22.812J **<u>Nuclear Energy Economics and Policy Analysis</u>** **S'04**

Classnote: The Time Value of Money

1. Motivating Example

To motivate the discussion, we consider a homeowner faced with a decision whether to install a solar hot water heater to replace an existing natural gas fired hot water heater.

First, we must design the solar heating system. A typical U.S. family of four in a three bedroom house uses about 100 gallons of hot water (at 150 ^{o}F) per day. A solar thermal system designed to provide this hot water is shown schematically in the Figure. Cold water at temperature T_c is pumped to solar collectors where it is heated to temperature T_h. This hot water is stored in a tank for later use. Auxiliary heating is provided for periods of no sun.

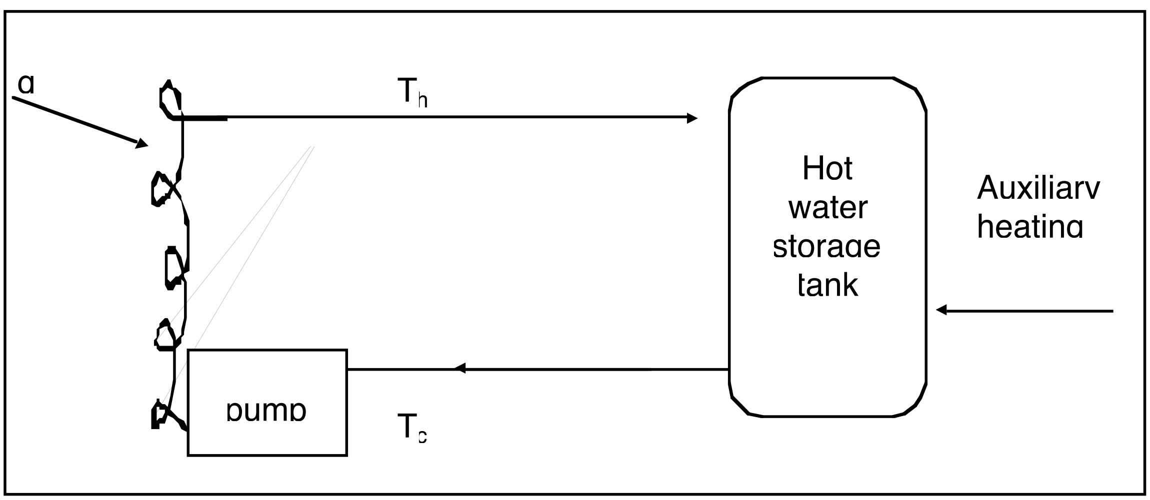


Figure 1. Schematic of residential solar hot water heating system.

Assume: that the cold water enters the solar collector at 40 ^{o}F and that the hot water in the storage tank should be at 150 ^{o}F. The daily BTU requirement for hot water heating is:

BTUs required/day = (100 gal/day)(8.33 lbs/gal)(1 BTU/lb-^{o}F) (150 - 40) (^{o}F)
= 91,630 BTU / day.

Thus the required heat load is about 90,000 BTU/day. Assuming a 50% collector efficiency this requires 180,000 BTU/day of solar radiation.

How much solar heat does the sun deliver each day? This obviously depends on both season and location. Typical values are given in the following table:

Solar BTU delivered per sq. ft. per day	January	June	Average
Boston,MA	500	2000	1000
Tucson, AZ	1000	2500	2000

If we decide to meet 50% of the required heat load from solar energy (see below), this means that, on average, we require about 90 sq. ft. of hot water solar collector area per home in Boston, Massachusetts and about 45 sq. ft. in Tucson, Arizona. These systems will deliver about 50 gallons of hot water per day at a temperature of 150 ^{O}F, assuming a feed water temperature of 40 ^{O}F. The systems thus provide:

$$Q = V.\square.\square T.c=45{,}000 \text{ BTU of heat per day}$$

where the volumetric flowrate V = 50 gallons per day, the density of water ρ =8.33 lbs/gal, the temperature increase ΔT=(150 - 40) ^{o}F, and the specific heat of water c =1 BTU/^{o}F-lb.

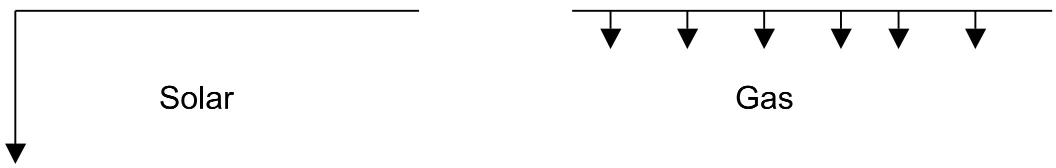
Economic analysis. Is it worthwhile to switch from gas to solar heat for the hot water this home requires? In order to answer this question we must compare the cost of the two alternatives.

For the solar hot water heating system, we assume that once the system is installed (and paid for), the continuing operating costs are negligible, i.e. the cost of electricity for the pump and the cost of maintenance are assumed to be very small. We also ignore the cost of the hot water tank, because this must be purchased for either the solar or conventional gas hot water heating system. The cost estimates for the main components of the solar system are presented in the following table.

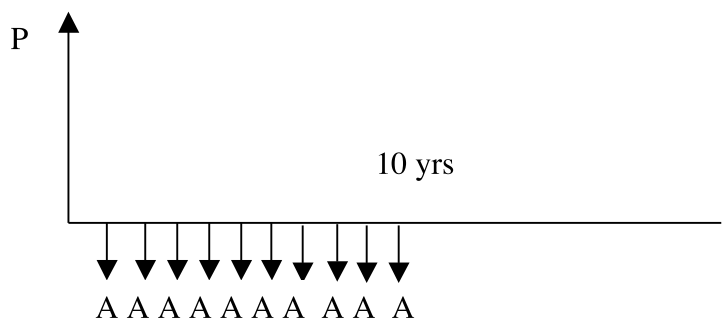
System Cost	Boston	Tuscon
1. Panels ($17/sq.ft.)	$1530	$765
2. Piping	500	500
3. Pump & controls	100	100
4. Installation	500	500
Total	$2630	$1865

Installation of the solar hot water heating system delivering 45,000 BTU per day (or, equivalently, 16.4 million BTU per year) will thus cost about $2600 in Boston and $1900 in Tucson.

How does this compare to the cost of conventional hot water heating by gas? To make this comparison, we must weigh a one-time investment in the solar heating system against the recurring expense of purchasing the gas. (We assume that there is no capital cost for the gas system, since the gas burner is integrated into the hot water storage tank.)



One way to make this comparison is to apportion the capital cost of the solar heating system uniformly to each year of its operating life, and then compare this annualized cost with the annual cost of the gas. But simply dividing the capital cost by the number of years of life – 10 years, say -- would understate the true cost of the investment, because this would ignore the interest cost of the invested capital. To see why, imagine that the homeowner borrows all the money to buy the solar heating system from the bank. Let us further assume that the loan is for a fixed 10-year term, at a constant interest rate of 10%/year. Each year, the homeowner must pay off part of the loan principal and pay interest on the portion of the principal that is still outstanding. To make the comparison easier, let us further assume that the terms of the loan require the borrower to make equal annual payments to the bank throughout the life of the loan – that is, the annual 'debt service', the sum of the principal repayment and the interest owed on the remaining principal, is the same in each year.



(Notice the convention on the cash flow diagram: Cash receipts (income streams) are denoted as upward arrows; cash outlays (expense streams) are denoted as downward arrows.)

We want to calculate the uniform loan annual payments A as a function of the initial loan P:

After the end of the first year, the homeowner pays interest at rate r on the principal, rP, and retires a portion of the principal, D_1, where

$$A = Pr + D_1 \qquad (1)$$

After the end of the second year, the homeowner pays interest on the residual principal of $(P-D_1)$ and retires a further portion of the principal, D_2, where

$$A = (P-D_1)r + D_2 \qquad (2)$$

And substituting for D_1 in (2) and solving for D_2 we have:

$$D_2 = (A - Pr)(1+r) \qquad (3)$$

After the end of the third year, the homeowner pays interest on the residual principal, $P - D_1 - D_2$, and retires a further portion of the principal D_3, where

$$A = (P-D_1-D_2)r + D_3 \qquad (4)$$

And substituting for D_1 and D_2 in (4) and solving for D_3, we have:

$$D_3 = (A - Pr)\,(1+r)^2 \qquad (5)$$

And, by induction,

$$D_n = (A - Pr)\,(1 + r)^{n-1} \qquad (6)$$

And since

$$\sum_{n=1}^{N} D_n = P$$

we can write

$$P = (A - Pr)(1 + (1+r) + (1+r)^2 + \ldots + (1+r)^{N-1})$$

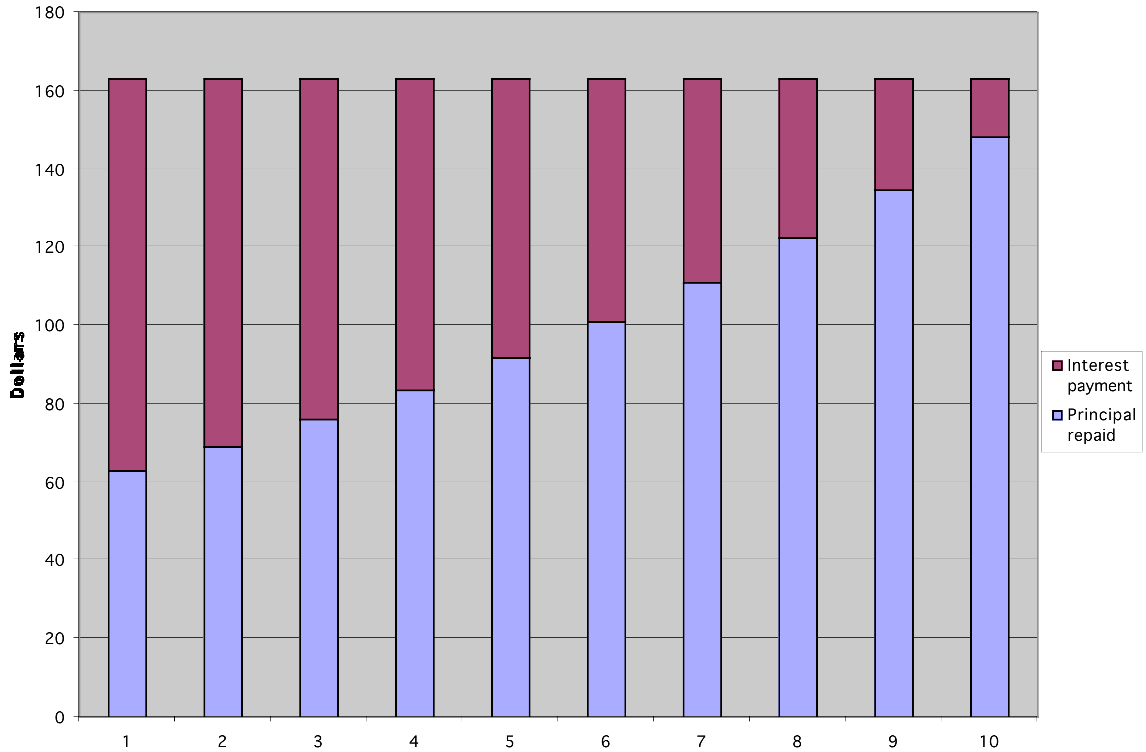
$$= (A - Pr)\left[\frac{(1 - (1+r)^N)}{-r}\right]$$

and solving for A, we have

$$\boxed{A = P\left[\frac{r(1+r)^N}{(1+r)^N - 1}\right] \qquad (1)}$$

The term in square parentheses in equation (1) is called the ‘annual capital charge rate’, □. As already noted, the annual payment is comprised partly of interest on the outstanding principal and partly of principal repayment. In the early years of the loan, interest accounts for the lion’s share of the payment; towards the end of the term, most of the payment goes towards repaying the principal.

And for a 10-year loan, with an interest rate of 10%/year, we have:



We can now calculate the minimum price of natural gas, p*, in dollars per thousand cubic feet ($/MCF), above which the annual cost of gas exceeds the annual capital charge for the solar heating system, as follows:

$$I_o \, \square = p^* \, Q$$

Where I_o is the capital cost of the solar heating system and Q is the annual gas requirement (in MCF) for delivering 45,000 BTU/day.

The heat content of natural gas is about 1 million BTU/MCF. If we assume an 80% heating efficiency for the gas, the annual gas requirement is given by:

$$Q = (16.4 \times 10^{6})/0.8 \text{ (BTU/yr)} \times 10^{-6} \text{ (MCF/BTU)}$$

$$= 20.5 \text{ MCF/yr}$$

The crossover price of natural gas above which solar heating is economic in Boston and Tucson is shown in the table below for different values of the interest rate, assuming the solar system is financed with a 10-year loan.

<u>Minimum delivered price of natural gas above which residential solar hot water heating is economical ($/MCF)</u>

		Threshold price of gas, p* ($/MCF)	
Loan interest rate r (%/yr)	Annual capital charge rate, □ (%/yr)	Boston (I=$2630)	Tucson (I=$1865)
3	11.7	15	10.5
6	13.6	17.4	12.5
10	1.3	20.9	14.8

As expected, solar hot water heating is competitive at a lower gas price in sunny Tucson than in Boston. But residential gas prices are considerably higher in Boston than in Tucson because of the higher cost of transporting gas to the northeastern U.S. If the difference in delivered gas prices between the two cities were large enough, solar hot water heating could in principle be economical in Boston and not in Tucson.

In fact, the average price of natural gas delivered to residential consumers in Massachusetts during 2001 was $13.35/MCF, and $10.34/MCF in Arizona. So, for a typical interest rate (of 6%/yr or more), we can conclude on the basis of this analysis that solar hot water heating is not economical in either location.

However, our analysis has not taken account of the effect of tax credits for residential solar installations, which in some instances have been granted for up to 50% of the installation cost. Such credits reduce the effective investment cost by 50%, which would be enough to make solar hot water heating economical for homeowners in some parts of the country, as shown in the following Table. Of course, the cost to society of the solar option would not have changed – it is the taxpayer who is effectively paying the difference.

Minimum delivered price of natural gas ($/MCF) above which residential solar hot water heating is economical, assuming 50% tax credit

Loan interest rate r (%/yr)	Annual capital charge rate, □ (%/yr)	Threshold price of gas, p^* ($/MCF)	
		Boston (I=$1315)	Tucson (I=$940)
3	11.7	7.5	5.25
6	13.6	8.7	6.25
10	1.3	10.45	7.4

Note: Why is the system designed for only a fraction of the load? At any given location, the solar flux varies with season. If the system were designed to satisfy 100% of the required heat load throughout the year, the solar collector area would have to be large enough to meet the load during the part of the year when the solar flux is smallest. For the rest of the year, the system would produce excess hot water. Reducing the collector area would lessen the cost, but would also mean that the heat load would only partly be met when the flux was lowest, requiring the purchase of backup heating during that time of the year. A further reduction in the collector area would lengthen the interval during which there would be insufficient hot water to meet the load, further increasing the requirement for backup energy. Beyond a certain point the collector area would be so small that there would be no time of the year when it would be capable of meeting 100% of the load. In general there is an optimal size for the collector, determined by the economic tradeoff between the capital cost of the collector and the cost of purchasing auxiliary heat energy. The situation is shown in the Figure below. For times $t_1 < t < t_2$, 100% of the heat load is met; for times $t < t_1$ and $t > t_2$ backup energy is required. In the example discussed above, the optimal collector was assumed for simplicity to deliver 50% of the required load.

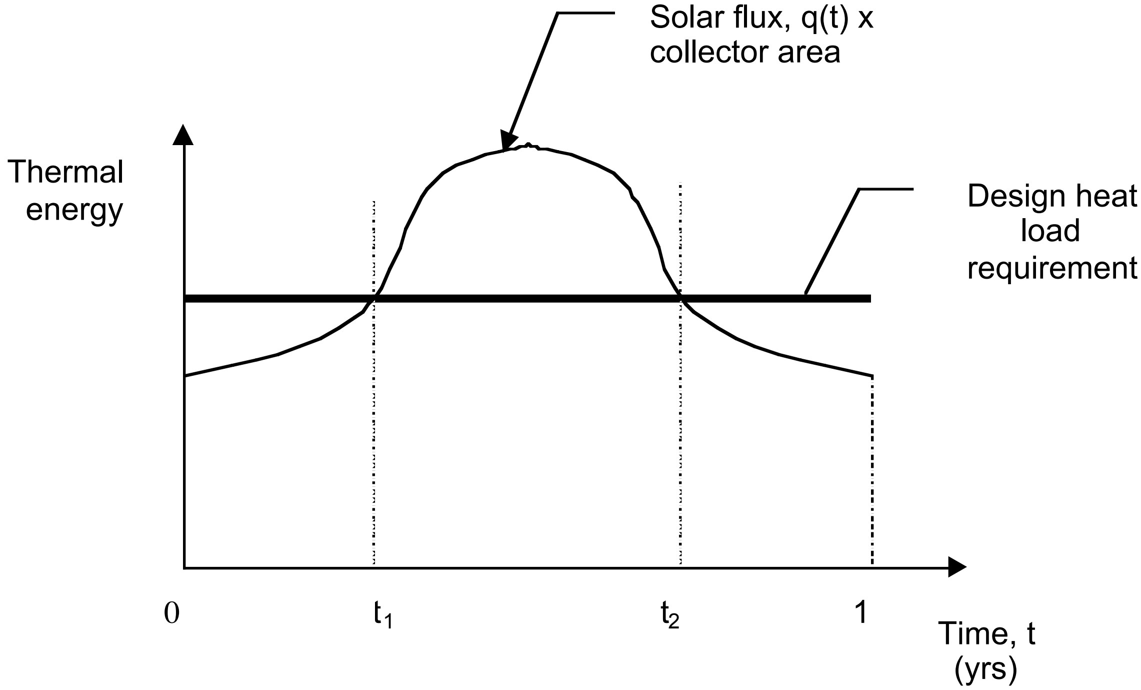


Figure: Solar flux, q(t) as a function of time during the year relative to design heat requirement

Key points from example:

- It is often necessary in economic and financial analysis to compare projects with cash flows occurring at different points in time.
- You cannot add or subtract cash flows occurring at different points in time without first correcting them to a common time base.

The basic principle is quite simple: a dollar received today is worth more than one received a year from today because one can invest today's dollar (e.g., put it in a bank) and accrue interest on it over the coming year. For example, at an effective annual interest rate of 6%, today's dollar would be worth $1.06 in one year. We speak of the equivalence of the two cash flows: $1 today and $1.06 a year from now.

(NOTE: This has nothing to do with the phenomenon of inflation – the decline in the purchasing power of money over time. There are two distinct phenomena: Earning power and purchasing power.)

Translating a cash flow to an equivalent amount at some future date is called finding the future worth. The equivalent amount of a future cash flow today is called the present worth.

The algebraic factors -- time value factors -- used to make these transformations are in such common use that a standard nomenclature and algebraic shorthand notation has developed.

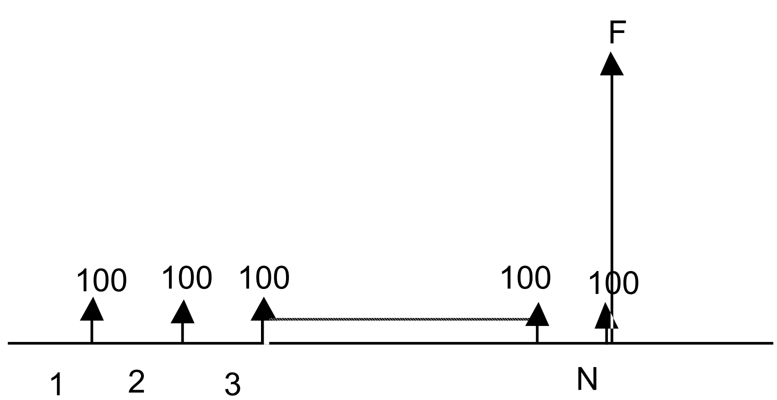
Examples:

"Future Worth Factor", (F/P, i%, N): F = P x (F/P, i%, N)

Future Worth Factor	Factor used to find the future worth, F, in N years of an amount, P today	(F/P,i%,N)	$(1+i)^N$
Present Worth Factor	Factor used to convert a future cash flow, F, N years from now, into an equivalent amount today, P	(P/F,i%,N)	$(1+i)^{-N}$

A single cash flow can also be translated into an equivalent annuity -- a uniform series of amounts of money occurring each year over a specified number of years. And conversely, can translate an annuity into an equivalent single amount as of either the beginning or the end of the series (or at any other time).

Example
Future worth, F, of an annuity of, say, $100, where the annuity is paid at the end of each year for N years:



$$F = 100(1+i)^{N-1} + 100(1+i)^{N-2} + \ldots. + 100$$

$$= 100[\sum_{n=1}^{N-1}(1+i)^n + 1]$$

$$= 100[\frac{(1+i)^N - 1}{i}]$$

= 100 x “uniform series compound amount factor”

= 100 x (F/A,i%,N)

Conversely, to obtain the value of the annuity, A, equivalent to a future amount F, we use the reciprocal factor

$$A = F[\frac{i}{(1+i)^N - 1}]$$

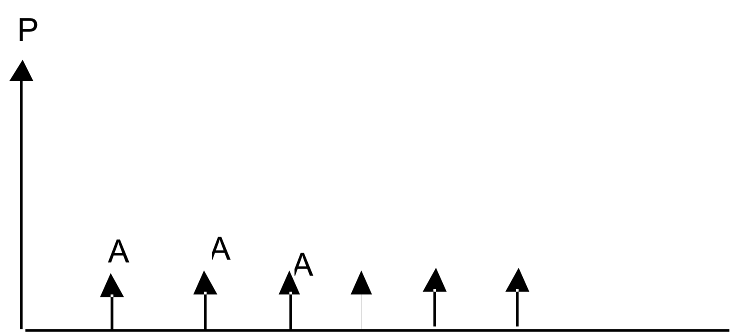
or

A = F x ‘sinking fund factor’

A = F x (A/F, i%, N)

<u>Example</u>

To calculate the present value, P, of an annuity, A, we can sum the PW factors of each cash flow:



$$P = A\,(1+i)^{-1} + A(1+i)^{-2} + \;\ldots\ldots\ldots$$

Alternatively, we can use the factors we have already developed, by observing that P is both the present value of the annuity itself but also the present value of the future worth of the annuity, which we have already calculated.

$$
\begin{aligned}
P &= A[\frac{(1+i)^N - 1}{i}].(1+i)^{-N} \\
&= A[\frac{(1+i)^N - 1}{i(1+i)^N}] \\
&= A \, x \, PW \, of \; annuity \; factor \\
&= A.(P/A, i\%, N)
\end{aligned}
$$

Conversely, the annuity A paid over N years that is equivalent in value to a present amount P is just the reciprocal

$$A = P[\frac{i(1+i)^N}{(1+i)^N - 1}]$$

= P x capital recovery factor

= P x (A/P, i%, N)

(This is, of course, identical to Equation (1) above.)

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There is no need to memorize the formulae. Spreadsheets and many calculators have these functions pre-programmed. Also, most texts on engineering economy have tables of values for these factors for a range of values of i% and N.

You should, however, be familiar with the concepts that these factors represent.
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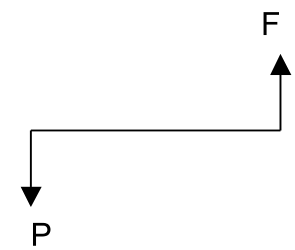
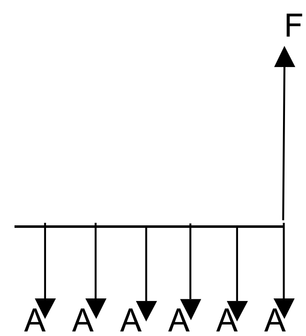
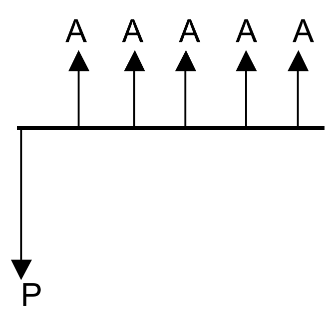
Factor Name	Factor Notation	Formula	Cash Flow Diagram
Future worth factor (compound amount factor)	(F/P, i, N)	$F=P(1+i)^N$	
Present worth factor	(P/F, i, N)	$P=F(1+i)^{-N}$	
Uniform series compound amount factor (aka future-worth-of-an-annuity factor)	(F/A, i, N)	$F = A\left[\frac{(1+i)^N - 1}{i}\right]$	
Sinking fund factor	(A/F, i, N)	$A = F\left[\frac{i}{(1+i)^N - 1}\right]$	
Present worth of an annuity factor	(P/A, i, N)	$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$	
Capital recovery factor	(A/P, i, N)	$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$	

Special properties of time value factors

When N --> ∞, the capital recovery factor, (A/P,i%, N) = i

This is just another way of saying that if you deposit a sum in a bank paying a constant interest rate i%/year, you can draw interest on it in perpetuity.

Conversely, (P/A, i%, N --> ∞) --> 1/i

i.e., the present worth of an annuity of <u>infinite</u> duration is <u>finite</u> and equal to the value of one payment divided by the interest rate.

<u>Non-Uniform Series</u>

It is also possible to derive time value factors for non-uniform cash flows. For example:

- arithmetic gradient series (or linear gradient series -- cash flows increase by a constant amount)

- geometric gradient series (cash flows increase by a constant percentage)

See Park and Sharp-Bette for these (p.51-57).