22.615 Final
Spring 2007

Overview

The purpose of the final exam is to calculate the MHD β limit in a high-beta toroidal tokamak against the dangerous $n = 1$ external ballooning-kink mode. Effectively, this corresponds to an approximate calculation of the Troyon limit.

The analysis is greatly simplified by several reasonably good approximations. First, the equilibrium is calculated using the high-beta tokamak ordering. Second the equilibrium profiles are a toroidal version of Shafranov's step-function model consisting of a plasma core with an approximately constant current density surrounded by a force free plasma region carrying no current. Surrounding the force free plasma is a vacuum out to infinity.

All the relevant equilibrium solutions have been derived in class and are contained in the MHD textbook. Most of the equilibrium work involves renormalizing the physical parameters appearing in the formulae taking into account the presence of the force free plasma. If you do the problem right, there is not much computing required. However, if you do not think about the problem in a mathematically convenient way there can be a large amount of iteration and computer usage. The procedure described below requires <u>no</u> iteration although as you will see it involves somewhat unintuitive mathematical parameters. In any event feel free to use whatever procedure you like – just make sure you get the correct final answer.

The stability analysis is carried out using a δW analysis. This can be a daunting task taking months to years of computational effort to get the "exact" answer, even using the simplifying high-beta tokamak ordering. For the final exam the analysis will be greatly simplified by using a trial function for the displacement with only a few parameters. I will give you what I think is a reasonably good trial function but you should feel free to do better if you want to try.

Part 1 – The Equilibrium Problem

The step-function, force-free model of interest is sketched on the last page. The final goal of the equilibrium calculation is to rewrite the solutions obtained in class in terms of more appropriate physical parameters as described below. There are two specific pieces of information that you need to calculate.

(1) A boundary curve in pressure-current space showing where equilibria are possible. Each equilibrium solution is subject to the constraint that the safety factor on axis q_0 is held constant.

(2) An evaluation of the safety factor at the edge of the plasma core $q_{a'}$ and at the edge of the force free plasma q_a.

The starting point for the calculation is the high-beta tokamak equilibrium for a circular cross section plasma. The flux function in the plasma core is given by Eq. 6.101,

$$\frac{\psi_{core}}{a'^2 B_0} = \frac{1}{2q'_*}\left[\frac{r^2}{a'^2} - 1 + \nu'\left(\frac{r^3}{a'^3} - \frac{r}{a'}\right)\cos\theta\right]$$

All quantities are identical to those given in the textbook and the class notes except that $a, q_*, \nu, \varepsilon, \beta_t$ are now written with a prime to indicate that they correspond to values on the boundary of the core and not the true outer plasma boundary which occurs at the edge of the force-free plasma region. The corresponding parameters at the edge of the force free region have no primes. Most of the work in this part of the exam is to determine the relationships between the primed and unprimed parameters.

The equilibrium flux function given in the textbook for the vacuum region actually is valid for both the force-free and vacuum regions in the exam problem. The reason is that by assumption zero current flows in the force free region so that the magnetic flux is the same as for a vacuum. This flux is given by Eq. 6.113,

$$\frac{\psi_{outer}}{a'^2 B_0} = \frac{1}{q'_*}\left[\ln\left(\frac{r}{a'}\right) + \frac{\nu'}{2}\left(\frac{r}{a'} - \frac{a'}{r}\right)\cos\theta\right]$$

To begin the analysis, choose one of the vacuum flux surfaces inside the separatrix and define this as the edge of the force free plasma. The left radius of this surface is denoted by a_1 while the right radius is denoted by a_2. In general $a_1 \neq a_2$. The actual radius of the plasma is denoted by a and this is a basic geometric quantity of physical interest. By definition

$$a = \frac{1}{2}(a_1 + a_2)$$

As you progress all quantities should be expressed in terms of a rather than a'.

To specify an equilibrium solution you will need to give numerical values for three parameters. For mathematical and computational convenience these parameters are denoted by ζ_1, α, q_0. The definitions are as follows. The quantity q_0 is the safety factor on the magnetic axis. We shall assume that $q_0 = 1.05$ for <u>all</u> equilibria under consideration in order to avoid the internal $m = 1, n = 1$ sawtooth mode. The quantity alpha is defined by $\alpha = a'/a$. It approximately measures the ratio of the core radius to the total plasma radius – it is inversely related to the peaking factor. Clearly $0 < \alpha < 1$. The last quantity of interest is $\zeta_1 = a_1/a$. It is related to the shift of the plasma and thus indirectly to the plasma beta. Physically we expect that $\zeta_1 > 1$ since the plasma core is shifted outwards with respect to the plasma edge.

Now carry out the following calculations:

1. **Assuming that values are given for ζ_1 and α, derive an expression for the value of ν'**
2. **Ultimately there are two equilibrium limits to evaluate. The first one is qualitatively similar to the one derived in class. It corresponds to the situation**

where the separatrix moves onto the outer edge of the force-free plasma. This condition yields another relation between $\zeta_1, \alpha,$ and ν'. Derive this relation.

3. **Using the results of parts (1) and (2) plot curves of ν' vs ζ_1 and α vs ζ_1. This is one set of curves that define the allowable equilibrium space. You should also find that solutions exist only for $\zeta_1 < \zeta_{crit}$. Find ζ_{crit}. (Hint: ζ_{crit} lies in the range $1 < \zeta_{crit} < 2$)**

You will evaluate the second equilibrium limit shortly. At this point, however, it is more convenient to derive relations between the physical parameters at the edge of the plasma core and at the edge of the force-free plasma. To do this you will need to evaluate the total cross sectional area bounded by the outer edge of the force free plasma. Do this as follows.

4. **Assume allowable values of ζ_1 and α are specified. Write a subroutine that evaluates the normalized area $\hat{A} = A/\pi a^2$. Note that $\hat{A} = \hat{A}(\zeta_1, \alpha)$. A small amount of algebra and computation is required to carry out this task.**

You should now be in a position to calculate the physical parameters of interest. Again, assume that values of ζ_1 and α are specified along with the value $q_0 = 1.05$. This can be done conveniently in the following order.

5. **Derive expressions for**
   a. ν' **(already done in part 1)**
   b. $q'_* = \dfrac{2\pi a'^2 B_0}{\mu_0 R_0 I}$
   c. $\dfrac{\beta'_t}{\varepsilon'} = \dfrac{2\mu_0}{B_0^2}\left(\dfrac{R_0}{a'}\right)\left(\dfrac{1}{\pi a'^2}\int prdrd\theta\right)$
   d. $q_* = \dfrac{2AB_0}{\mu_0 R_0 I}$
   e. $\dfrac{\beta_t}{\varepsilon} = \dfrac{2\mu_0}{B_0^2}\left(\dfrac{R_0}{a}\right)\left(\dfrac{1}{A}\int prdrd\theta\right)$
   f. $\nu = \dfrac{\beta_t q_*^2}{\varepsilon}$

You should be able to obtain numerical values for each of these quantities once ζ_1 and α are specified.

6. **Using these relations plot a curve of β_t/ε vs $1/q_*$ for the special equilibrium beta limit constraint given in part (3).**

The next part of the problem involves the second equilibrium limit mentioned above. This is a limit on the smallest allowable value of q_* (i.e. largest allowable value of $1/q_*$). If we want to hold q_0 fixed we cannot decrease q_* indefinitely because in general $q_0 < q_*$. Since $q_* \propto A/I$ the smallest q_* at a fixed current occurs for the smallest A. Clearly the smallest A occurs when the outer surface of the force-free plasma moves inward until it coincides with the surface of the plasma core.

7. **Assume the outer surface of the force-free plasma coincides with the surface of the plasma core. Using this constraint you should be able to determine a relationship between β_t/ε and $1/q_*$. Determine this relation and superimpose it on the plot of β_t/ε vs $1/q_*$ obtained in part (6). If you did everything right you should now have a closed region in β_t/ε - $1/q_*$ space where equilibria are allowable.**

The last part of the equilibrium problem involves the calculation of safety factor q on certain surfaces. This information will be needed to carry out the stability analysis.

8. **Derive (but do not evaluate) a general integral expression for the safety factor on any flux surface, either in the core or the force free plasma, using the high beta tokamak ordering. Now, explicitly evaluate $q_{a'} = q_{a'}(\zeta_1, \alpha)$, the safety factor at the surface of the plasma core. This is easy – we did it in class. Finally, write a subroutine that evaluates $q_a = q_a(\zeta_1, \alpha)$, the safety factor at the outer edge of the force-free plasma. A small amount of computation is required for this task.**

NOW YOU ARE DONE WITH THE EQUILIBRIUM CALCULATION