ_3. That means, given σ_1, σ_2, and σ_3, one can calculate I_1, I_2, and I_3, and given I_1, I_2, and I_3, one can calculate σ_1, σ_2, and σ_3. One set of these quantities uniquely determines the other set. Furthermore, any three independent combinations of these invariants will obviously also be invariants and can specify the state of stress just as the principal stresses σ_1, σ_2, and σ_3 do.

Comparison with the equations derived in the foregoing section shows that the invariant I_1 is related to the hydrostatic state of stress and that the invariant I_2 is particularly important in the distortion energy theory of yielding since it is related to the deviatoric state of stress.

To obtain the invariants of the stress deviator tensor, replace σ_p by $\sigma_p' + \frac{1}{3}I_1$ in Eq. (23). The results in

$$\sigma_p'^{\,3} - J_1\sigma_p'^{\,2} - J_2\sigma_p' - J_3 = 0 \tag{25}$$

where

$$J_1 = 0 \tag{26a}$$

$$J_2 = \frac{1}{3}\left(I_1^2 + 3I_2\right) \tag{26b}$$

$$J_3 = \frac{1}{27}\left(2I_1^3 + 9I_1I_2 + 27I_3\right). \tag{26c}$$

One advantage of using the stress deviator tensor is apparent: the first invariant of this tensor is always zero.

The invariants J_2 and J_3 can, of course, be written in terms of the stress components σ_{ij} or in terms of the principal stresses σ_1, σ_2, and σ_3; in the latter case:

$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] . \qquad (27)$$

In terms of the stress deviators defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

the invariants take the following form:

$$J_1 = s_{ii} \equiv 0 \qquad (28a)$$

$$J_2 = \frac{1}{2}s_{ij}^2 \qquad (28b)$$

$$J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki} . \qquad (28c)$$

Because J_2 is frequently used in structural mechanics practice, several alternative forms of Eq. (28b) are worth recording:

$$J_2 = \frac{1}{2}\left(s_{11}^2 + s_{22}^2 + s_{33}^2\right) + s_{12}^2 + s_{23}^2 + s_{31}^2 \qquad (28b)'$$

$$= \frac{1}{2}\left(s_1^2 + s_2^2 + s_3^2\right) \qquad (28b)''$$

$$= -\left(s_1 s_2 + s_2 s_3 + s_3 s_1\right) . \qquad (28b)'''$$

For example, in the uniaxial state of tension, the stress tensor is

$$\sigma_{ij} \equiv \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the stress deviator takes the form

$$s_{ij} \equiv \begin{bmatrix} \frac{2}{3}\sigma_1 & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_1 & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_1 \end{bmatrix}$$

Thus the second invariant of the stress deviator is

$$J_2 = \frac{1}{2}s_{ij}^2 = \frac{1}{2}\left(\frac{4}{9}\sigma_1^2 + \frac{1}{9}\sigma_1^2 + \frac{1}{9}\sigma_1^2\right) = \frac{1}{3}\sigma_1^2 .$$

Relating the consideration of J_2 to the distortion energy theory of yield (von Mises yield criterion), it can be stated that yielding will occur at J_2 = const., $dJ_2 = 0$.

The equivalent or effective stress, Eq. (19), may be expressed in terms of the second invariant of the stress deviator, Eq. (27), as follows:

$$\bar{\sigma} = \sqrt{3J_2} . \qquad (29)$$

3. Inelastic Deformation

Post yielding behavior will now be examined.

The investigation of deformation in the plastic range under combined loading conditions requires the formulation of plastic flow rules.

The plastic flow rules as formulated by Levy and von Mises relate the increments of total strain (total strain rates) to the stress deviation:

$$d\varepsilon_{ij} = s_{ij} d\lambda \ , \tag{30}$$

where s_{ij} is the stress deviator tensor and $d\lambda$ is a non-negative constant, which may vary throughout the loading history.

While elastic deformation is determined by the state of stress alone without regard to the history of loading, inelastic deformation depends on strain history. This explains why in an analysis of inelastic stresses it is required to relate the stresses to strain increments with respect to some independent variable which, for convenience, is usually chosen as time, thus

$$\dot{\varepsilon}_{ij} = s_{ij} d\lambda \ , \tag{30a}$$

where the superposed dot indicates the differential with respect to time.

The Levy-von Mises equations, Eq. (30), assume the total strain increments to be equal to the plastic strain increments, the elastic strains being ignored. Thus these equations can be only applied to problems of large plastics flow, i.e. to problems in which the plastic part of the strains is large as compared with the elastic part.

For stress-strain analysis in the elasto-plastic range relations between increments of plastic strain and the instantaneous stress deviation are needed, which read

$$d\varepsilon_{ij}^{p} = s_{ij} d\lambda \ . \tag{31}$$

Equations (31), which are known as the Prandtl-Reuss equations, state that the increments of plastic strain depend on the current values of the deviatoric stress rate, (not on the stress increment to reach this state). They also imply that the principal axes of stress tensors and of plastic strain increment tensors coincide.

These equations, however, merely give a relationship between the ratios of plastic strain increments and stresses in the different directions, e.g.

$$\frac{\dot{\varepsilon}_1 - \dot{\varepsilon}_2}{\sigma_1 - \sigma_2} = \frac{\dot{\varepsilon}_2 - \dot{\varepsilon}_3}{\sigma_2 - \sigma_3} = \frac{\dot{\varepsilon}_3 - \dot{\varepsilon}_1}{\sigma_3 - \sigma_1} = d\lambda \tag{32}$$

stating the proportionality of the rates of the principal shear strains to the principal shear stresses.

Considering principal directions of increments of plastic strains and principal directions of the instantaneous stress deviation stress deviation tensor, Eq. (31) can be written in the following ways:

$$\frac{d\varepsilon_1^p}{s_1} = \frac{d\varepsilon_2^p}{s_2} = \frac{d\varepsilon_3^p}{s_3} = d\lambda \tag{33a}$$

or $\quad d\varepsilon_1^p = s_1 d\lambda \ , \ d\varepsilon_2^p = s_2 d\lambda \ , \ d\varepsilon_3^p = s_3 d\lambda \ ,$

or
$$\frac{d\varepsilon_1^p - d\varepsilon_2^p}{s_1 - s_2} = \frac{d\varepsilon_2^p - d\varepsilon_3^p}{s_2 - s_3} = \frac{d\varepsilon_3^p - d\varepsilon_1^p}{s_3 - s_1} = d\lambda \ . \tag{33b}$$

To determine the actual magnitudes of the increments of plastic strain, in addition to the above equations reference to a yield criterion is needed. The von Mises yield criterion is to be utilized which implies the assumption of the incompressibility of material under plastic flow conditions, i.e.

$$\varepsilon_1^p + \varepsilon_2^p + \varepsilon_3^p = 0 \ , \tag{34}$$

where $d\varepsilon_1^p$, $d\varepsilon_2^p$, $d\varepsilon_3^p$ are the principal components of plastic strain. Referring to the stress-strain relations given by Eqs.(3) this implies that in plastic flow Poisson's ratio is $\nu = \frac{1}{2}$, thus,

$$
\begin{aligned}
d\varepsilon_1^p &= \left[\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right] d\lambda, \\
d\varepsilon_2^p &= \left[\sigma_2 - \frac{1}{2}(\sigma_3 + \sigma_1)\right] d\lambda, \\
d\varepsilon_3^p &= \left[\sigma_3 - \frac{1}{2}(\sigma_1 + \sigma_2)\right] d\lambda.
\end{aligned} \tag{35}
$$

Similarly to the definition of equivalent or effective stress, Eq. (19), an equivalent or effective plastic strain increment can be defined which in terms of principal plastic strain increments is:

$$d\bar{\varepsilon}^p = \frac{\sqrt{2}}{3}\left[\left(d\varepsilon_1^p - d\varepsilon_2^p\right)^2 + \left(d\varepsilon_2^p - d\varepsilon_3^p\right)^2 + \left(d\varepsilon_3^p - d\varepsilon_1^p\right)^2\right]^{\frac{1}{2}} \tag{36}$$

or in general tensorial notation

$$d\bar{\varepsilon}^p = \left[\frac{2}{3} d\varepsilon_{ij}^{p\,2}\right]^{\frac{1}{2}}. \tag{37}$$

For the special case of uniaxial tensile loading in the principal direction 1, the tensor of plastic strain increments has the form

$$d\varepsilon_{ij}^p \equiv \begin{bmatrix} d\varepsilon_1 & 0 & 0 \\ 0 & -\frac{1}{2}d\varepsilon_1 & 0 \\ 0 & 0 & -\frac{1}{2}d\varepsilon_1 \end{bmatrix}$$

So that in this case Eq. (37) reduces to:

$$d\bar{\varepsilon}^p = d\varepsilon_1^p. \tag{38}$$

The effective plastic strain increment $d\bar{\varepsilon}^p$ is related to the effective stress by

$$d\lambda = \frac{3}{2}\frac{d\bar{\varepsilon}^p}{\bar{\sigma}} \tag{39}$$

It follows from Eqs. (31) and (39), under consideration of Eq. (16), that for a perfectly plastic material the Prandtl-Reuss equations take the form

$$d\varepsilon_{ij}^p = \frac{3}{2}\frac{d\bar{\varepsilon}^p}{\sigma_o} s_{ij} \tag{40}$$

4. References

D.C. Drucker: Introduction to Mechanics of Deformable Solids, McGraw-Hill Book Company, New York, 1967.

A. Mendelson: Plasticity: Theory and Application. The Macmillan Company, New York, 1968.