

# 3.185 Problem Set 5

## Advanced Heat Transfer

### Solutions

1. Poirier & Geiger problem 6.11 (p. 217). (15)

To calculate the thermal conductivity of these aluminum-titanium alloys, you can use Wiedmann-Franz law:

$$k_{el} = \frac{\pi^2}{3} \left( \frac{k_B}{e} \right)^2 \sigma_e T$$

where  $k_{el}$  is the electrical component of the thermal conductivity,  $k_B$  and  $e$  are Boltzmann's constant and the electron charge,  $\sigma_e$  is the electrical conductivity and  $T$  the temperature. The constant  $\frac{\pi^2}{3} \left( \frac{k_B}{e} \right)^2$  is written as  $L$  and equal to  $2.45 \times 10^{-8} \frac{\text{Wohm}}{\text{K}^2}$ .

On pages 197-198 of G&P (last year's text), there is another equation specific to titanium-aluminum alloys:

$$k = AL\sigma_e T + B$$

where  $A$  and  $B$  are experimentally-fitted constants, equal to 0.997 and  $2.7 \frac{\text{W}}{\text{m}\cdot\text{K}}$  respectively. Because this applies to alpha and alpha+beta titanium alloys, and aluminum is an alpha stabilizer, it's a good bet this works for these alloys. (You weren't required to use last year's text, this just illustrates that the experimentally-fitted  $A$  and  $B$  constants give close to the same answer as the plain Wiedmann-Franz law.)

Al content, a/o	$\rho_e, \mu\text{ohm} \cdot \text{cm} (= 10^{-8} \text{ohm} \cdot \text{m})$	$L\sigma_e T, \frac{\text{W}}{\text{m}\cdot\text{K}}$	$AL\sigma_e T + B, \frac{\text{W}}{\text{m}\cdot\text{K}}$
0	112	17.5	20.1
3	140	14.0	16.7
6	165	11.9	14.6
11	190	10.3	13.0
33	210	9.3	12.0

It's worth noting how much the electrical resistivity rises, and the thermal conductivity falls, when aluminum is added, even for a low-conductivity metal like titanium. Copper and aluminum are even more sensitive to the presence of foreign elements, which is why people go to great lengths to purify copper for electrical and thermal applications requiring high conductivity.

2. Spreadsheet finite difference model of 1-D unsteady conduction

(a) The Biot number is given by:

$$\frac{hL}{k} = \frac{9000 \frac{\text{W}}{\text{m}^2\cdot\text{K}} \cdot 5 \times 10^{-3} \text{m}}{0.21 \frac{\text{W}}{\text{m}\cdot\text{K}}} = 214$$

Because this is well above 100, we can use the constant-temperature boundary condition approximation.

Note: even if you used half the thickness for  $L$ , your Biot number would still be above 100.

(b) The 1-D explicit finite difference method begins with:

$$\frac{T_{i,n+1} - T_{i,n}}{\Delta t} = \alpha \frac{T_{i-1,n} - 2T_{i,n} + T_{i+1,n}}{(\Delta x)^2}$$

where the subscripts give the  $x$  node number and timestep number respectively. Solving for  $T_{i,n+1}$  gives:

$$T_{i,n+1} = T_{i,n} + \frac{\alpha \Delta t}{(\Delta x)^2} (T_{i-1,n} - 2T_{i,n} + T_{i+1,n})$$

The quantity  $\alpha \Delta t / (\Delta x)^2$  is the mesh Fourier number  $\text{Fo}_M$ , so we can rewrite this as:

$$T_{i,n+1} = (1 - 2\text{Fo}_M)T_{i,n} + \text{Fo}_M(T_{i-1,n} + T_{i+1,n})$$

(c) Since  $n = 5$ , the  $x$ -spacing between nodes is one-fifth of the thickness, or 1mm. The maximum allowed mesh Fourier number is  $1/2$ , so this gives us the maximum allowed timestep as follows:

$$\frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

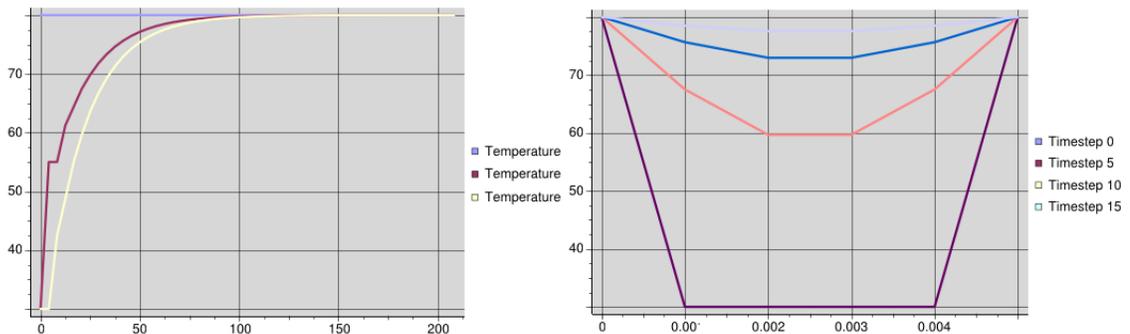
$$\Delta t \leq \frac{(\Delta x)^2}{2\alpha}$$

Here  $\alpha = k / (\rho c_p) = 1.2 \times 10^{-7} \frac{\text{m}^2}{\text{s}}$ , so this maximum timestep is:

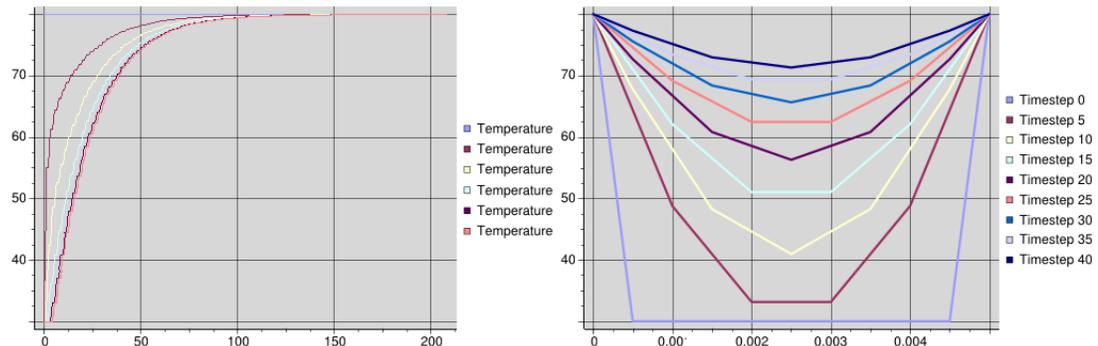
$$\Delta t \leq \frac{(10^{-3} \text{m})^2}{2 \cdot 1.2 \times 10^{-7} \frac{\text{m}^2}{\text{s}}} = 4.2 \text{s}$$

When  $n = 10$ ,  $\Delta x$  drops to  $1/2 \text{mm}$ , which is half its previous value, so the maximum timestep falls to  $1/4$  of what it was, or about 1 second.

(d) You could either graph this in terms of time with three plots at  $x_0$ ,  $x_1$  and  $x_2$  (since the other three are symmetric), or with respect to  $x$  at several timesteps. Both are shown below:



(e) Same as the last part, though there are more data:



3. Electron beam centrifugal atomization of metal (35)

- (a) If the chamber is black (meaning it absorbs all incident radiation) and cold (indicating it doesn't radiate significant heat back to the ingot), then the heat flux from the top surface is given by:

$$q_{rad} = \epsilon\sigma T^4$$

We are given 0.55 for  $\epsilon$ , and the temperature is the titanium melting point which is 1940K, so this heat flux is about 442 kW/m<sup>2</sup>.

- (b) The vapor pressure is given by the Clausius-Clapeyron equation, whose form and constants were given in the problem:

$$\log_{10} p_v(\text{torr}) = -\frac{23200}{T} + 11.74 - 0.66 \log_{10} T(+0 \cdot T) = -2.39$$

$$p_v = 4.1 \times 10^{-3} \text{ torr}$$

To convert from torr to J/m<sup>3</sup> (a.k.a. N/m<sup>2</sup>, or Pascals), which are the required units for the Langmuir equation, note that 760 torr = 101300 Pascals = 1 atm, so

$$p_v = 4.1 \times 10^{-3} \text{ torr} \times \frac{101300 \frac{\text{J}}{\text{m}^3}}{760 \text{ torr}} = 0.545 \frac{\text{J}}{\text{m}^3}$$

Now we put this into the Langmuir equation:

$$J_{vap} = \frac{p_v}{\sqrt{2\pi MRT}} = \frac{0.545 \frac{\text{J}}{\text{m}^3}}{\sqrt{2\pi \cdot 0.0479 \frac{\text{kg}}{\text{mol}} \cdot 8.314 \frac{\text{J}}{\text{K} \cdot \text{mol}} \cdot 1940\text{K}}} = \frac{0.545 \frac{\text{kg}}{\text{m} \cdot \text{s}^2}}{\sqrt{4854 \frac{\text{kg}^2 \text{m}^2}{\text{s}^2 \text{mol}^2}}} = 7.8 \times 10^{-3} \frac{\text{mol}}{\text{m}^2 \cdot \text{s}}$$

The heat flux at this evaporation rate is simply:

$$q_{vap} = J_{vap} \Delta H_{vap} = 7.8 \times 10^{-3} \frac{\text{mol}}{\text{m}^2 \cdot \text{s}} \cdot 440 \frac{\text{kJ}}{\text{mol}} = 3.44 \frac{\text{kW}}{\text{m}^2}$$

- (c) The total required power density is the sum of:

- Heat loss to radiation as calculated in part 3a:  $q_{rad} = 442 \frac{\text{kW}}{\text{m}^2}$ .
- Heat loss to evaporation as calculated in part 3b:  $q_{vap} = 3.44 \frac{\text{kW}}{\text{m}^2}$ .
- Heat required to raise the titanium temperature from 300K to its melting point of 1940K, which we can call  $q_{c_p}$ . Assuming constant heat capacity, and neglecting solid-solid phase transformations (since no information about them was given), the required heat per unit volume is:

$$H_{c_p} = \rho c_p \Delta T$$

If we assign the variable  $u$  to the melt interface speed, then the power required per unit area to heat the titanium to the melting point at that speed is:

$$q_{c_p} = u H_{c_p} = u \rho c_p \Delta T$$

Inserting our parameters gives:

$$q_{c_p} = 0.01 \frac{\text{m}}{\text{min}} \cdot \frac{1 \text{ min}}{60 \text{ s}} \cdot 4700 \frac{\text{kg}}{\text{m}^3} \cdot 700 \frac{\text{J}}{\text{kg} \cdot \text{K}} \cdot (1940\text{K} - 300\text{K}) = 900 \frac{\text{kW}}{\text{m}^2}$$

- Heat required to melt the titanium  $q_m$ . The heat per unit volume is the heat of fusion multiplied by the density  $\rho \Delta H_f$ . The power per unit area required to melt at linear velocity  $u$  is simply:

$$q_m = u \rho \Delta H_f = 0.01 \frac{\text{m}}{\text{min}} \cdot \frac{1 \text{ min}}{60 \text{ s}} \cdot 4700 \frac{\text{kg}}{\text{m}^3} \cdot 300 \frac{\text{kJ}}{\text{kg}} = 235 \frac{\text{kW}}{\text{m}^2}$$

No single use of heat dominates, but the evaporation loss is relatively insignificant. The total power required comes to:

$$q_{total} = q_{rad} + q_{evap} + q_{c_p} + q_m = 1580 \frac{\text{kW}}{\text{m}^2}$$

This sounds like a large amount of power, but since a typical atomization unit might use an ingot of 15 cm diameter, the power required for this small area is only about 28 kW.

- (d) The  $q_{c_p}$  and  $q_m$  terms are both proportional to the velocity  $u$ , and thus proportional to the amount of material used. The  $q_{rad}$  and  $q_{evap}$  terms, on the other hand, are independent of  $u$ . So if we go faster, we produce more powder using more  $q_{c_p}$  and  $q_m$  but the same amount of  $q_{rad}$  and  $q_{evap}$ . On a “per unit of material produced” basis, we use the same amount of energy per unit of material to heat and melt it, but *less* is wasted on radiation and evaporation.

It is therefore more efficient to run faster.

- (e) The steady-state temperature distribution as derived in class is:

$$\frac{T - T_i}{T_m - T_i} = \exp\left(-\frac{uz}{\alpha}\right)$$

where  $z$  is the distance from the melt interface. This is valid if the length of the ingot is much greater than the lengthscale over which the temperature is significantly different from the initial temperature:  $L \gg \alpha/u$ . That lengthscale is given by:

$$\frac{\alpha}{u} = \frac{k}{u\rho c_p} = \frac{20 \frac{\text{W}}{\text{m}\cdot\text{K}}}{0.01 \frac{\text{m}}{\text{min}} \cdot \frac{1\text{min}}{60\text{s}} \cdot 4700 \frac{\text{kg}}{\text{m}^3} \cdot 700 \frac{\text{J}}{\text{kg}\cdot\text{K}}} = 0.036\text{m}$$

So only the top few centimeters closest to the melt interface are significantly hotter than the initial temperature of the ingot. Since the ingot is initially 1 meter long, this can be considered semi-infinite for most of the duration of the atomization process.

- (f) The heat flux is given by:

$$q_z = -k \frac{\partial T}{\partial z}$$

For the temperature distribution given in part 3e, this is:

$$q_z = -k \frac{\partial}{\partial z} \left[ T_i + (T_m - T_i) \exp\left(-\frac{uz}{\alpha}\right) \right] = \frac{k(T_m - T_i)u}{\alpha} \exp\left(-\frac{uz}{\alpha}\right)$$

Note that the ratio  $k/\alpha$  is equal to  $\rho c_p$  according to the definition of  $\alpha$ , so at  $z = 0$ , this is equal to:

$$q_z|_{z=0} = u\rho c_p(T_m - T_i)$$

This is identically equal to  $q_{c_p}$  in part 3c, which was  $900 \frac{\text{kW}}{\text{m}^2}$ . This is because it is this heat flux which provides the heat to the solid titanium ingot to raise it to the melting point.

- (g) If the ingot is horizontal and rotating like a rolling pin, a fixed electron beam hitting its top will generate a narrow stream of atomized liquid droplets on a tangent line to the spinning ingot. In this way, it is better than the vertical arrangement, which sends droplets flying all over the place. However, as the metal melts, atomizes and leaves, the horizontal cylinder is left with less material there, so the local diameter changes in that part of the ingot. This looks something like a cylinder which is being peeled while rotating on a lathe. Because droplet size is a function of the ingot diameter and rotation rate, one must change the rotation speed as the ingot gets chewed away by the atomization of the droplets in order to maintain a consistent droplet size. In the vertical arrangement, the diameter never changes, so the droplet size doesn't either; this is an advantage of the vertical over horizontal centrifugal atomizer.

Another possibility, suggested by a student, would be to have the ingot rotating horizontally, but with the beam scanning back and forth along a line on top of it. This would result in a wide stream

of droplets coming off roughly in a plane tangent to the cylinder, which could be a disadvantage. On the other hand, the diameter of the cylinder decreases roughly uniformly as it is atomized, so it may be possible to have slightly more uniform droplet size than in the horizontal atomizer with fixed beam position.

As you can see, this can get as complicated as you want to make it; any of these aspects of the various designs gets full credit for this 4-point subpart of the problem.

#### 4. Radiative Cooling of an Aluminum Cube

- (a) The radiative flux from the surface in a cold black enclosure is simply

$$q_{rad} = e = \epsilon\sigma T^4.$$

That was too easy!

- (b) With an “environment temperature” of zero, we set:

$$q_{rad} = h_{rad}(T - 0) = \epsilon\sigma T^4 \Rightarrow h_{rad} = \epsilon\sigma T^3.$$

- (c) Since the size and thermal conductivity are constant, the “maximum Biot number” over this temperature range corresponds to the Biot number for maximum radiative “heat transfer coefficient”  $h_{rad}$ . Since  $h_{rad}$  is an increasing function of temperature, this will be at the highest temperature, 1000K.

$$\text{Bi}_{rad} = \frac{0.85 \cdot 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \cdot \text{K}^4} \cdot (1000\text{K})^3 \cdot 0.1\text{m}}{238 \frac{\text{W}}{\text{m} \cdot \text{K}}} = 0.020$$

Since the maximum Biot number is well below 0.1, the uniform temperature assumption is valid.

- (d) This setup begins as a typical Newtonian cooling problem, but you’ve never quite seen it in this form before, so you have to solve a new differential equation:

accumulation = –out

$$V\rho c_p \frac{dT}{dt} = Aq = A\epsilon\sigma T^4$$

$$\frac{V\rho c_p}{A\epsilon\sigma} \frac{dT}{T^4} = dt$$

$$\frac{V\rho c_p}{A\epsilon\sigma} \int_{T_{init}}^{T_{final}} \frac{dT}{T^4} = \int_{t_{init}=0}^{t_{final}} dt$$

$$\frac{V\rho c_p}{A\epsilon\sigma} \left[ -\frac{1}{3T^3} \right]_{T_{init}}^{T_{final}} = t_{final} - 0$$

$$t_{final} = \frac{V\rho c_p}{3A\epsilon\sigma} \left( \frac{1}{T_{final}^3} - \frac{1}{T_{init}^3} \right)$$

$$t_{final} = \frac{(0.1\text{m})^3 \cdot 2700 \frac{\text{kg}}{\text{m}^3} \cdot 917 \frac{\text{J}}{\text{kg} \cdot \text{K}}}{3 \cdot 5 \cdot (0.1\text{m})^2 \cdot 0.85 \cdot 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \cdot \text{K}^4}} \left( \frac{1}{(400\text{K})^3} - \frac{1}{(1000\text{K})^3} \right) \simeq 5000\text{seconds}$$

Note that at 400K, if the surroundings are at room temperature (around 300 K) and the room and cube can be considered “grey” surfaces, they return  $(3/4)^4 = 32\%$  of the radiative flux back to the cube (it’s this simple because  $A_1 F_{12} = A_2 F_{21}$  always, and  $\epsilon_1 = \alpha_1$  in the grey body approximation), so the flux is quite a bit lower—and cooling is quite a bit slower—by radiation alone. On the other hand, convection becomes more important at the lower temperatures.

Note that this analysis captures the enormous reduction in heat loss rate as the cube cools, which is not accounted for in the constant- $h$  Newtonian cooling formulation. Pretty neat, huh?