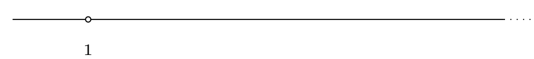
Banach-Tarski: Preliminaries

1 The Theorem

Banach-Tarski Theorem It is possible to decompose a ball into a finite number of pieces and reassemble the pieces (without changing their size or shape) so as to get two balls, each of the same size as the original.

1.1 Warm-Up Case 1: A Line

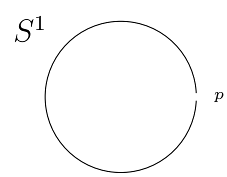
It is possible to decompose $[0, \infty) - \{1\}$ into two distinct parts, and reassemble the parts (without changing their size or shape) so as to get back $[0, \infty)$.



- Decompose $[0, \infty) - \{1\}$ into: (i) $\{2, 3, 4, \ldots\}$ and (ii) everything else.
- Translate $\{2, 3, 4, \ldots\}$ one unit to the left.

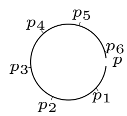
1.2 Warm-Up Case 2: A Circle

It is possible to decompose $S^1 - \{p\}$ into two distinct parts, and reassemble the parts (without changing their size or shape) so as to get back S^1.



- Decompose $S^1 - \{p\}$ into: (i) B and (ii) everything else.
- Rotate B one unit counter-clockwise.

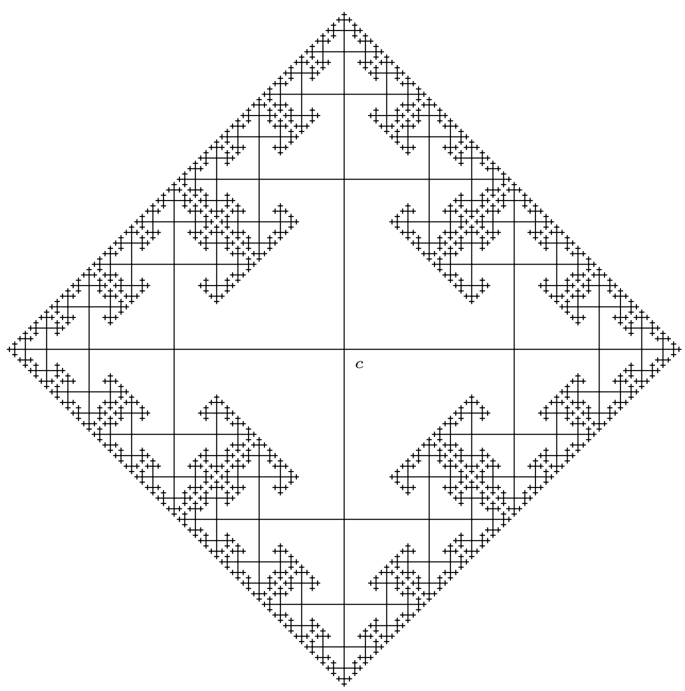
$$B = \left\{x \in S^1 : x \text{ is } n \text{ units clockwise from } p \ (n \in \mathbb{Z}^+)\right.$$



The first six members of B.

1.3 Warm-Up Case 3: The Cayley Graph

It is possible to decompose (the set of endpoints of) the Cayley Graph[1] into four distinct parts, and reassemble the parts (albeit changing their size) so as to get back *two copies* of the same size as the original.



- Decompose C^e into quadrants: L^e, R^e, U^e, D^e.

[1] A Cayley Path is a finite sequence of steps starting from c, where no step follows its inverse. The Cayley Graph C is the set of Cayley Paths. X^e is the set of endpoints of Cayley paths in X.

- Make first copy by expanding R^e and translating left to meet L^e.
- Make second copy by expanding U^e and translating down to meet D^e.

1.4 A more abstract description of the procedure

> *Notation:* if X is a set of Cayley Paths, let $\overleftarrow{X}$ be the set that results from eliminating the first step from each of the Cayley Paths in X.

By the definition of Cayley Paths:

(α) $C = \overleftarrow{R} \cup L$

(β) $C = \overleftarrow{D} \cup U$

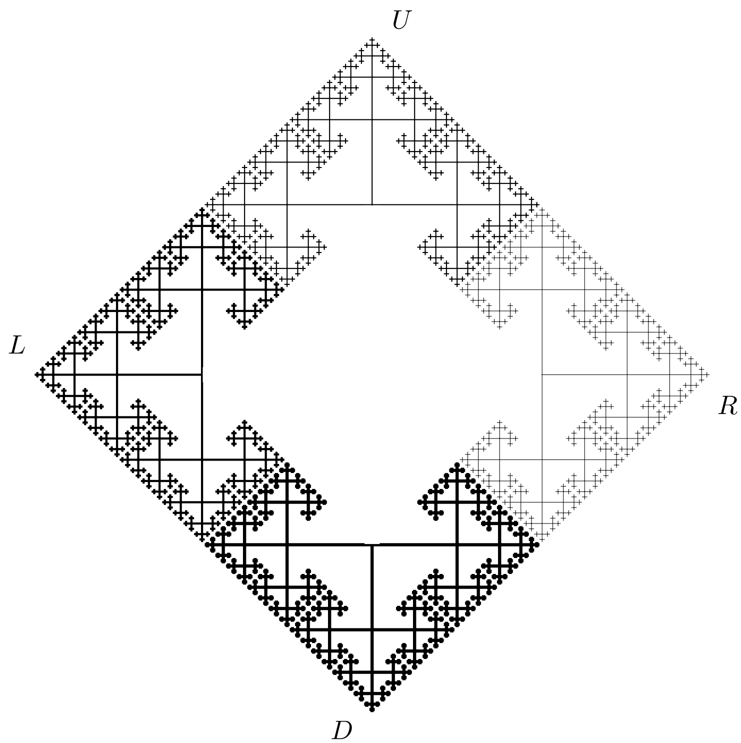
Since every Cayley Path has a unique endpoint, (α) and (β) entail:

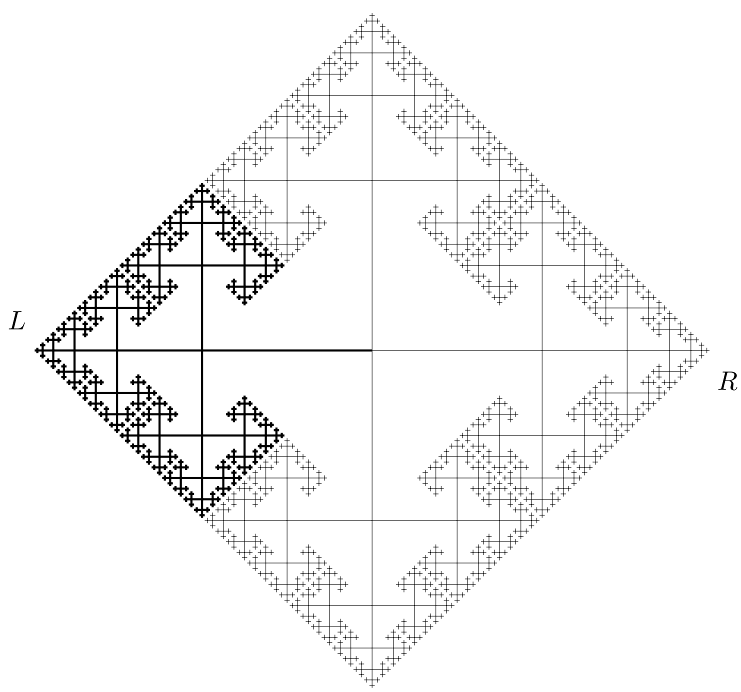
(α') $C^e = \left(\overleftarrow{R}\right)^e \cup L^e$

(β') $C^e = \left(\overleftarrow{D}\right)^e \cup U^e$

On our two-dimensional interoperation of the Cayley Graph, this delivers the intended result because:

1. C^e is decomposed into U^e, D^e, L^e and R^e (ignoring the central vertex)
2. One can get from R^e to $\left(\overleftarrow{R}\right)^e$, and from D^e to $\left(\overleftarrow{D}\right)^e$, by performing a translation together with an expansion.


U
L
R
D


L
R