

Subject 24.242. Logic II. HW3 Sample Answers

For each term τ , we have defined a code number $\ulcorner \tau \urcorner$, according to the following prescription:

$$\begin{aligned}\ulcorner 0 \urcorner &= \text{Pair}(1,1). \\ \ulcorner x_i \urcorner &= \text{Pair}(2,i). \\ \ulcorner s\tau \urcorner &= \text{Pair}(4, \ulcorner \tau \urcorner) \\ \ulcorner \tau + \rho \urcorner &= \text{Pair}(5, \text{Pair}(\ulcorner \tau \urcorner, \ulcorner \rho \urcorner)). \\ \ulcorner \tau \cdot \rho \urcorner &= \text{Pair}(6, \text{Pair}(\ulcorner \tau \urcorner, \ulcorner \rho \urcorner)). \\ \ulcorner \tau \in \rho \urcorner &= \text{Pair}(7, \text{Pair}(\ulcorner \tau \urcorner, \ulcorner \rho \urcorner)).\end{aligned}$$

$\text{Pair}(x,y)$ is, you will recall, $\frac{1}{2}(x+y)(x+y+1) + x$.

1. Give the Arabic numeral for $\ulcorner (0 + 0) \urcorner$.
 $\text{Triple}(5, \ulcorner 0 \urcorner, \ulcorner 0 \urcorner) = \text{Pair}(5, \text{Pair}(\ulcorner 0 \urcorner, \ulcorner 0 \urcorner)) = \text{Pair}(5, \text{Pair}(4,4)) = \text{Pair}(5,40) = 1040$.
2. Show that a set of natural numbers is decidable if and only if it is either finite or the range of an increasing calculable total function. (A total function f is *increasing* iff, for any x and y , if $x < y$, then $f(x) < f(y)$.)
(\Rightarrow) If S is infinite, it is the range of the following increasing total function:

$$\begin{aligned}f(0) &= \text{the least element of } S. \\ f(n+1) &= \text{the least element of } S \text{ greater than } f(n).\end{aligned}$$

If S is decidable, f can be calculated by testing the natural numbers, one after another, for membership in S .

(\Leftarrow) A finite set is obviously decidable, just by incorporating a list of the set into the program. If S is the range of an increasing, calculable total function f , we can test whether n is an element of S by calculating $f(0), f(1), f(2)$, and so on, until we reach an i with $f(i) \geq n$. If $f(i) = n$, then n is in S . If $f(i) > n$, then $n \notin S$.