Subject 24.242. Logic II. Answers to the last homework assignmetn

Recall that a *normal modal system* for the modal sentential calculus is a set of formulas Γ that meets the following conditions:

(TC) Every tautological consequence of Γ is in Γ.
(Nec) If ϕ is in Γ, so in $\Box\phi$.
(K) All instances of the schema $(\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi))$ are in Γ.

1. A binary relation *R* on a set *W* is *symmetric* iff, for every *v* and *w* in *W*, if *Rwv* then *Rvw*. Let KB be the smallest normal modal system that contains all instances of the schema
(B) $(\Diamond\Box\phi \rightarrow \phi)$
Show that a sentence is in KB if and only if it's valid for the class of frames <*W,R,I*>, with *R* symmetric.

First, we show that (B) is valid for the class of symmetric frames. Suppose that R is symmetric and that $\Diamond\Box\phi$ is true at the world w in the frame <W,R,I>. Then there is a world v accessible from w in which $\Box\phi$ is true. So ϕ is true in every world accessible from v. In particular, ϕ is true in w, since, by symmetry, w is accessible from v. So $(\Diamond\Box\phi \rightarrow \phi)$ is valid in <W,R,I>.

Let Γ be the set of sentences valid for the class of symmetric frames. Γ is a normal modal system that includes (B), and so Γ includes KB. We need to show that, if a sentence ϕ isn't in KB, it isn't in Γ. That is, we need to show that, if ϕ isn't in KB, then there is a symmetric frame in which there is a world in which ϕ is false. We know that the canonical frame for KB contains a world in which ϕ is false; so it will be enough to show that the canonical frame for KB is symmetric.

Suppose that w and v are worlds in the canonical frame for KB and that Rwv. We need to see that Rvw, that is, we need to see that, whenever $\Box\psi$ is in v, ψ is in w. Since $\Box\psi$ is true in v, $\Diamond\Box\psi$ is true in every world that has access to v; in particular, $\Diamond\Box\psi$ is true in w. Since $(\Diamond\Box\psi \rightarrow \psi)$ is true in w, it follows that ψ is true in w, and so $\psi \in$ w.

5. Prove de Jongh's theorem that all instances of schema
(4) $(\Box\phi \rightarrow \Box\Box\phi)$
are elements of the smallest normal modal system that includes all instances of the schema:
(L) $(\Box(\Box\phi \rightarrow \phi) \rightarrow \Box\phi)$.
[Hint: The instance of schema (L) that you'll use is $(\Box(\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi)) \rightarrow \Box(\phi \wedge \Box\phi))$.]

1.	$((\phi \wedge \Box\phi) \rightarrow \phi)$	**(TC)**
2.	$\Box((\phi \wedge \Box\phi) \rightarrow \phi)$	**(Nec), 1**
3.	$(\Box((\phi \wedge \Box\phi) \rightarrow \phi) \rightarrow (\Box(\phi \wedge \Box\phi) \rightarrow \Box\phi))$	**(K)**
4.	$(\Box(\phi \wedge \Box\phi) \rightarrow \Box\phi)$	**(TC), 2, 3**
5.	$(\phi \rightarrow (\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi)))$	**(TC), 4**
6.	$\Box(\phi \rightarrow (\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi)))$	**(Nec), 5**

7. $(\Box(\phi \rightarrow (\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi))) \rightarrow (\Box\phi \rightarrow \Box(\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi))))$ **(K)**

8. $(\Box\phi \rightarrow \Box(\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi)))$ **(TC), 6, 7**

9. $(\Box(\Box(\phi \wedge \Box\phi) \rightarrow (\phi \wedge \Box\phi)) \rightarrow \Box(\phi \wedge \Box\phi))$ **(L)**

10. $(\Box\phi \rightarrow \Box(\phi \wedge \Box\phi))$ **(TC), 8, 9**

11. $((\phi \wedge \Box\phi) \rightarrow \Box\phi)$ **(TC)**

12. $\Box((\phi \wedge \Box\phi) \rightarrow \Box\phi)$ **(Nec), 11**

13. $(\Box((\phi \wedge \Box\phi) \rightarrow \Box\phi) \rightarrow (\Box(\phi \wedge \Box\phi) \rightarrow \Box\Box\phi))$ **(K)**

14. $(\Box(\phi \wedge \Box\phi) \rightarrow \Box\Box\phi)$ **(TC), 12, 13**

15. $(\Box\phi \rightarrow \Box\Box\phi)$ **(TC), 10, 14**