

Subject 24.242. Answers to HW5 Sample

Here are some theorems of PA that we derived either in class or in the notes. You may want to use them in the problems below: (For reference, I'll refer to these theorems as (D1) through (D6):

- (D1)  $(\forall x)(\forall y)(x + y) = (y + x)$
- (D2)  $(\forall x)(\forall y)(\forall z)((x + y) + z) = (x + (y + z))$
- (D3)  $(\forall x)(x \cdot [1]) = x$
- (D4)  $(\forall x)(\forall y)(x \cdot y) = (y \cdot x)$
- (D5)  $(\forall x)(\forall y)(\forall z)((x \cdot y) \cdot z) = (x \cdot (y \cdot z))$
- (D6)  $(\forall x)(\forall y)(\forall z)((x + y) \cdot z) = ((x \cdot z) + (y \cdot z))$

1. Show that " $(\forall x)(\forall y)(\forall z)(x E (y+z)) = ((xEy) \cdot (xEZ))$ " is a theorem of PA. We prove this by induction. The basis clause is " $(x E (y + 0)) = ((xEy) \cdot (xE0))$ ," proved as follows:

$$\begin{aligned} (x E (y + 0)) & \\ = (x E y) & \quad \text{[by (Q3)]} \\ = ((x E y) \cdot [1]) & \quad \text{[by (D3)]} \\ = ((x E y) \cdot (x E 0)) & \quad \text{[by (Q7)]} \end{aligned}$$

Next, for the induction step. Assume as inductive hypothesis:

$$(x E (y + z)) = ((x E y) \cdot (x E z)).$$

We want to show:

$$(x E (y + sz)) = ((x E y) \cdot (x E sz)).$$

$$\begin{aligned} (x E (y + sz)) & \\ = (x E s(y + z)) & \quad \text{[by (Q4)]} \\ = ((x E (y + z)) \cdot x) & \quad \text{[by (Q8)]} \\ = (((x E y) \cdot (x E z)) \cdot x) & \quad \text{[by IH]} \\ = ((x E y) \cdot ((x E z) \cdot x)) & \quad \text{[by (D5)]} \\ ((x E y) \cdot (x E sz)) & \quad \text{[by (Q8)]} \end{aligned}$$

6. Show that, where  $\mathfrak{A}$  is a nonstandard model of PA, there isn't any formula  $\phi(x)$  of the language arithmetic that is satisfied by all the standard numbers in  $\mathfrak{A}$  but isn't satisfied by any of the nonstandard elements of  $\mathfrak{A}$ .

If  $\phi(x)$  were such a formula, that, because 0 is standard,  $\phi(0)$  would hold in  $\mathfrak{A}$ , and because the successor of a standard element is always standard,  $(\forall x)(\phi(x) \rightarrow \phi(sx))$  would also hold in  $\mathfrak{A}$ . But because  $\mathfrak{A}$  contains nonstandard elements,  $(\forall x)\phi(x)$  doesn't hold in  $\mathfrak{A}$ . So the induction axiom  $((\phi(0) \wedge (\forall x)(\phi(x) \rightarrow \phi(sx))) \rightarrow (\forall x)\phi(x))$  fails in  $\mathfrak{A}$ .