

3.21 Kinetics of Materials—Spring 2006

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Lecture 9: Solutions to the Diffusion Equation—III.

References

1. Balluffi, Allen, and Carter, *Kinetics of Materials*, Sections 5.2.4–5.2.6.

Key Concepts

- Separation of variables is very useful technique for solving diffusion problems in *periodic or finite geometries*. In this method, solutions are sought that can be expressed as *products* of functions in a single variable: e.g., for one-dimensional diffusion, solutions of the form $c(x, t) = X(x)T(t)$.
- Substitution of this relation into the diffusion equation yields separate differential equations for the functions $X(x)$ and $T(t)$, and the solutions of most interest are to the equations $\frac{dT}{dt} = -\lambda DT$ and $\frac{d^2X}{dx^2} = -\lambda X$, where λ is a positive constant.
- The spatial part of the solution takes the form $X(x) = A \sin(\sqrt{\lambda}x) + B \cos(\sqrt{\lambda}x)$. $X(x)$ must satisfy boundary conditions to the particular problem being solved. For diffusion out of a slab with $c(0, t) = 0$ and $c(L, t) = 0$ this leads to eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$, where n is an integer. The general solution for the eigenfunctions $X(x)$ is then of the form $X_n(x) = a_n \sin(n\pi x/L)$ for these b.c.s.
- The time dependent part of the solution takes the form $T(t) = T^\circ \exp(-\lambda Dt)$ and λ is the same constant required for the spatial part of the solution, so $T_n(t) = T_n^\circ \exp(-n^2\pi^2 Dt/L^2)$.
- Because n in the above equations can take on any integer values, the general form of the solution is the infinite series $c(x, t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L) \exp(-n^2\pi^2 Dt/L^2)$. The constants $A(n)$ in this expression are evaluated from the initial conditions and are seen in this problem to be the Fourier sine series representation of the initial condition $c(x, 0)$ on the interval $x = (0, L)$. For different geometries, the initial and boundary conditions may require a full Fourier series (sine and cosine terms) in the solution.
- The *method of Laplace transforms* is most useful when a problem has *time-dependent* or *prescribed-flux* boundary conditions. The Laplace transform of a function $f(x, t)$ is defined as $\mathcal{L}[f(x, t)] = \hat{f}(x, p) = \int_0^{\infty} e^{-pt} f(x, t) dt$. The Laplace transform thus transforms f from the (x, t) domain to the (x, p) domain.
- The general method of solving the diffusion equation by Laplace transforms involves transforming the diffusion equation, transforming the initial and boundary conditions, finding the solution to the transformed equations in the (x, p) domain, and finally back-transforming the solution to the (x, t) domain. The last step, usually the most challenging, is generally accomplished with the aid of tabulated Laplace transforms (see *KoM* Table 5.3).

Related Exercises in *Kinetics of Materials*

Review Exercises 5.10–5.13, pp. 124–129.