In most of these proofs, when things have got a little complicated, I have numbered the steps I'm taking in the hope that this makes things clearer. You aren't required to do this when you answer, but I think it's probably a good idea — it helps you get clear on what, exactly, you are doing, and it helps me understand what you are doing (if I can't follow your proof, that's bad).

Section 5.3E, Question 14

In what follows, I will refer to the following sentence:

- $\Gamma \vdash \mathbf{P}$ in *SD* if and only if $\Gamma \models \mathbf{P}$.

as 'S&C' (short for 'Soundness and Completeness' — you'll see why I use this name in the near future).

Part (a)

Let α be an argument of *SL* such that the set of assumptions that begin α is Γ and the conclusion of α is $\mathbf{P}$ (I'm using 'α' so you don't confuse it with a sentence letter of *SL*, but you can use whatever you like).

1. α is valid in SD iff there is an *SD* derivation that has the members of Γ as primary assumptions and $\mathbf{P}$ in the scope of those assumptions only (by definition of 'valid in SD').

2. There is an *SD* derivation that has the members of Γ as primary assumptions and $\mathbf{P}$ in the scope of those assumptions only iff $\Gamma \vdash \mathbf{P}$ in *SD* (by definition of '$\vdash$').

3. $\Gamma \vdash \mathbf{P}$ in *SD* iff $\Gamma \models \mathbf{P}$ (S&C).

4. $\Gamma \models \mathbf{P}$ iff there is no truth-value assignment such that every member of Γ is true and $\mathbf{P}$ is false (by definition of '$\models$').

5. There is no truth-value assignment such that every member of Γ is true and $\mathbf{P}$ is false iff α is truth-functionally valid (by definition of 'truth-functionally valid').

So, assuming S&C, an argument of *SL* is valid in *SD* if and only if the argument is truth-functionally valid.
Q.E.D.

Part (b)

1. A sentence $\mathbf{P}$ of *SL* is a theorem in *SD* iff $\emptyset \vdash \mathbf{P}$ in *SD* (by definition of theoremhood).

2. $\emptyset \vdash \mathbf{P}$ in *SD* iff $\emptyset \models \mathbf{P}$ (by S&C).

3. $\emptyset \models$ **P** iff there is no truth-value-assignment that makes every member of $\emptyset$ true and **P** false (by definition of '$\models$').

4. There is no truth-value-assignment that makes every member of $\emptyset$ true and **P** false iff there is no truth-value assignment that makes **P** false (as every truth-value assignment makes every member of $\emptyset$ true).

5. There is no truth-value assignment that makes **P** false iff **P** is truth-functionally true (definition of 'truth-functionally true').

So, assuming S&C, a sentence **P** of *SL* is a theorem in *SD* if and only if **P** is truth-functionally true.

Q.E.D.

Part (c)

1. Sentences **P** and **Q** of *SL* are equivalent in *SD* iff $\{\mathbf{P}\} \vdash \mathbf{Q}$ in *SD* and $\{\mathbf{Q}\} \vdash \mathbf{P}$ in *SD* (definition of 'equivalent in *SD*').

2. $\mathbf{P} \vdash \mathbf{Q}$ in *SD* and $\mathbf{Q} \vdash \mathbf{P}$ in *SD* iff $\mathbf{P} \models \mathbf{Q}$ and $\mathbf{Q} \models \mathbf{P}$ (by S&C).

3. $\mathbf{P} \models \mathbf{Q}$ and $\mathbf{Q} \models \mathbf{P}$ iff there is no truth-value assignment such that **P** is true and **Q** is false, and vice-versa. (by definition of '$\models$').

4. There is no truth-value assignment such that **P** is true and **Q** is false, and vice-versa, iff **P** and **Q** are truth-functionally equivalent (by definition of 'truth-functionally equivalent').

So, assuming S&C, sentences **P** and **Q** of *SL* are equivalent in *SD* if and only if **P** and **Q** are truth-functionally equivalent.

Q.E.D.

Section 6.1E, Question 1

Part (b)

To show:

CLAIM: Every sentence of *SL* that contains no binary connectives is truth-functionally indeterminate.

CLAIM follows from the following...

BASIS CLAUSE: Every atomic sentence of *SL* is truth-functionally indeterminate.

INDUCTIVE STEP: If every sentence of *SL* containing (a) no binary connectives and (b) n or fewer negations is truth-functionally indeterminate, then so is every sentence of *SL* containing no binary connectives and $n+1$ negations.

. . . as every sentence of *SL* that contains no binary connectives is a sentence of *SL* that contains no binary connectives and n negations, for some natural number n.

The proof of Basis Clause is immediate — every atomic sentence of SL is such that there is a truth-value assignment that makes it true and a truth-value assignment that makes it false, so every atomic sentence of *SL* is truth-functionally indeterminate. It remains to prove Inductive Step.

Proof of Inductive Step:

1. Suppose every sentence **P** of *SL* containing (a) no binary connectives and (b) n or fewer negations is truth-functionally indeterminate (i.e., suppose the antecedent of Inductive Step).

2. Then for all such **P**, there exists a truth-value assignment that make **P** true and a truth-value assignments that makes **P** false.

3. So, by the definition of '∼', there is truth-value assignment that makes ⌜∼ **P**⌝ false and a truth-value assignment that make ⌜∼ **P**⌝ true, for all such **P**.

4. But every sentence of *SL* containing no binary connectives and $n + 1$ negations is of the form ⌜∼ **P**⌝, for some such **P**.

5. So for every sentence of *SL* containing no binary connectives and $n + 1$ negations, there is truth-value assignment that makes it true and a truth-value assignment that makes it false.

6. So every sentence of *SL* containing no binary connectives and $n + 1$ negations is truth-functionally indeterminate.

So, if every sentence of *SL* containing (a) no binary connectives and (b) n or fewer negations is truth-functionally indeterminate, then so is every sentence of *SL* containing no binary connectives and $n + 1$ negations.

Q.E.D.

Part (e)

Where **P** is a sentence of *SL* and **Q** is a sentential component of **P**, let $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ be a sentence that is the result of replacing at least one occurrence of **Q** in **P** with the sentence $\mathbf{Q}_1$.

To show:

Claim: If **Q** and $\mathbf{Q}_1$ are truth-functionally equivalent, then **P** and $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ are truth-functionally equivalent.

Clearly, Claim follows from the following. . .

Basis Clause: Claim is true when **P** is atomic.

Inductive Step: If Claim is true for all **P** containing n or fewer connectives, it is true for all **P** containing $n+1$ connectives.

...as every sentence of *SL* contains n connectives, for some natural number n.

Proof of Basis Clause:

1. $[\mathbf{P}](\mathbf{Q}_1//\mathbf{P})$ is just $\mathbf{Q}_1$.
2. So if $\mathbf{Q}_1$ is truth-functionally equivalent to **P**, then obviously $[\mathbf{P}](\mathbf{Q}_1//\mathbf{P})$ is truth-funcitonally equivalent to **P**.
3. But when **P** is atomic, it's only sentential component is **P**.
4. So, for all sentential components **Q** of **P**, if **Q** and $\mathbf{Q}_1$ are truth-functionally equivalent, then **P** and $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ are truth-functionally equivalent, when **P** is atomic.

So Claim is true when **P** is atomic.
Q.E.D.

Proof of Inductive Step: Every sentence of *SL* containing $n+1$ connectives is either of the form $\ulcorner\sim \mathbf{P}\urcorner$, for some **P** containing n connectives, or is of the form $\mathbf{P}\cdot\mathbf{R}$, where $\mathbf{P},\mathbf{R}$ contain n or fewer connectives (and '·' is a variable that ranges over binary connectives of *SL*). I prove Inductive Step for each case in turn.

Case 1: Consider a sentence of *SL* of the form $\ulcorner\sim \mathbf{P}\urcorner$, where **P** is a sentence containing n connectives. Every sentential component **Q** of $\ulcorner\sim \mathbf{P}\urcorner$ is either

(a) $\ulcorner\sim \mathbf{P}\urcorner$ itself, or

(b) is a sentential component of **P**.

I prove each sub-case in turn.

Sub-case (a): When **Q** is $\ulcorner\sim \mathbf{P}\urcorner$ itself, $[\ulcorner\sim \mathbf{P}\urcorner](\mathbf{Q}_1//\mathbf{Q})$ is truth-funcitonally equivalent to $\ulcorner\sim \mathbf{P}\urcorner$, for $\mathbf{Q}_1$ truth-functionally equivalent to **Q**, by the same argument as in the proof of Basis Clause.

Sub-case (b):

1. Suppose **Q** is a sentential component of **P**.
2. Suppose, also, that **P** and $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ are truth-functionally equivalent when $\mathbf{Q}_1$ truth-functionally equivalent to **Q** (i.e., suppose the antecedent of Inductive Step for the case of **P**).
3. Then $\ulcorner\sim \mathbf{P}\urcorner$ and $\ulcorner\sim ([\mathbf{P}](\mathbf{Q}_1//\mathbf{Q}))\urcorner$ are truth-functionally equivalent (by the definition of '∼').
4. And $\ulcorner\sim ([\mathbf{P}](\mathbf{Q}_1//\mathbf{Q}))\urcorner$ is identical to $[\ulcorner\sim \mathbf{P}\urcorner](\mathbf{Q}_1//\mathbf{Q})$, when **Q** is a sentential component of **P**.

5. So, if $\mathbf{P}$ and $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ are truth-functionally equivalent, then $\ulcorner\sim \mathbf{P}\urcorner$ is truth-functionally equivalent to $[\ulcorner\sim \mathbf{P}\urcorner](\mathbf{Q}_1//\mathbf{Q})$, for $\mathbf{Q}_1$ truth-functionally equivalent to $\mathbf{Q}$, when $\mathbf{Q}$ is a sentential component of $\mathbf{P}$.

So, if CLAIM is true for a sentence $\mathbf{P}$ containing n connectives, it is true for $\ulcorner\sim \mathbf{P}\urcorner$. That concludes the proof for Case 1.

Case 2: Consider a sentence of *SL* of the form $\mathbf{P} \cdot \mathbf{R}$, where $\mathbf{P}, \mathbf{R}$ are sentences containing n or fewer connectives. Every sentential component of $\mathbf{P} \cdot \mathbf{R}$ is either

(a) $\mathbf{P} \cdot \mathbf{R}$ itself,

(b) a sentential component of $\mathbf{P}$ or a sentential component of $\mathbf{R}$ (or both).

I prove each sub-case in turn.

Sub-case (a): The proof here is the same as the proof of BASIS CLAUSE and sub-case (a) of Case 1, *mutatis-mutandis*.

Sub-case (b):

1. Suppose $\mathbf{Q}$ a sentential component of $\mathbf{P}$ or $\mathbf{R}$ or both.

2. Suppose, also, that $\mathbf{P}$ and $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ are truth-functionally equivalent, and $\mathbf{R}$ and $[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$ are truth-functionally equivalent, when $\mathbf{Q}_1$ is truth-functionally equivalent to $\mathbf{Q}$ (i.e., suppose the antecedent of INDUCTIVE STEP for the cases of $\mathbf{P}$ and $\mathbf{R}$).

3. Then $\mathbf{P} \cdot \mathbf{R}$ is truth-functionally equivalent to $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q}) \cdot [\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$ (by the relevant binary-connective's defintion).

4. And $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q}) \cdot [\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$ is identical to $[\mathbf{P} \cdot \mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$, when $\mathbf{Q}$ is a sentential component of $\mathbf{P}$ or $\mathbf{R}$.

5. So, if $\mathbf{P}$ and $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ are truth-functionally equivalent, and $\mathbf{R}$ and $[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$ are truth-funcitonally equivalent, then $\mathbf{P} \cdot \mathbf{R}$ is truth-functionally equivalent to $[\mathbf{P} \cdot \mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$, for $\mathbf{Q}_1$ truth-functionally equivalent to $\mathbf{Q}$, when $\mathbf{Q}$ is a sentential component of $\mathbf{P}$ or $\mathbf{R}$.

So, if CLAIM is true for sentences $\mathbf{P}, \mathbf{R}$ containing n or fewer connectives, it is true for $\mathbf{P} \cdot \mathbf{R}$.

That concludes the proof for Case 2.

So that concludes the proof for INDUCTIVE STEP.

Q.E.D.

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