24.118: Paradox and Infinity, Spring 2019
Problem Set 2: The Higher Infinite

How these problems will be graded:

- In Part I there is no need to justify your answers. Assessment will be based on whether your answers are correct.
- In Part II you must justify your answers. Assessment will be based both on whether you give the correct answer and on how your answers are justified. (In some problem sets I will ask you to answer questions that don't have clear answers. In those cases, assessment will be based entirely on the basis of how your answer is justified. Even if it is unclear whether your answer is correct, it should be clear whether or not the reasons you have given in support of your answer are good ones.)
- *No answer may consist of more than 150 words.* Longer answers will not be given credit. (Showing your work in a calculation, a proof, or a computer program does not count towards the word limit.)
- You may consult published literature and the web. You must, however, credit all sources. Failure to do so constitutes plagiarism and can have serious consequences. For advice about when and how to credit sources see: https://integrity.mit.edu/

Part I

1. Determine whether each of the following statements is true or false. (3 points each)

   (a) $\wp^n(\mathbb{N}) \subseteq \wp^m(\mathbb{N})$, for $n < m$ and $n, m \in \mathbb{N}$.[1]

   (b) $|\wp^n(\mathbb{N})| < |\wp^m(\mathbb{N})|$, for $n < m$ and $n, m \in \mathbb{N}$.

   (c) $\wp^n(\mathbb{N}) \subseteq \{\wp^m(\mathbb{N}) : m \in \mathbb{N}\}$, $n \in \mathbb{N}$.

   (d) $\wp^n(\mathbb{N}) \subseteq \bigcup\{\wp^m(\mathbb{N}) : m \in \mathbb{N}\}$, $n \in \mathbb{N}$.

   (e) $|\wp^n(\mathbb{N})| < |\{\wp^m(\mathbb{N}) : m \in \mathbb{N}\}|$, $n \in \mathbb{N}$

   (f) $|\wp^n(\mathbb{N})| < |\bigcup\{\wp^m(\mathbb{N}) : m \in \mathbb{N}\}|$, $n \in \mathbb{N}$

   Notation:

   - $\mathbb{N}$ is the set of natural numbers.
   - $\wp^n(A) = \underbrace{\wp(\wp(\ldots\wp(}_{n \text{ times}} A \underbrace{)\ldots))}_{n \text{ times}}$ $(n \in \mathbb{N})$.

[1] You may assume that the natural numbers are not sets.

- $\bigcup A = \{x : x \in B \text{ for } B \in A\}$.

2. Determine whether each of the following statements is true or false. (2 points each)

   (a) $0' + 0''' = 0''' + 0'$
   (b) $0' \times 0''' = 0''' \times 0'$
   (c) $0' + \omega = \omega' + 0$
   (d) $0' + \omega = 0 + \omega'$
   (e) $0''' \times \omega = (\omega + \omega) + \omega$
   (f) $\omega \times 0''' = \omega + (\omega + \omega)$
   (g) $(\omega \times 0'') + \omega <_o (\omega \times \omega) + 0''$
   (h) $\omega \times \omega <_o \omega \times (0'' \times \omega)$
   (i) $\omega \times (\omega + \omega) = (\omega \times \omega) + (\omega \times \omega)$
   (j) $\alpha + 0' = \alpha \cup \{\alpha\}$ (α an ordinal)

Part II

3. Draw a diagram (or use prose[2]) to give an informal characterization of the well-ordering types represented by each of the following ordinals. (4 points each)

   (a) $(\omega \times \omega) + 0'''$
   (b) $(\omega + 0''') \times 0'''$
   (c) $(0''' \times \omega) \times 0'''$

   [Correction: This problem should have been listed under Part I. So there is no need to justify your answer. Simply draw your diagram or write down your prose.]

4. Using an example not found in the course material:

   (a) Specify (i) a set whose members are not numbers and (ii) an ordering on that set that is not a (strict) total ordering. (5 points)
   (b) Specify (i) a set of any kind and (ii) a (strict) total ordering on that set that is not a well-ordering. (5 points)

5. If α and β are ordinals, does $\alpha <_o \beta$ entail $|\alpha| < |\beta|$? If so explain, why. If not give a counterexample. (10 points)

[2] What does it mean to use prose to give an informal characterization of a well-order type? Suppose, for example, that the well-order type in question corresponded to ω. Then you might say something like "A countably infinite sequence of items which is ordered like the natural numbers, with a first member, a second member, and so forth—but no last member."

6. Recall the Ordinal Construction Principle:

   Construction Principle At each stage, we introduce a new ordinal, namely: the set of all ordinals that have been introduced at previous stages.

   Use this Principle to show that if α is an ordinal with infinitely many members, then either $\alpha = \omega$ or $\omega <_o \alpha$. (10 points)

7. Give an example of a set whose cardinality is greater than the cardinality of each $\wp^n(\mathscr{U})$ ($n \in \mathbb{N}$), where $\mathscr{U} = \bigcup\{\mathbb{N}, \wp(\mathbb{N}), \wp(\wp(\mathbb{N})), \dots\}$. (10 points; don't forget to justify your answer.)

8. Give an example of a set of cardinality "much greater" than $|\mathfrak{B}_{\omega\times\omega}|$, where

$$\mathfrak{B}_\alpha = \begin{cases} \mathbb{N}, \text{ if } \alpha = 0 \\ \wp(\mathfrak{B}_\beta), \text{ if } \alpha = \beta' \\ \bigcup\{\mathfrak{B}_\gamma : \gamma <_o \alpha\} \text{ if } \alpha \text{ is a limit ordinal greater than } 0 \end{cases}$$

   What does "much greater" mean here? It must be possible to show that there are infinitely many sizes of infinity between the set you identify and $|\mathfrak{B}_{\omega\times\omega}|$. (10 points; don't forget to justify your answer.)