Subject 24.242. Logic II. Answers to the last homework assignmetn

Recall that a *normal modal system* for the modal sentential calculus is a set of formulas $\Gamma$ that meets the following conditions:

(TC) Every tautological consequence of $\Gamma$ is in $\Gamma$.
(Nec) If $\phi$ is in $\Gamma$, so in $\Box\phi$.
(K) All instances of the schema $(\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi))$ are in $\Gamma$.

1. A binary relation *R* on a set *W* is *symmetric* iff, for every *v* and *w* in *W*, if *Rwv* then *Rvw*. Let KB be the smallest normal modal system that contains all instances of the schema
(B) $(\Diamond\Box\phi \rightarrow \phi)$
Show that a sentence is in KB if and only if it's valid for the class of frames <*W,R,I*>, with *R* symmetric.

**First, we show that (B) is valid for the class of symmetric frames. Suppose that R is symmetric and that $\Diamond\Box\phi$ is true at the world w in the frame <W,R,I>. Then there is a world v accessible from w in which $\Box\phi$ is true. So $\phi$ is true in every world accessible from v. In particular, $\phi$ is true in w, since, by symmetry, w is accessible from v. So $(\Diamond\Box\phi \rightarrow \phi)$ is valid in <W,R,I>.**

**Let $\Gamma$ be the set of sentences valid for the class of symmetric frames. $\Gamma$ is a normal modal system that includes (B), and so $\Gamma$ includes KB. We need to show that, if a sentence $\phi$ isn't in KB, it isn't in $\Gamma$. That is, we need to show that, if $\phi$ isn't in KB, then there is a symmetric frame in which there is a world in which $\phi$ is false. We know that the canonical frame for KB contains a world in which $\phi$ is false; so it will be enough to show that the canonical frame for KB is symmetric.**

**Suppose that w and v are worlds in the canonical frame for KB and that Rwv. We need to see that Rvw, that is, we need to see that, whenever $\Box\psi$ is in v, $\psi$ is in w. Since $\Box\psi$ is true in v, $\Diamond\Box\psi$ is true in every world that has access to v; in particular, $\Diamond\Box\psi$ is true in w. Since $(\Diamond\Box\psi \rightarrow \psi)$ is true in w, it follows that $\psi$ is true in w, and so $\psi \in$ w.**

5. Prove de Jongh's theorem that all instances of schema
(4) $(\Box\phi \rightarrow \Box\Box\phi)$
are elements of the smallest normal modal system that includes all instances of the schema:
(L) $(\Box(\Box\phi \rightarrow \phi) \rightarrow \Box\phi)$.
[Hint: The instance of schema (L) that you'll use is $(\Box(\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi)) \rightarrow \Box(\phi \wedge \Box\phi))$.]

| | | |
|---|---|---|
| **1.** | $((\phi \wedge \Box\phi) \rightarrow \phi)$ | **(TC)** |
| **2.** | $\Box((\phi \wedge \Box\phi) \rightarrow \phi)$ | **(Nec), 1** |
| **3.** | $(\Box((\phi \wedge \Box\phi) \rightarrow \phi) \rightarrow (\Box(\phi \wedge \Box\phi) \rightarrow \Box\phi))$ | **(K)** |
| **4.** | $(\Box(\phi \wedge \Box\phi) \rightarrow \Box\phi)$ | **(TC), 2, 3** |
| **5.** | $(\phi \rightarrow (\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi)))$ | **(TC), 4** |
| **6.** | $\Box(\phi \rightarrow (\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi)))$ | **(Nec), 5** |

| | | |
|---|---|---|
| 7. | $(\Box(\phi \rightarrow (\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi))) \rightarrow (\Box\phi \rightarrow \Box(\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi))))$ | (K) |
| 8. | $(\Box\phi \rightarrow \Box(\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi)))$ | (TC), 6, 7 |
| 9. | $(\Box(\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi)) \rightarrow \Box(\phi \wedge \Box\phi))$ | (L) |
| 10. | $(\Box\phi \rightarrow \Box(\phi \wedge \Box\phi))$ | (TC), 8, 9 |
| 11. | $((\phi \wedge \Box\phi) \rightarrow \Box\phi)$ | (TC) |
| 12. | $\Box((\phi \wedge \Box\phi) \rightarrow \Box\phi)$ | (Nec), 11 |
| 13. | $(\Box((\phi \wedge \Box\phi) \rightarrow \Box\phi) \rightarrow (\Box(\phi \wedge \Box\phi) \rightarrow \Box\Box\phi))$ | (K) |
| 14. | $(\Box(\phi \wedge \Box\phi) \rightarrow \Box\Box\phi)$ | (TC), 12, 13 |
| 15. | $(\Box\phi \rightarrow \Box\Box\phi)$ | (TC), 10, 14 |