Mohr's Circle
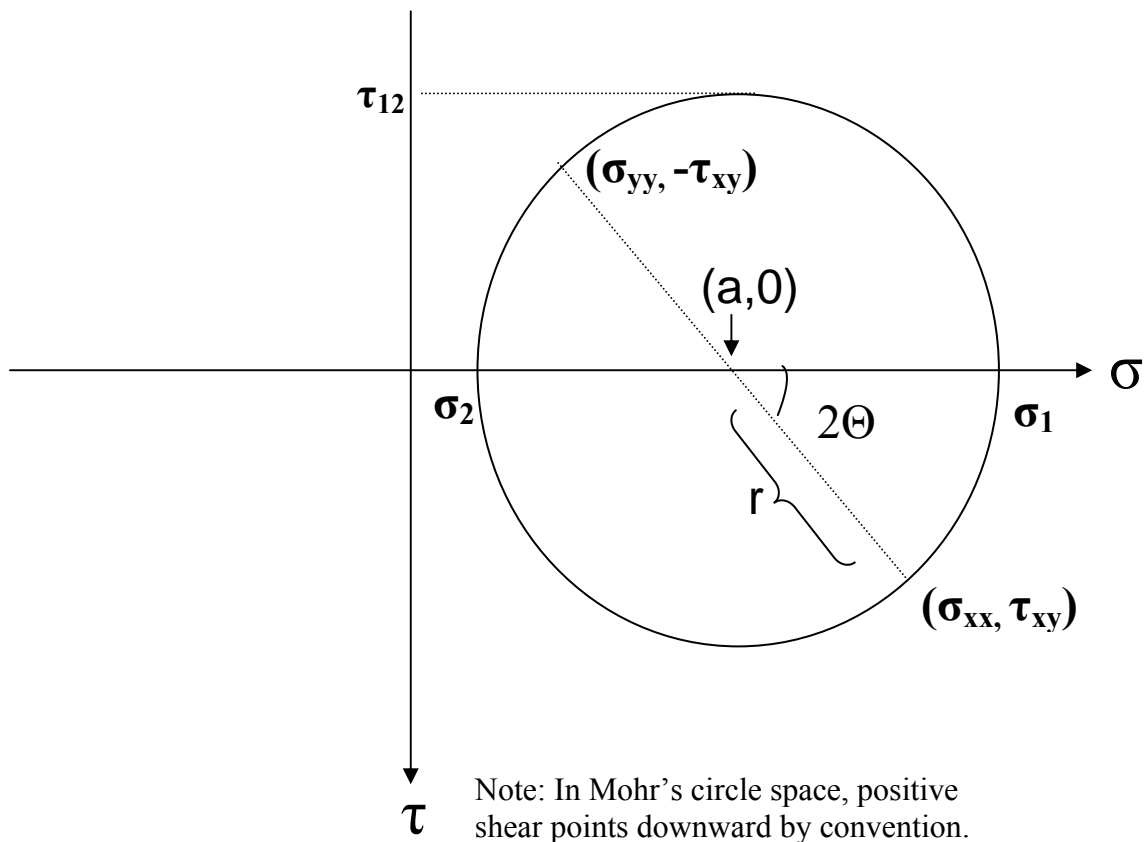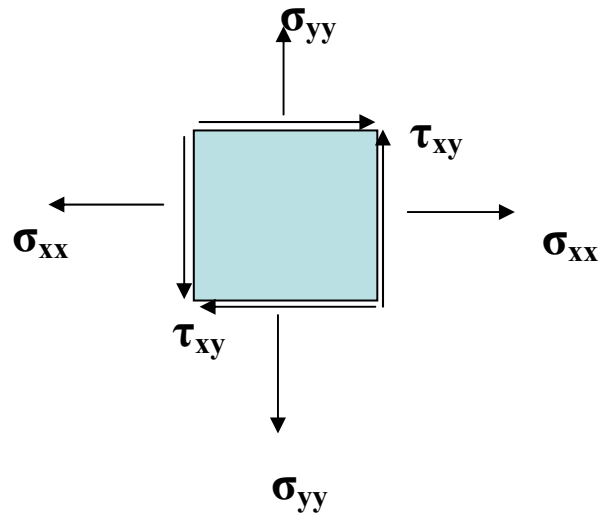
-> A geometrical tool to help us transform stress states w/o heavy duty math.

$$\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}$$

Note: In Mohr's circle space, positive shear points downward by convention.

Four Simple Steps:

1) Plot point 1: $\left(\sigma_{xx}, \tau_{xy}\right)$;   point 2: $\left(\sigma_{yy}, -\tau_{xy}\right)$

Note: shear stress on the y
face tends to rotate element in
the clockwise direction,
therefore negative by
convention.

2) Construct a circle using these two points.

Center: $(a, 0) = \left(\dfrac{\sigma_{xx} + \sigma_{yy}}{2},\ 0 \right)$

Radius: $r = \sqrt{\left(\dfrac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \tau_{xy}^{\ 2}}$

3) Determine the principal stresses:

$$\sigma_1 = a + r = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \tau_{xy}^{\ 2}}$$

$$\sigma_2 = a - r = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \tau_{xy}^{\ 2}}$$

$$\tau_{12} = r = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \tau_{xy}^{\ 2}}$$
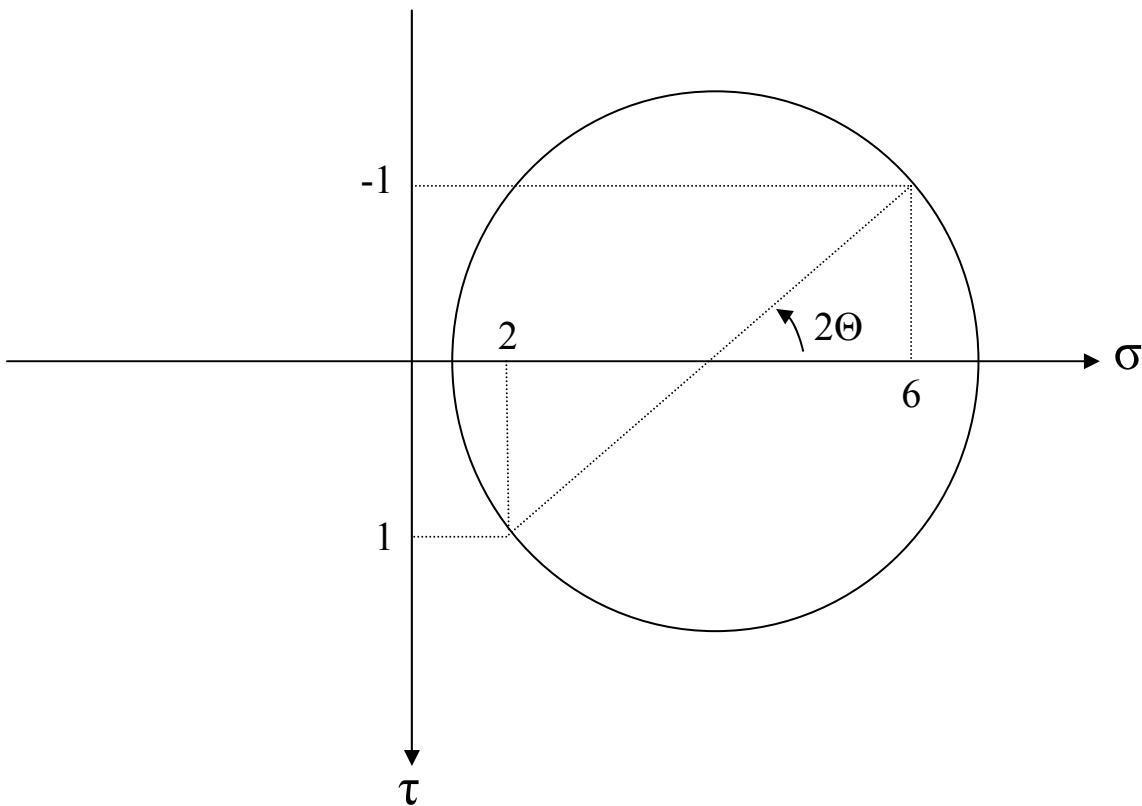
4) Determine the principal directions.
   Note: Mohr's circle angle is twice the actual angle of rotation.

$$\tan(2\theta) = \frac{y}{x} = \left(\frac{\tau_{xy}}{\left(\dfrac{\sigma_{xx} - \sigma_{yy}}{2} \right)} \right)$$

$$\theta = \frac{1}{2}\arctan\left(\frac{\tau_{xy}}{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)}\right)$$

Examples:

$$\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}$$

1. Plot $(2,1)$; $(6, -1)$

2. Center $(a,0) = \left(\frac{2+6}{2},0\right) = (4,0)$

    Find the radius: $r = \sqrt{\left(\frac{2-6}{2}\right)^2 + 1^2} = \sqrt{5}$
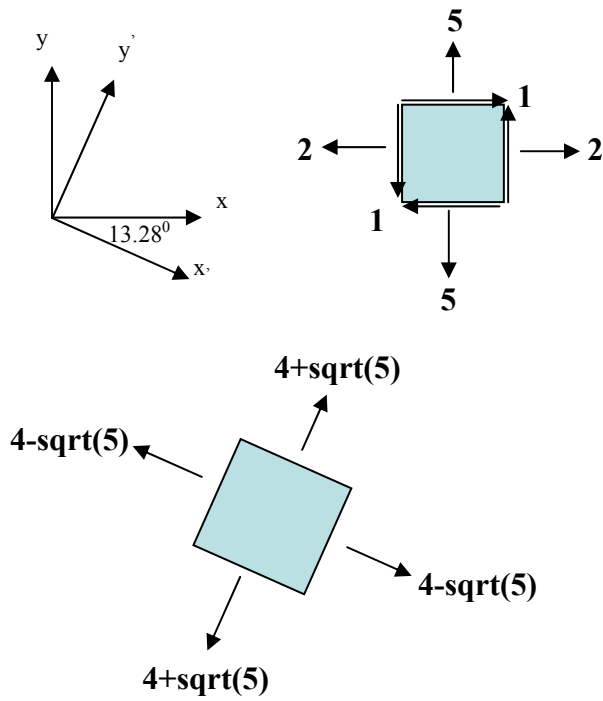
3, 4. Determine principal stresses and theta:

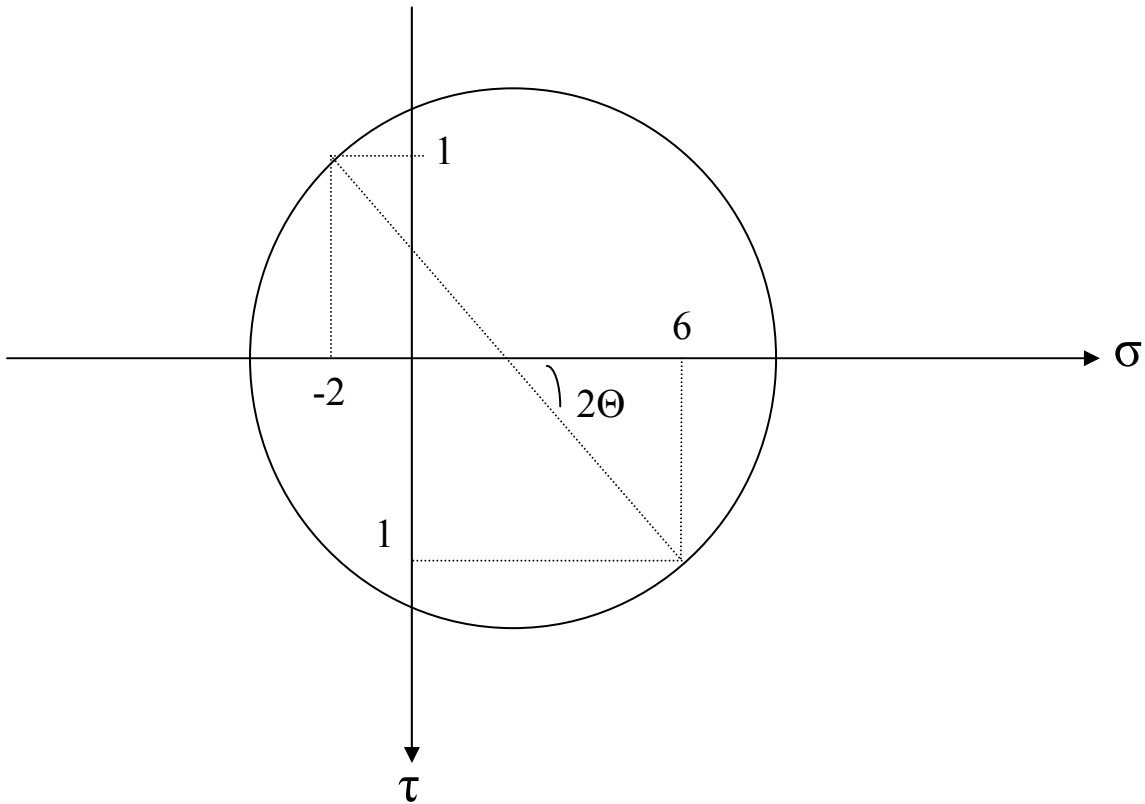$$\sigma_1 = a + r = 4 + \sqrt{5}$$

$$\sigma_2 = a - r = 4 - \sqrt{5}$$

$$\tau_{12} = r = \sqrt{5}$$

$$\Theta = \frac{1}{2}\arctan\left(\frac{1}{-2}\right) \cong -13.28^0$$

Example 2

$$\sigma = \begin{pmatrix} 6 & 1 \\ 1 & -2 \end{pmatrix}$$

1. Plot $(6,1)$; $(-2, -1)$

2. Center $(a,0) = \left(\dfrac{6 + (-2)}{2}, 0 \right) = (2,0)$

   Find the radius: $r = \sqrt{\left(\dfrac{6 - (-2)}{2} \right)^2 + 1^2} = \sqrt{17}$

3, 4. Determine principal stresses and theta:

$$\sigma_1 = a + r = 2 + \sqrt{17}$$

$$\sigma_2 = a - r = 2 - \sqrt{17}$$

$$\tau_{12} = r = \sqrt{17}$$

$$\Theta = \frac{1}{2}\arctan\left(\frac{1}{4}\right) \cong 7.0^0$$