22.616. Plasma Transport theory

Problem #3 Solutions

1. Moment~~um~~ Equation Structure

Vlasov Equation

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f + \frac{q}{m}\left(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B}\right)\cdot\frac{\partial}{\partial \underline{v}} f = 0 \qquad \text{---(1)}$$

Assume

$$f \simeq f^M = \frac{n}{(2\pi T/m)^{3/2}} e^{-\frac{m(\underline{v}-\underline{V})^2}{2T}} \qquad \text{---(2)}$$

Let $\boxed{\underline{v}' = \underline{v} - \underline{V},}$ then $\underline{v} = \underline{v}' + \underline{V}$ --(3)

$$f^M = \frac{n}{(2\pi T/m)^{3/2}} \exp\left(-\frac{m v'^2}{2T}\right) \qquad \text{---(4)}$$

1° Take the density moment of Eq (1).

$$\frac{\partial}{\partial t}\int d^3v f + \nabla\cdot\int d^3v\, \underline{v} f + \int d^3v \frac{q}{m}\left(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B}\right)\cdot\frac{\partial}{\partial \underline{v}} f = 0$$

Where

$$\int d^3v f = \int d^3v (f - f_M) + \int d^3v f_M$$

$$= \int d^3v (f - f_M) + n \simeq n \qquad \text{---(5)}$$

$$\int d^3v\, \underline{v} f = \int d^3v' (\underline{v}' + \underline{V}) f$$

$$= \int d^3v' \underline{v}' f + \int d^3v' f\, \underline{V} \simeq n\underline{V} \quad \text{---} (6)$$

Assume $\int d^3v' \underline{v}' f = \int d^3v' \underline{v}' (f - f_M^M) = 0 \quad \text{---} (7)$

∂^0 ~~Take the~~

$$\frac{q}{m} \int (\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}) \cdot \frac{\partial}{\partial \underline{v}} f \, d^3v$$

Integrate by parts $= -\frac{q}{m} \int f \nabla_{\underline{v}} \cdot (\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}) \, d^3v$

$\underline{E} = \underline{E}(\underline{x}, t)$ so $\nabla_{\underline{v}} \cdot \underline{E} = 0$

$$\nabla_{\underline{v}} \cdot (\underline{v} \times \underline{B}) = \frac{\partial}{\partial v_i} (\varepsilon_{ijk} v_j B_k) = \varepsilon_{ijk} B_k \delta_{ij} = \varepsilon_{iik} B_k = 0$$

Therefore $\frac{q}{m} \int (\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}) \cdot \frac{\partial}{\partial \underline{v}} f \, d^3v = 0 \quad \text{---} (8)$

Then we have

$$\boxed{\frac{\partial n}{\partial t} + \nabla \cdot (n \underline{V}) = 0} \quad \text{---} (9)$$

2° Take the momentum moment of Eq (1)

$$\underbrace{\frac{\partial}{\partial t}\int d^3v\, m\underline{v} f}_{①} + \nabla\cdot \underbrace{\int d^3v\, m\underline{v}\underline{v} f}_{②} + \underbrace{\int d^3v\, \underline{v}\, q(\underline{E}+\frac{1}{c}\underline{v}\times\underline{B})\cdot\frac{\partial}{\partial \underline{v}} f}_{③} = 0$$

$$① = \int d^3v\, m\underline{v} f = nm\underline{V} \quad \text{(proved at eq (6))}$$

$$② = m\int d^3v' (\underline{v}'+\underline{V})(\underline{v}'+\underline{V}) f$$

$$= m\int d^3v' (\underline{v}'\underline{v}' + \underline{v}'\underline{V} + \underline{V}\underline{v}' + \underline{V}\underline{V}) f$$

$$= m\int d^3v'\, \underline{v}'\underline{v}' f + mn\underline{V}\underline{V}$$

$$= m\int d^3v'\, \underline{v}'\underline{v}' (f - f^M) + m\int d^3v'\, \underline{v}'\underline{v}' f^M + nm\underline{V}\underline{V}$$

$$= \underline{\underline{\Pi}} + nT\underline{\underline{I}} + nm\underline{V}\underline{V} \qquad \text{---- (11)}$$

$$③ = -\int d^3v\, q(\underline{E}+\frac{1}{c}\underline{v}\times\underline{B})\cdot\underline{\underline{I}} f$$

$$= -qn\underline{E} - \frac{q}{c}\left(\int d^3v\, \underline{v} f\right)\times\underline{B}$$

$$= -qn\underline{E} - \frac{q}{c}n\underline{V}\times\underline{B} \qquad \text{--- (12)}$$

Therefore we have

$$\frac{\partial}{\partial t}(nm\underline{V}) + \nabla\cdot(\underline{\underline{\Pi}} + nT\underline{\underline{I}} + nm\underline{V}\,\underline{V}) - qn\underline{E} - qn\frac{1}{c}\underline{V}\times\underline{B} = 0$$

i.e.

$$nm\frac{\partial \underline{V}}{\partial t} + m\underline{V}\frac{\partial n}{\partial t} + \nabla\cdot\underline{\underline{\Pi}} + \nabla p + nm\underline{V}\cdot\nabla\underline{V} + m\underline{V}\nabla\cdot(n\underline{V})$$

$$= qn(\underline{E} + \frac{1}{c}\underline{V}\times\underline{B})$$

Plug in the density moment equation (9). we have

$$\boxed{\begin{aligned} &nm\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = -\nabla p + qn(\underline{E} + \frac{1}{c}\underline{V}\times\underline{B}) - \nabla\cdot\underline{\underline{\Pi}} \qquad \text{--- (13)} \\ &\text{with} \\ &\underline{\underline{\Pi}} \equiv m\int d^3v\,\underline{v}'\underline{v}'(f - f^M) \qquad \text{(Stress tensor)} \qquad \text{--- (14)} \end{aligned}}$$

3°. Take the Energy moment of Eq (1). we get

$$\overbrace{\frac{\partial}{\partial t}\int d^3v\,\frac{1}{2}mv^2 f}^{①} + \nabla\cdot\Big(\overbrace{\int d^3v\,\frac{1}{2}mv^2\underline{v}f}^{②}\Big) + \Big(\overbrace{\int d^3v\,\frac{q}{m}(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B})\cdot\frac{\partial f}{\partial \underline{v}}}^{③}$$

$$\times\frac{1}{2}mv^2\Big) = 0$$

$$① = \frac{1}{2}m \int d^3v' (\underline{v}' + \underline{V}) \cdot (\underline{v}' + \underline{V}) f$$

$$= \frac{1}{2}m \int d^3v' (v'^2 + V^2 + 2\underline{v}' \cdot \underline{V}) f$$

$$= \frac{1}{2}m \int d^3v' (v'^2 + V^2) f$$

$$\simeq \frac{1}{2}m \int d^3v' v'^2 f^M + \frac{nm}{2} V^2 + \frac{1}{2}m \int d^3v' v'^2 (f - f^M)$$

$$= \frac{3}{2}nT + \frac{1}{2}nmV^2 \qquad \text{--- (15)}$$

(Assume $\frac{1}{2}m \int d^3v' v'^2 (f - f^M) = 0$)

②

Define

$$\underline{Q} \equiv \int d^3v \frac{1}{2} m v^2 \underline{v} f$$

$$= \frac{m}{2} \int d^3v' (\underline{v}' + \underline{V})^2 (\underline{v}' + \underline{V}) f$$

$$= \frac{m}{2} \int d^3v' (v'^2 \underline{v}' + v'^2 \underline{V} + V^2 \underline{v}' + V^2 \underline{V} + 2\underline{v}' \cdot \underline{V} \underline{v}' + 2\underline{v}' \cdot \underline{V} \underline{V}) f$$

$$= \frac{m}{2} \int d^3v' (v'^2 \underline{v}' + v'^2 \underline{V} + V^2 \underline{V} + 2\underline{v}' \cdot \underline{V} \underline{v}') f \qquad \text{--- (16)}$$

let $\underline{q} \equiv \int d^3v' \frac{m}{2} v'^2 \underline{v}' f$

As we defined in Eq <14>

$$\underline{\underline{\Pi}} = \int d^3v \, m \underline{v}' \underline{v}' (f - f^M)$$

$$= n \int d^3v' \underline{v}' \underline{v}' f - nT \underline{\underline{I}} \qquad \text{---} \langle 17 \rangle$$

Therefore

$$\underline{Q} = \underline{q} + \frac{m}{2} \int d^3v' v'^2 \underline{V} f + \frac{nm}{2} V^2 \underline{V} + (\underline{\underline{\Pi}} + nT \underline{\underline{I}}) \cdot \underline{V}$$

$$= \underline{q} + \frac{m\underline{V}}{2} \int d^3v' v'^2 (f - f^M) + \frac{3}{2} nT \underline{V} + \frac{nmV^2}{2} \underline{V}$$

$$+ \underline{\underline{\Pi}} \cdot \underline{V} + nT \underline{V}$$

$$\Rightarrow \boxed{\underline{Q} = \underline{q} + \frac{5}{2} nT \underline{V} + \frac{nmV^2}{2} \underline{V} + \underline{\underline{\Pi}} \cdot \underline{V}} \qquad \text{---} \langle 18 \rangle$$

$$③ = \frac{q}{m} \int d^3v \, \frac{1}{2} m v^2 \frac{\partial}{\partial \underline{v}} \cdot \left[(\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}) f \right]$$

Integrate by parts $\Rightarrow$

$$③ = - \int d^3v \, q \, \underline{v} \cdot (\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}) f$$

$$= -\int d^3v \frac{q}{m} \underline{v} \cdot \underline{E} f$$

$$= - q n \underline{V} \cdot \underline{E} \qquad \text{---(19)}$$

Therefore the energy moment equation turns out to be

$$\underline{\frac{\partial}{\partial t}\left(\frac{3}{2}nT + \frac{1}{2}mnV^2\right) + \nabla\cdot\underline{Q} = qn\underline{V}\cdot\underline{E}} \qquad \text{---(20)}$$

Where

$$\nabla\cdot\underline{Q} = \nabla\cdot\left(q + \frac{5}{2}nT\underline{V} + \frac{nmV^2}{2}\underline{V} + \underline{\underline{\Pi}}\cdot\underline{V}\right)$$

$$= \nabla\cdot\left(q + \frac{5}{2}nT\underline{V}\right) + \underline{V}\cdot\nabla\left(\frac{1}{2}nmV^2\right) + \frac{1}{2}nmV^2\nabla\cdot\underline{V}$$

$$+ \underline{V}\cdot(\nabla\cdot\underline{\underline{\Pi}}) + \underline{\underline{\Pi}}:\nabla\underline{V}$$

$$= \nabla\cdot\left(q + \frac{5}{2}nT\underline{V}\right) + \frac{1}{2}nm\underline{V}\cdot\nabla V^2 + \frac{1}{2}mV^2\left(\underline{V}\cdot\nabla n + n\nabla\cdot\underline{V}\right)$$

$$+ \underline{V}\cdot(\nabla\cdot\underline{\underline{\Pi}}) + \underline{\underline{\Pi}}:\nabla\underline{V}$$

$$= \nabla\cdot\left(q + \frac{5}{2}nT\underline{V}\right) + \frac{1}{2}nm\underline{V}\cdot\nabla V^2 - \frac{1}{2}mV^2\frac{\partial n}{\partial t}$$

$$+ \underline{V}\cdot(\nabla\cdot\underline{\underline{\Pi}}) + \underline{\underline{\Pi}}:\nabla\underline{V} \qquad \text{---(21)}$$

Plug Eq(21) into Eq(20). We obtain

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) + \nabla\cdot\left(\frac{5}{2}nT\underline{V} + \underline{q}\right) + \underline{V}\cdot\nabla\underline{\underline{\Pi}} + \underline{\underline{\Pi}}:\nabla\underline{V}$$

$$+ \frac{1}{2}nm\underline{V}\cdot\nabla V^2 - \frac{1}{2}mV^2\frac{\partial n}{\partial t} + \frac{\partial}{\partial t}\left(\frac{1}{2}nmV^2\right) = qn\underline{V}\cdot\underline{E}$$

i.e.

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) + \nabla\cdot\left(\frac{5}{2}nT\underline{V} + \underline{q}\right) + \underline{V}\cdot\nabla\underline{\underline{\Pi}} + \underline{\underline{\Pi}}:\nabla\underline{V}$$

$$+ \frac{1}{2}nm\underline{V}\cdot\nabla V^2 + nm\underline{V}\cdot\frac{\partial\underline{V}}{\partial t} = qn\underline{V}\cdot\underline{E} \qquad \text{---(22)}$$

Then evaluate $\underline{V}\cdot$ Eq.(13) (the Momentum moment Equation)

$$nm\left(\underline{V}\cdot\frac{\partial\underline{V}}{\partial t} + \underline{V}\cdot(\underline{V}\cdot\nabla\underline{V})\right) = -\underline{V}\cdot\nabla p + qn\underline{V}\cdot\underline{E} - \underline{V}\cdot(\nabla\cdot\underline{\underline{\Pi}})$$

$$\text{---(23)}$$

Notice

$$\underline{V}\cdot(\underline{V}\cdot\nabla\underline{V}) = V_i V_j\frac{\partial}{\partial x_j}V_i = V_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}V_iV_i\right)$$

$$= \underline{V}\cdot\nabla\frac{1}{2}V^2 = \frac{1}{2}\underline{V}\cdot\nabla V^2.$$

So Eq(23) is

$$nm\underline{V}\cdot\frac{\partial\underline{V}}{\partial t} + \frac{1}{2}nm\underline{V}\cdot\nabla V^2 = -\underline{V}\cdot\nabla p + qn\underline{V}\cdot\underline{E} - \underline{V}\cdot(\nabla\cdot\underline{\underline{\Pi}})$$

$$\text{---(24)}$$

combine Eq.(22) & Eq.(24), we finally get

$$\boxed{\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right)+\nabla\cdot\left(\frac{5}{2}nT\underline{V}+\underline{q}\right)=\underline{V}\cdot\nabla P-\underline{\underline{\Pi}}:\nabla\underline{V}}$$

---(25)

Notice

$$\nabla\cdot\left(\frac{5}{2}nT\underline{V}\right)=\nabla\cdot(P\underline{V})+\frac{3}{2}\nabla\cdot(nT\underline{V})$$

$$=\underline{V}\cdot\nabla P+P\nabla\cdot\underline{V}+\frac{3}{2}n\underline{V}\cdot\nabla T+\frac{3}{2}T\nabla\cdot(n\underline{V})$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right)=\frac{3}{2}n\frac{\partial T}{\partial t}+\frac{3}{2}T\frac{\partial n}{\partial t}$$

So Eq.(25) $\Rightarrow$

$$\frac{3}{2}n\left(\frac{\partial}{\partial t}+\underline{V}\cdot\nabla\right)T+\frac{3}{2}T\underbrace{\left(\frac{\partial n}{\partial t}+\nabla\cdot(n\underline{V})\right)}_{=0}$$

$$+P\nabla\cdot\underline{V}=-\nabla\underline{q}-\underline{\underline{\Pi}}:\nabla\underline{V}$$

So we obtain

$$\frac{3}{2}n\left(\frac{\partial}{\partial t}+\underline{V}\cdot\nabla\right)T+P\nabla\cdot\underline{V}=-\nabla\cdot\underline{q}-\underline{\underline{\Pi}}\cdot\nabla\underline{V}$$

---(26)

If $\underline{\underline{\Pi}}=0$ (zero viscous stress), $\underline{q}=0$ (no heat flux)

Eq(26) can be written as

$$\frac{3}{2}n\left(\frac{\partial}{\partial t}+\underline{V}\cdot\nabla\right)T + P\nabla\cdot\underline{V} = 0 \qquad ---(27)$$

from the density moment eq.

$$n\nabla\cdot\underline{V} = -\frac{Dn}{Dt}$$

So Eq(27) change to

$$\frac{3}{2}n\frac{DT}{Dt} - T\frac{Dn}{Dt} = 0$$

$$\Rightarrow \qquad \frac{1}{T}\frac{DT}{Dt} - \frac{2}{3}\frac{1}{n}\frac{Dn}{Dt} = 0$$

i.e. $\frac{D}{Dt}\left(\ln\left(Tn^{-2/3}\right)\right) = 0$

~~since~~ $\Rightarrow$ $\frac{D}{Dt}\left(\ln p\, n^{-5/3}\right) = 0$

i.e. $\boxed{\frac{D}{Dt}\left(pn^{-5/3}\right) = 0}$ $\qquad$ --- (28)