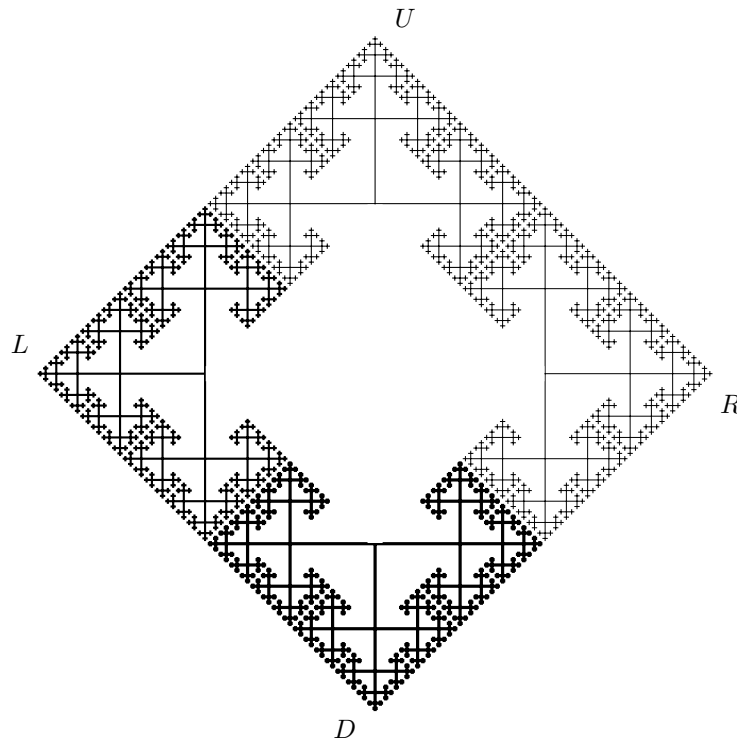# Banach-Tarski: The Theorem

1 The Theorem

Banach-Tarski Theorem It is possible to decompose a ball into a finite number of pieces and reassemble the pieces (without changing their size or shape) so as to get two balls, each of the same size as the original.

2 The basic idea

By the definition of Cayley Paths:[1]

(α) $C = \overleftarrow{R} \cup L$

(β) $C = \overleftarrow{D} \cup U$

So we have:

(α′) $C^e = \left(\overleftarrow{R}\right)^e \cup L^e$

(β′) $C^e = \left(\overleftarrow{D}\right)^e \cup U^e$

On our two-dimensional interoperation of the Cayley Graph, we have:

- Different Cayley Paths have different endpoints.

  [So C^e is decomposed into U^e, D^e, L^e and R^e, ignoring the center.]

- One can get from R^e to $\left(\overleftarrow{R}\right)^e$, and from D^e to $\left(\overleftarrow{D}\right)^e$, by performing a translation together with an expansion.

  [So our duplication requires only translations and expansions.]

Now we're going to work with a **different interpretation** of the Cayley Graph, on which the graph is wrapped around the surface of a ball.

On this other interpretation, we'll have:

- Different Cayley Paths have different endpoints.[2]

  [So C^e is decomposed into U^e, D^e, L^e and R^e, ignoring the center.]

- One can get from R^e to $\left(\overleftarrow{R}\right)^e$, and from D^e to $\left(\overleftarrow{D}\right)^e$, by performing a *rotation*.

  [So our duplication requires only rotations.]

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[1] If X is a set of Cayley Paths, X^e is the set of endpoints of paths in X and $\overleftarrow{X}$ is the set that results from eliminating the first step from each of the Cayley Paths in X.

[2] Or close enough... See below!

3   An external coordinate system

- The x-axis runs from your right to your left through the center of the ball.

- The y-axis runs is orthogonal to the x-axis. It runs from the wall in front of you to the wall in behind of you, through the center of the ball.

- The the z-axis is orthogonal to the other two. It runs from the ground to the sky, through the center of the ball.

4   A spherical interpretation of the Graph

- The "center" of our graph is interpreted as an arbitrary point c on the surface of our ball.

- A "step" is interpreted as the result of performing a *rotation* on the sphere, by a certain angle θ:

    - An "up" rotation is a counterclockwise rotation of θ degrees about the x axis. (When you're holding the ball in front of you, you perform this rotation by rotating the ball from bottom to top.)
    - A "down" rotation is a clockwise rotation of θ degrees about the x axis. (When you're holding the ball in front of you, you perform this rotation by rotating the ball from top to bottom.)
    - A "right" rotation is a counterclockwise rotation of θ degrees about the z axis. (When you're holding the ball in front of you, you perform this rotation by rotating the ball from left to right.)
    - A "left" rotation is a clockwise rotation of θ degrees about the z axis. (When you're holding the ball in front of you, you perform this rotation by rotating the ball from right to left.)

5   The endpoints of Cayley Paths

- To each rotation ρ corresponds a function f_ρ, which takes each point p on the surface of the ball to the point on the surface of the ball whose current location (relative to an external reference frame) would come to be occupied by p were rotation ρ to be performed.

- The "endpoint" of Cayley Path $\langle \rho_1, \rho_2, \ldots, \rho_n \rangle$ is the point

$$f_{\rho_1}(f_{\rho_2}(\ldots f_{\rho_n}(c)\ldots))$$

6 The key results

When the rotation θ is chosen properly,[3] we get a nice result:

- **Result 1:** Different Cayley Paths always have different endpoints.[4]

  In other words: if $\langle \rho_1, \rho_2, \ldots, \rho_n \rangle$ and $\langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle$ are distinct Cayley Paths, then

  $$f_{\rho_1}(f_{\rho_2}(\ldots f_{\rho_n}(c)\ldots)) \neq f_{\sigma_1}(f_{\sigma_2}(\ldots f_{\sigma_m}(c)\ldots))$$

- **Result 2:** One can get from R^e to $\left(\overleftarrow{R}\right)^e$, and from D^e to $\left(\overleftarrow{D}\right)^e$, by performing a rotation.

7 Duplicating the Surface of the Ball

- Partition the surface of the ball into cells, with two points are in same cell if they are linked by a Cayley Path.

- Choose a "center" for each cell.[5]

  (This delivers a partition of the surface of the ball into Cayley Graphs.)

- Perform the duplication procedure to each Cayley Graph symultaneously.

7.1 A complication

Wait! For $\langle \rho_1, \rho_2, \rho_3, \ldots \rangle$ a Cayley Path:

- $f_{\rho_1}(f_{\rho_2}(f_{\rho_r}(\ldots)))$ is a rotation (by Euler's Theorem).

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[3]For instance: $\theta = \arccos(1/3) \approx 70.53°$

[4]Annoyingly, this result holds for *almost* every choice of our "central" point c but not quite every choice. I'll come back to this. . .

[5]Rather than specifying a center for each cell, we use the Axiom of Choice to prove that a set of cell-centers exists.

- So $f_{\rho_1}(f_{\rho_2}(f_{\rho_r}(\dots)))$ does not change the location of the points intersecting its axis of rotation.

- So whenever c is such that some such problem point is the endpoint of some Cayley Path, we will have different Cayley Paths sharing an endpoint.

The good news:

- There are only countably many problem points.

- One can deal with these points separately, by applying a sophisticated version of the trick we used in Warm-Up Case 2.

8   Duplicating the Ball

- Use the same procedure as before, but rather than working with points on the surface of the ball, work with the lines that connect the center of the ball with each point.

- Wait! What about the center of the ball?

9   A region with no volume

Some of the Banach-Tarski pieces must be non-measurable must therefore lack definite volumes!

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