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MWG

22.38 PROBABILITY AND ITS APPLICATIONS TO RELIABILITY, QUALITY CONTROL AND RISK ASSESSMENT

Fall 2004

CONVERGENCE OF BINOMIAL AND POISSON DISTRIBUTIONS IN LIMITING CASE OF n LARGE, p << 1

The binomial distribution for m successes out of n trials, where p = probability of success in a single trial:

$$P(m,n) = \binom{n}{m} p^{m} (1-p)^{(n-m)} \ .$$

For n large and n >> m,

$$\binom{n}{m} = \frac{n(n-1)\ldots(n-m+1)(n-m)!}{m!(n-m)!} = \frac{n(n-1)\ldots(n-m+1)}{m!} \cong \frac{n^m}{m!} \ .$$

For p << 1, m << n,

$$(1-p)^{(n-m)} = 1 - np + \frac{n(n-1)}{2!} p^2 + \ldots$$

$$\cong 1 - np + \frac{(np)^2}{2!} + \ldots \cong e^{-(np)}$$

$$P(m,n) \cong \frac{n^m p^m}{m!} e^{-(np)} = \frac{(np)^m e^{-np}}{m!} = \text{Prob.}_{\text{Poisson}}(m,n) \ .$$

This results is in the form $P(m,n) \cong \dfrac{\mu^m e^{-\mu}}{m!} = \underset{\text{Poisson}}{P(m,t,\mu)}$.

Recall $\mu_{\text{Binomial}} = np$, and $\mu = \mu_{\text{Poisson}} = \lambda t$, or

$$\text{Prob.}_{\text{Poisson}} = \frac{(\lambda t)^m e^{-\lambda t}}{m!} \ .$$