Course 22.314 “Structural Mechanics in Nuclear Power Technology”

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THERMOELASTIC STRESS-STRAIN REATIONS FOR AXISYMMETRIC PLANE STRAIN CONDITIONS AND ISOTROPIC MATERIAL

1. General Relations

For linear elastic isotropic material with temperature-independent materials properties the thermoelastic stress-strain relations for axisymmetric plane strain (polarsymmetric) conditions in the cylindrical coordinate system correspond to the set of Eqs. [M-11A/(24)] with the shear strain components vanishing, thus:

$$\varepsilon_r = \frac{\partial u_r}{\partial r} = \frac{1}{E}\left[\sigma_r - \nu\left(\sigma_\theta + \sigma_z\right)\right] + \alpha T(r), \qquad (1a)$$

$$\varepsilon_\theta = \frac{u_r}{r} = \frac{1}{E}\left[\sigma_\theta - \nu\left(\sigma_r + \sigma_z\right)\right] + \alpha T(r), \qquad (1b)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_r + \sigma_\theta\right)\right] + \alpha T(r) = \text{const.} \qquad (1c)$$

where u_r and u(r) and $T(r) = T_1(r) - T_o$. The equilibrium conditions are reduced to:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0.$$

Solving Eqs. (1) for the stresses gives the following stress-displacement relations:

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{du_r}{dr} + \nu\left(\frac{u_r}{r} + \varepsilon_z\right) - (1+\nu)\alpha T(r)\right],$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{u_r}{r} + \nu\left(\frac{du_r}{dr} + \varepsilon_z\right) - (1+\nu)\alpha T(r)\right], \qquad (3)$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu\left(\frac{u_r}{r} + \frac{du_r}{dr}\right) - (1+\nu)\alpha T(r)\right].$$

Substitution of these expressions into the equilibrium equations Eq. (2) renders an ordinary differential equation for the radial displacements $u_r = u(r)$;

$$\frac{d^2u_r}{dr} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(u_r r\right)\right] = \frac{1+\nu}{1-\nu}\alpha\frac{dT(r)}{dr} \qquad (4)$$

which has the general solution

$$u(r) = \frac{1+\nu}{1-\nu}\alpha\frac{1}{r}\int rT(r)dr + C_1 r + \frac{C_2}{r}, \qquad (5)$$

where C_1 and C_2 are constants of integration.
Utilizing Eq. (5) in Eqs. (1) and (3) yields the following expressions for the strain components and the stress components:

$$\varepsilon_r = \frac{1+\nu}{1-\nu}\alpha\left[T - \frac{1}{r^2}\int_{r=r_i}^{r} rT(r)dr\right] + C_1 - \frac{C_2}{r^2}, \tag{6a}$$

$$\varepsilon_\theta = \frac{1+\nu}{1-\nu}\alpha\frac{1}{r^2}\int_{r=r_i}^{r} rT(r)dr + C_1 + \frac{C_2}{r^2}; \tag{6b}$$

$$\sigma_r(r) = -\frac{\alpha E}{1-\nu}\frac{1}{r^2}\int_{r=r_i}^{r} rT(r)dr + \frac{E}{1+\nu}\left[\frac{C_1}{1-2\nu} - \frac{C_2}{r^2} + \frac{\nu\varepsilon_z}{1-2\nu}\right], \tag{7a}$$

$$\sigma_\theta(r) = \frac{\alpha E}{1-\nu}\frac{1}{r^2}\int_{r=r_i}^{r} rT(r)dr - \frac{\alpha E}{1-\nu}T(r) + \frac{E}{1+\nu}\left[\frac{C_1}{1-2\nu} + \frac{C_2}{r^2} + \frac{\nu\varepsilon_z}{1-2\nu}\right], \tag{7b}$$

$$\sigma_z(r) = -\frac{\alpha E}{1-\nu}T(r) + \frac{E}{(1+\nu)(1-2\nu)}\left[2\nu C_1 + (1-\nu)\varepsilon_z\right]. \tag{7c}$$

The constants C_1 and C_2 have to be determined from the boundary conditions for the radial stresses, respectively the radial displacements at the curved cylinder surfaces. The value of the constant ε_z [see Eq. (1c)] follows from the boundary conditions at the cylinder ends.

2. <u>Axial Boundary Conditions</u>

a) In the case of rigid support of the cylinder ends in z-direction: w = 0 and $\varepsilon_z = 0$, so that Eq. (7c) reduces to:

$$\sigma_z = -\frac{\alpha E}{1-\nu}T(r) + \frac{2\nu E}{(1+\nu)(1-2\nu)}C_1. \tag{8}$$

b) If in the case of free cylinder ends there are no axial external forces acting on the cylinder, the resultant of the axial stresses must vanish at the cylinder ends, i.e. the axial stresses σ_z must form an equilibrium system:

$$\int_{(A)}\sigma_z dA = 2\pi\int_{r=r_i}^{r}\sigma_z r dr = 0 \tag{9}$$

utilizing Eq. (7c), this condition becomes:

$$-\frac{2\pi\alpha E}{1-\nu}\int_{r=r_i}^{r_o} rT(r)dr + \frac{\pi E\left(r_o^2 - r_i^2\right)}{(1+\nu)(1-2\nu)}\left[2\nu C_1 + (1-\nu)\varepsilon_z\right] = 0. \tag{10}$$

c) If the cylinder body is subjected to tensile forces acting along the cylinder axis, the following relation holds:

$$\int_{(A)}\sigma_z dA = P. \tag{11}$$

In all three cases a), b), and c) the radial stresses σ_r and the tangential stresses σ_θ in the central region of a long cylinder remain unaffected by the boundary conditions at the end faces of the cylinder.

3. Hollow Cylinder Subjected to Polarsymmetric Temperature Field and Uniformly Distributed Pressure Loading of the Cylindrical Surfaces

The radial stress boundary conditions for a hollow cylinder, $r_i \le r \le r_a$, subjected to uniformly distributed loading of the internal and external curved surfaces for pressure loading are:

$$(\sigma_r)_{r=r_i} = -p_i, \qquad (\sigma_r)_{r=r_o} = -p_o; \tag{12}$$

for tensile loading positive signs are valid. Substitution of these boundary conditions into Eqs. (7a) renders the required relations for the constants of integration. The boundary condition $(\sigma_r)_{r=r_i} = -p_i$ yields:

$$C_2 = \frac{p_i(1+\nu)r_i^2}{E} + \frac{C_1 r_i^2}{1-2\nu} + \frac{\nu\varepsilon_z r_i^2}{1-2\nu}, \tag{13}$$

from the boundary condition $(\sigma_r)_{r=r_o} = -p_o$ there follows:

$$\frac{EC_1}{(1+\nu)(1-2\nu)} = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} + \frac{\alpha E}{(1-\nu)\left(r_o^2 - r_i^2\right)}\int_{r_i}^{r_o} rT(r)dr - \frac{\nu E\varepsilon_z}{(1+\nu)(1-2\nu)}, \tag{14}$$

and thus

$$\frac{EC_2}{1+\nu} = \left(p_i - p_o\right)\frac{r_i^2 r_o^2}{\left(r_o^2 - r_i^2\right)} + \frac{\alpha E r_i^2}{(1-\nu)\left(r_o^2 - r_i^2\right)}\int_{r_i}^{r_o} rT(r)dr. \tag{15}$$

Substituting the relations obtained for the constants C_1 and C_2, Eqs. (14) and (15), into the expression for the radial displacement, Eq. (5), gives:

$$\begin{aligned}\frac{Eu_r}{1+\nu} &= \frac{\alpha E}{(1-\nu)r}\int_{r_i}^{r} rT(r)dr + \frac{\alpha E}{(1-\nu)\left(r_o^2 - r_i^2\right)}\left[(1-2\nu)r + \frac{r_i^2}{r}\right]\int_{r_i}^{r_o} rT(r)dr \\ &+ \frac{p_i r_i^2}{\left(r_o^2 - r_i^2\right)}\left[(1-2\nu)r + \frac{r_0^2}{r}\right] - \frac{p_o r_o^2}{\left(r_o^2 - r_i^2\right)}\left[(1-2\nu)r + \frac{r_i^2}{r}\right] - \frac{\nu E\varepsilon_z r}{1+\nu}.\end{aligned} \tag{16}$$

This solution is being introduced into Eqs. (1) in order to obtain relations for the stress components:

$$\begin{aligned}\sigma_r &= -\frac{\alpha E}{(1-\nu)r^2}\int_{r_i}^{r} rT(r)dr + \frac{\alpha E}{(1-\nu)\left(r_o^2 - r_i^2\right)}\left[1 - \frac{r_i^2}{r^2}\right]\int_{r_i}^{r_o} rT(r)dr + \frac{p_i r_i^2}{\left(r_o^2 - r_i^2\right)}\left[1 - \frac{r_0^2}{r^2}\right] \\ &- \frac{p_o r_o^2}{\left(r_o^2 - r_i^2\right)}\left[1 - \frac{r_i^2}{r^2}\right]\end{aligned} \tag{17}$$

$$\begin{aligned}\sigma_\theta &= \frac{\alpha E}{(1-\nu)r^2}\int_{r_i}^{r} rT(r)dr - \frac{\alpha E T(r)}{(1-\nu)} + \frac{\alpha E}{(1-\nu)\left(r_o^2 - r_i^2\right)}\left[1 + \frac{r_i^2}{r^2}\right]\int_{r_i}^{r_o} rT(r)dr \\ &+ \frac{p_i r_i^2}{\left(r_o^2 - r_i^2\right)}\left[1 + \frac{r_0^2}{r^2}\right] - \frac{p_o r_o^2}{\left(r_o^2 - r_i^2\right)}\left[1 + \frac{r_i^2}{r^2}\right]\end{aligned} \tag{18}$$

$$\sigma_z = E\varepsilon_z - \frac{\alpha E T(r)}{(1-\nu)} + \frac{2\nu\alpha E}{(1-\nu)\left(r_o^2 - r_i^2\right)}\int_{r_i}^{r_o} rT(r)dr + 2\nu\frac{\left(p_i r_i^2 - p_o r_o^2\right)}{\left(r_o^2 - r_i^2\right)}. \tag{19}$$

4. <u>Special Case: Hollow Cylinder Subjected to Pressure Loading in Isothermal Condition</u>

In the special case

$$(\sigma_r)_{r=r_i} = -p_i, \qquad (\sigma_r)_{r=r_o} = -p_o, \qquad T(r) = 0, \tag{20}$$

the following expressions for the stress components are obtained from Eqs. (17), (18), and (19), (e.g. TIMOSHENKO/GOODIER, p. 50; KANTOROWITSCH, p. 25)

$$\sigma_r = \frac{1}{r_o^2 - r_i^2}\left[(p_o - p_i)\frac{r_i^2 r_o^2}{r^2} + p_i r_i^2 - p_o r_o^2\right], \tag{21}$$

$$\sigma_r = \frac{1}{r_o^2 - r_i^2}\left[-(p_o - p_i)\frac{r_i^2 r_o^2}{r^2} + p_i r_i^2 - p_o r_o^2\right]. \tag{22}$$

Specially for $p_i \neq 0$, $p_o = 0$:

$$\left.\begin{aligned}\sigma_r &= p_i \frac{r_i^2}{r_o^2 - r_i^2}\left[1 - \left(\frac{r_o}{r}\right)^2\right] = \frac{p_i}{\kappa_o^2 - 1}\left[1 + \frac{r_o^2}{r^2}\right]\\ \sigma_\theta &= p_i \frac{r_i^2}{r_o^2 - r_i^2}\left[1 - \left(\frac{r_o}{r}\right)^2\right] = \frac{p_i}{\kappa_o^2 - 1}\left[1 + \frac{r_o^2}{r^2}\right]\end{aligned}\right\} \tag{23}$$

where $\kappa_o = \frac{r_o}{r_i}$; and for $p_i = 0$, $p_o \neq 0$:

$$\left.\begin{aligned}\sigma_r &= -p_o \frac{r_o^2}{r_o^2 - r_i^2}\left[1 - \left(\frac{r_i}{r}\right)^2\right] = -\frac{p_o \kappa_o^2}{\kappa_o^2 - 1}\left[1 + \frac{r_i^2}{r^2}\right]\\ \sigma_\theta &= p_o \frac{r_o^2}{r_o^2 - r_i^2}\left[1 + \left(\frac{r_i}{r}\right)^2\right] = -\frac{p_o \kappa_o^2}{\kappa_o^2 - 1}\left[1 + \frac{r_i^2}{r^2}\right]\end{aligned}\right\} \tag{24}$$

For the axial stresses σ_z, according to the end conditions the following relations hold:

a) for the hollow cylinder with open ends:

$$\sigma_z = 0 \; ; \tag{25a}$$

b) for the hollow cylinder with closed ends:

$$\sigma_z = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}; \tag{25b}$$

c) for the hollow cylinder with open ends and axially rigidly restrained end faces:

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) = 2\nu \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}. \tag{25c}$$

In these three cases the following expressions for the radial displacement are obtained:

a) for the hollow cylinder with open ends:

$$u_r = r\varepsilon_\theta = \frac{r}{E}(\sigma_\theta - \nu\sigma_r) = \frac{1-\nu}{E}\frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} + \frac{1+\nu}{E}\frac{(p_i - p_o) r_i^2 r_o^2}{(r_o^2 - r_i^2) r}; \tag{26a}$$

Specially for $p_i \neq 0$, $p_o = 0$:

$$u_r = \frac{(1-\nu)}{E}\frac{p_i r_i^2}{r_o^2 - r_i^2} + \frac{(1+\nu)}{E}\frac{p_i r_i^2 r_o^2}{(r_o^2 - r_i^2) r}, \tag{26a'}$$

$$(u_r)_{r=r_i} = \frac{p_i r_i^2}{E(r_o^2 - r_i^2)}\left[(1-\nu)r_i^2 + (1+\nu)r_o^2\right], \qquad (u_r)_{r=r_o} = \frac{2 p_i r_i^2 r_o}{E\left(r_o^2 - r_i^2\right)}.$$

Since

$$\frac{(u_r)_{r=r_i}}{(u_r)_{r=r_o}} = (1-\nu)\frac{r_i}{2r_o} + (1+\nu)\frac{r_o}{2r_i} > 1,$$

there is $(u_r)_{r=r_i} > (u_r)_{r=r_o}$, which means that the wall thickness is decreasing.

Specially for $p_i = 0$, $p_o \neq 0$:

$$u_r = -\frac{1-\nu}{E}\frac{p_o r_o^2}{r_o^2 - r_i^2} r - \frac{1+\nu}{E}\frac{p_o r_i^2 r_o^2}{\left(r_o^2 - r_i^2\right)r},$$

$$(u_r)_{r=r_i} = -\frac{2 p_o r_i r_o^2}{E(r_o^2 - r_i^2)}, \qquad (u_r)_{r=r_o} = -\frac{p_o r_o}{E(r_o^2 - r_i^2)}\left[(1-\nu)r_o^2 + (1+\nu)r_i^2\right]. \tag{26a'}$$

b) for the hollow cylinder with closed ends:

$$u_r = \frac{r}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right] = \frac{1-2\nu}{E}\frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} r + \frac{1+\nu}{E}\frac{(p_i - p_o) r_i^2 r_o^2}{\left(r_o^2 - r_i^2\right)r}; \tag{26b}$$

c) for the hollow cylinder with open ends and axially restrained end faces:

$$u_r = \frac{r}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right] = \frac{1-\nu-2\nu^2}{E}\frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} r + \frac{1+\nu}{E}\frac{(p_i - p_o) r_i^2 r_o^2}{\left(r_o^2 - r_i^2\right)r}. \tag{26c}$$

5. Special Case: Hollow Cylinder Subjected to Polarsymmetric Temperature Field T(r)

5.1 General Relations

In the special case

$$(\sigma_r)_{r=r_i, r_o} = 0, \qquad T(r) = f(r), \tag{27}$$

Eqs. (17) and (18) reduce to (see e.g. BOLEY/WEINER, p.290):

$$\sigma_r = \frac{\alpha E}{(1-\nu)r^2}\left[\frac{r^2 - r_i^2}{r_o^2 - r_i^2}\int_{r_i}^{r_o} r T(r)dr - \int_{r_i}^{r} r T(r)dr\right], \tag{28}$$

$$\sigma_\theta = \frac{\alpha E}{(1-\nu)r^2}\left[\frac{r^2 + r_i^2}{r_o^2 - r_i^2}\int_{r_i}^{r_o} r T(r)dr + \int_{r_i}^{r} r T(r)dr - T(r)r^2\right]. \tag{29}$$

The axial stresses obtained from Eq. (21) for the case of axially rigidly constrained cylinder ends, i.e. w = 0, $\varepsilon_z = 0$, are:

$$\sigma_z = \frac{2\nu\alpha E}{(1-\nu)\left(r_o^2 - r_i^2\right)}\int_{r_i}^{r_o} r T(r)dr - \frac{\alpha E}{1-\nu}T(r) \tag{30a}$$

and for the case of free cylinder end faces:

$$\sigma_z = \sigma_r + \sigma_\theta = \frac{\alpha E}{1-\nu}\left[\frac{2}{r_o^2 - r_i^2}\int_{r_i}^{r_o} r T(r)dr - T(r)\right]. \tag{30b}$$

The expression for the radial displacement u_r follows from the general relation Eq. (16):

$$u_r = \frac{\alpha}{r(1-\nu)}\left[(1+\nu)\int_{r_i}^{r} rT(r)\,dr + \frac{r^2(1-3\nu)+r_i^2(1+\nu)}{r_o^2-r_i^2}\int_{r_i}^{r_o} rT(r)\,dr\right]. \tag{31}$$

Herewith the values for the radial displacement at the curved surfaces are:

$$\left(u_r\right)_{r=r_i} = \frac{2\alpha r_i}{r_o^2-r_i^2}\int_{r_i}^{r_o} rT(r)\,dr, \qquad \left(u_r\right)_{r=r_o} = \frac{2\alpha r_o}{r_o^2-r_i^2}\int_{r_i}^{r_o} rT(r)\,dr. \tag{32}$$

In the case of axially rigidly constrained cylinder end faces, i.e. w = 0, $\varepsilon_z = 0$, the radial displacement is:

$$u_r = \frac{1+\nu}{1-\nu}\frac{\alpha}{r}\left[\int_{r_i}^{r} rT(r)\,dr + \frac{r^2(1-2\nu)+r_i^2}{r_o^2-r_i^2}\int_{r_i}^{r_o} rT(r)\,dr\right]. \tag{33}$$

Introducing an average temperature value

$$\hat{T} = \frac{\int_0^{2\pi}\int_{r_i}^{r_o} rT(r)\,dr\,d\theta}{\int_0^{2\pi}\int_{r_i}^{r_o} r\,dr\,d\theta} = \frac{2}{r_o^2-r_i^2}\int_{r_i}^{r_o} rT(r)\,dr\,; \tag{34}$$

and a temperature averaged over the interval from r_i to r, designated #### yields simplified forms of the thermal stress relations Eqs. (28), (29) and (30):

$$\sigma_r = \frac{\alpha E}{2(1-\nu)}\left(1-\frac{r_i^2}{r^2}\right)\left(\hat{T}-\hat{T}_r\right), \tag{35}$$

$$\sigma_\theta = \frac{\alpha E}{2(1-\nu)}\left[\left(\hat{T}+\hat{T}_r\right)+\frac{r_i^2}{r^2}\left(\hat{T}-\hat{T}_r\right)-2T_r\right], \tag{36}$$

$$\sigma_z = \frac{\alpha E}{(1-\nu)}\left(\hat{T}-\hat{T}_r\right). \tag{37}$$

5.2 <u>Simplified Equations for the Thin-Walled Hollow Cylinder</u>

Introducing

$$t = r_o - r_i, \tag{38}$$

designating the wall thickness of the hollow cylinder, the following relations hold

$$r_o = r_i + t, \qquad r = r_i + x \quad (0 \le x \le t), \qquad \mathrm{dr} = \mathrm{dx}. \tag{39}$$

Considering a thin-walled hollow cylinder, $t \ll r_i$, and neglecting terms of the order t^2 compared with terms of order (r_i t), gives the following approximate expressions:

$$\left.\begin{aligned} r^2 - r_i^2 &= r_i^2 + 2r_i x + x^2 - r_i^2 = 2r_i x + x^2 \approx 2r_i x,\\ r^2 + r_i^2 &= r_i^2 + 2r_i x + x^2 + r_i^2 = 2r_i^2 + 2r_i x + x^2 \approx 2r_i^2,\\ r_o^2 - r_i^2 &= r_i^2 + 2r_i t + t^2 - r_i^2 = 2r_i t + t^2 \approx 2r_i t. \end{aligned}\right\} \tag{40}$$

Further, replacing all radii by a mean radius R, i.e. let

$$r_i \approx r_i + x \approx r_o \approx R, \tag{41}$$

reduces the exact relations, Eqs. (28), (29) and (30b), to the approximate expressions:

$$\sigma_r \approx \frac{\alpha E}{1-\nu}\left[\frac{x}{tR}\int_0^t T\,dx - \frac{1}{R}\int_0^x T\,dx\right], \tag{42a}$$

$$\sigma_\theta = \frac{\alpha E}{1-\nu}\left[\frac{1}{t}\int_0^t T\,dx + \frac{1}{R}\int_0^x T\,dx - T\right], \tag{42b}$$

$$\sigma_z = \frac{\alpha E}{1-\nu}\left[\frac{1}{t}\int_0^t T\,dx - T\right]. \tag{42c}$$

Neglecting in these expressions terms with the factor $\frac{t}{R}$ which are small as compared with T, renders the approximate thermoelastic equations for the stress components in a thin-walled cylindrical shell:

$$\sigma_r = 0, \qquad \sigma_\theta = \sigma_z = \frac{\alpha E}{1-\nu}\left[\frac{1}{t}\int_0^t T\,dx - T\right]. \tag{43}$$

6. Hollow Cylinder Consisting of Coaxial Layers, Subjected to a Polarsymmetric Temperature Field, as an Approximate Model for the Radially Inhomogeneous Hollow Cylinder

A hollow cylinder, $r_i \le r \le r_o$, having radially varying thermoelastic properties E(r), α(r), ν(r), in practical engineering analysis of the thermoelastic stress-strain field can be approximated by considering the hollow cylinder to consist of n coaxial layers each having constant material properties.

Thereby, the differential equation for the radial displacement in the axisymmetric plane (polarsymmetric) case, incorporating material properties as functions of the radial coordinate,

$$\frac{d^2u}{dr} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \frac{1+\nu(r)}{1-\nu(r)}\frac{d}{dr}\left[\alpha(r)T(r)\right] + \frac{1}{E(r)}\left[\frac{du}{dr} + \frac{1+\nu(r)}{1-\nu(r)}\left(\frac{u}{r} + \varepsilon_z\right)\right]\frac{dE(r)}{dr}, \tag{44}$$

reduces to n coupled differential equations for the individual layers in each of which the materials properties have constant values. For the k-t layer the differential equation for the radial displacement reads:

$$\frac{d^2u_k}{dr} + \frac{1}{r}\frac{du_k}{dr} - \frac{u_k}{r^2} = \frac{1+\nu_k}{1-\nu_k}\frac{d}{dr}\left[\alpha_k T(r)\right], \qquad \text{k=1, 2, ….n.} \tag{45}$$

Corresponding to Eqs. (5), (6), and (7) one obtains:

$$u_k(r) = \frac{1+\nu_k}{1-\nu_k}\frac{\alpha_k}{r^2}\int_{r_i}^{r} rT(r)dr + C_{1k}r + \frac{C_{2k}}{r}, \tag{46}$$

$$\varepsilon_{r_k}(r) = \frac{1+\nu_k}{1-\nu_k}\left[\alpha_k T(r) - \frac{\alpha_k}{r^2}\int_{r_i}^{r} rT(r)dr\right] + C_{1k} - \frac{C_{2k}}{r^2}, \tag{47a}$$

$$\varepsilon_{\theta_k}(r) = \frac{1+\nu_k}{1-\nu_k}\frac{\alpha_k}{r^2}\int_{r_i}^{r} rT(r)dr + C_{1k} + \frac{C_{2k}}{r^2}; \tag{47b}$$

$$\sigma_{r_k}(r) = \left[-\frac{\alpha_k}{(1-\nu_k)r^2}\int_{r_i}^{r} rT(r)dr + \frac{1}{1+\nu_k}\left(\frac{C_{1k} + \nu_k\varepsilon_z}{1-2\nu_k} - \frac{C_{2k}}{r^2}\right)\right]E_k, \tag{48a}$$

$$\sigma_{\theta k}(r) = \left[\frac{\alpha_k}{(1-\nu_k)r^2} \int_{r_i}^{r} r T(r)\,dr - \frac{\alpha_k T(r)}{1-\nu_k} + \frac{1}{1+\nu_k}\left(\frac{C_{1k} + \nu_k \varepsilon_z}{1-2\nu_k} + \frac{C_{2k}}{r^2} \right) \right] E_k , \tag{48b}$$

$$\sigma_{zk}(r) = \left[-\frac{\alpha_k T(r)}{1-\nu_k} + \frac{2\nu_k C_{1k} + (1-\nu_k)\varepsilon_z}{(1+\nu_k)(1-2\nu_k)} \right] E_k . \tag{48c}$$

The 2n constants of integration in the above equations are determined means of the boundary conditions for the total hollow cylinder

$$p_i = -\left[\frac{1}{1+\nu_1}\left(\frac{C_{11} - \nu_1 \varepsilon_z}{1-2\nu_1} - \frac{C_{21}}{r_{1i}^2} \right) \right] E_1 , \tag{49a}$$

$$p_o = -\left[-\frac{\alpha_k}{(1-\nu_n) r_{no}^2} \int_{r_{ni}}^{r_{no}} r T(r)\,dr + \frac{1}{1+\nu_n}\left(\frac{C_{1n} + \nu_n \varepsilon_z}{1-2\nu_n} - \frac{C_{2n}}{1-2\nu_n} \right) \right] E_n , \tag{49b}$$

and by means of the continuity conditions for the radial stresses at the interfaces

$$\left[-\frac{\alpha_k}{(1-\nu_k) r_{ko}^2} \int_{r_{ki}}^{r_{ko}} r T(r)\,dr + \frac{1}{1+\nu_k}\left(\frac{C_{1k} + \nu_k \varepsilon_z}{1-2\nu_k} - \frac{C_{2k}}{r_{ko}^2} \right) \right] E_k$$

$$- \left[\frac{1}{1+\nu_{k+1}}\left(\frac{C_{1(k+1)} - \nu_{k+1} \varepsilon_z}{1-2\nu_{k+1}} - \frac{C_{2(k+1)}}{r_{(k+1)i}^2} \right) \right] E_{k+1} = 0 \tag{50a}$$

as well as by the corresponding continuity conditions for the deformation

$$C_{1k} r_{ko} + \frac{C_{2k}}{r_{ko}} + \frac{(1+\nu_k)\alpha_k}{(1-\nu_k) r_{ko}} \int_{r_{ki}}^{r_{ko}} r T(r)\,dr - C_{1(k+1)} r_{(k+1)i} - \frac{C_{2(k+1)}}{r_{(k+1)i}} = 0 , \tag{50b}$$

k = 1, 2, ….(n-1)

As far as the boundary conditions at the cylinder end faces are concerned in the case of rigid axial restraint of the end faces, there is the axial strain condition: $\varepsilon_z = 0$; in the case of free end faces, loaded by an axial resultant force P_z there is the equilibrium condition:

$$2\pi \sum_{k=1}^{n} \int_{r_{ki}}^{r_{ko}} \sigma_z r\,dr = \pi \sum_{k=1}^{n} E_k \left[-\frac{2\alpha_k}{1-\nu_k} \int_{r_{ki}}^{r_{ko}} r T(r)\,dr + \frac{(r_{ko}^2 - r_{ki}^2)(2\nu_k C_{1k} + (1-\nu_k)\varepsilon_z)}{(1+\nu_k)(1-2\nu_k)} \right] = 0 . \tag{51}$$

7. Bibliography

See e.g.

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