

## SOLUTIONS

3.1) I have two solutions to this problem. Which do you prefer?

FIRST: 3.1) The Lorentz force,  $F = q(\mathbf{v} \times \mathbf{B})$ , is everywhere directed along the radius vector. The torque on the orbital motion,  $\mathbf{T} = d\mathbf{L}/dt$ , is given by  $\mathbf{r} \times \mathbf{F}$ , which is zero. Hence, the Lorentz force cannot change the angular momentum or the angular frequency ( $\mathbf{L} = I\boldsymbol{\omega}$ ). Alternatively, the circular orbit generates an angular momentum and orbital magnetic moment that are directed parallel or antiparallel to the applied field. The torque due to the  $\mathbf{B}$  field acting on this moment,  $\boldsymbol{\mu}_m \times \mathbf{B}$  is also zero.

However, the presence of the magnetic field does add a new potential energy to the problem that should change something other than  $\boldsymbol{\omega}$ . If the electron is described classically (which is the limit of our treatment so far), it can assume an orbit of a different radius as  $B$  is applied. (If it is treated quantum mechanically, it may not incrementally change its radius, but can only do so in quantized steps that correspond to discrete energy differences.) So the energy of the system may be written:

$$E = \frac{1}{2} m\omega^2 r^2 - \frac{Ze^2}{4\pi\epsilon_0 r} \pm \mu_m B$$

and  $\mu_m = IA = e\omega\pi r^2$  where  $\omega$  and  $r$  are the Bohr frequency ( $\omega = [2E_K/m]^{1/2}/r = 4.2 \times 10^{16} \text{ s}^{-1}$ ) and radius ( $r_0 = 0.52 \text{ \AA}$ ) respectively. Thus,

$$r^3 = \frac{Ze^2}{4\pi\epsilon_0 [(m\omega^2/2) \mp \pi e\omega B]}$$

How significant is the magnetic field term in the denominator?

$$\frac{2\pi e B}{m\omega} \approx 2.6 \times 10^{-5} B$$

Even classically, the change in orbit radius is insignificant. These contradicting results underscore the limitations of the classical Bohr model.

SECOND: The Lorentz force,  $F = q(\mathbf{v} \times \mathbf{B})$ , is everywhere directed along the radius vector. A circular orbit would result from a central force  $-Ze^2/(4\pi\epsilon_0 r^2)$ ; its total energy,  $E = T + V = (1/2)mv^2 - Ze^2/(4\pi\epsilon_0 r)$ , is conserved with application of  $B$ . If the electron is described classically (which is the limit of our treatment so far), it can assume an orbit of a different radius as  $B$  is applied. (If it is treated quantum mechanically, it may not incrementally change its radius, but can only do so in quantized steps that correspond to discrete energy differences.)

The force balance describing the classical, field-induced change in the electron orbit is

$$\frac{mv^2}{r} - \frac{Ze^2}{4\pi\epsilon_0 r^2} - e v B = 0$$

To see how the angular frequency  $\omega$  varies with  $B$  we write the force balance as

$$m\omega^2 r = \frac{Ze^2}{4\pi\epsilon_0 r^2} + m\omega\omega_L r.$$

where  $\omega_L = eB/m$  which is of order  $10^{11} B \text{ s}^{-1}$  ( $B$  in T). At  $B = 0$ , this equation suggests the definition:

$$\omega_0^2 = \frac{Ze^2}{m4\pi\epsilon_0 r_0^3},$$

which is the frequency of the Bohr orbit if  $r_0 = 0.52 \text{ \AA}$ , the Bohr radius. The magnitude of  $\omega_0$  is about  $10^{17} \text{ s}^{-1}$ , optical frequency, as expected. Thus  $\omega_L$  is very small compared to  $\omega_0$ . At  $B \neq 0$  we therefore have

$$\omega^2 - \omega\omega_L - \omega_0^2 \left( \frac{r_0}{r} \right)^3 = 0$$

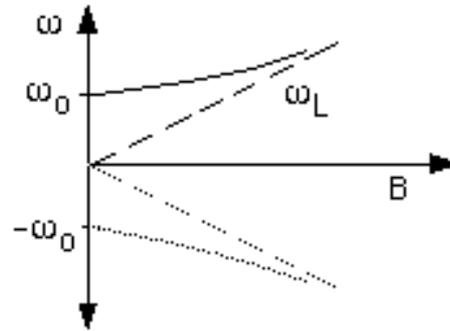
If we assume the radius is unchanged with application of  $B$ , then this equation is quadratic in  $\omega$ , and has the solution

$$\omega = \frac{\omega_L}{2} \left( 1 \pm \sqrt{1 + 4\omega_0^2 / \omega_L^2} \right)$$

or, considering the smallness of  $\omega_L/\omega_0 \approx 10^{-6}$ ,

$$\omega = \frac{\omega_L}{2} \pm \omega_0 \sqrt{1 + \omega_L^2/4\omega_0^2} \approx \frac{\omega_L}{2} \pm \omega_0 \left(1 + \omega_L^2/8\omega_0^2\right).$$

At  $B = 0$  this reduced to  $\omega = \pm\omega_0$ . For increasing  $B$ ,  $\omega$  increases or decreases as shown at right and asymptotically approaches the straight line  $\omega = \omega_L = eB/m$  for very large  $B$ . For experimentally achievable values of  $B$ ,  $\omega = \omega_0$ .



If we do not assume  $r = r_0$ , then we must solve the equation of motion for  $r$ . We assume  $r = r_0(1+\delta)$ , giving:

$$\omega^2 - \omega\omega_L - \frac{\omega_0^2}{(1+\delta)^3} \approx \omega^2 - \omega\omega_L - \frac{\omega_0^2}{(1+3\delta+3\delta^2\dots)} = 0,$$

which, if we take  $\omega = \omega_0$ , gives:

$$\delta = \frac{\omega_L}{3\omega}.$$

Thus, application of a field increases or decreases the radius of the orbit depending on the sign of the  $B$  field.

$$3.6) \text{ a) } \langle \cos \theta \rangle = \frac{\int_1^{-1} x e^{sx} dx}{\int_1^{-1} e^{sx} dx} \quad \text{where } x = \cos \theta \text{ and } s = \mu_m B/k_B T. \text{ Use table of}$$

integrals or the following more elegant derivation. If we let  $f(s) = \int_1^{-1} e^{sx} dx$ , then it is

clear that  $\langle \cos \theta \rangle = \frac{f'(s)}{f(s)} = \frac{d}{ds} \ln[f(s)]$ . Evaluating the integral in  $f(s)$  gives

$$f(s) = \frac{2 \sinh(s)}{s} \text{ so } \langle \cos \theta \rangle = \frac{d}{ds} \ln[f(s)] = \frac{\cosh(s)}{\sinh(s)} - \frac{1}{s} = L(s). \text{ Q.E.D.}$$

b) Use  $\coth(s) = 1/x + s/3 + \dots$ . The result is the classical Curie law for magnetization at weak fields/high temperatures:  $\langle \mu_m \rangle = \mu_m \langle \cos \theta \rangle = \mu_m^2 B / (3k_B T)$ .

c) Both  $\sinh$  and  $\cosh$  approach infinity for infinite argument, so  $\coth(s \rightarrow \infty) = 1$ . This describes saturation, at which point the thermally averaged moment in the field direction is equal to the magnitude of the moment:  $\langle \mu_m \rangle = \mu_m \langle \cos \theta \rangle = \mu_m$ .

3.7)  $N_v \mu_o \mu_m^2 / k_B T$  has units  $\text{m}^{-3} (\text{H/m}) (\text{J/T})^2 / \text{J}$ , but since the unit  $\text{T} = \text{W/m}^2 = \text{HA/m}^2$  and  $\text{A}^2 = \text{J}/(\text{s}\Omega)$  and  $\text{H/m} = \Omega \text{s}$  (considering that  $L \partial^2 q / \partial t^2$  has the same units as  $R \partial q / \partial t$ ), the MKS unit  $N_v \text{m}^3$  results. Cgs form of  $\chi$  gives units  $N(\text{erg/Oe})^2 / \text{erg}$  (here  $N$  may be either the number per unit volume,  $N_v$ , or Avogadro's number,  $N_A$ ) but since  $\text{Oe}^2 = \text{erg/cm}^3$ , the unit  $N \text{cm}^3$  results. Thus  $\chi$  (MKS) =  $4\pi\chi$ (cgs) but  $10^6 \chi_{\text{mol}}$  (MKS) =  $4\pi\chi$ (cgs).

For diamagnetic susceptibility,  $\chi$  has units  $N \mu_o \epsilon_o \cdot \text{Jm}^2 / \text{kg}$  since  $e^2 / (4\pi \epsilon_o r)$  is in joules. But  $\mu_o \epsilon_o = c^{-2}$  so our units on  $\chi$  are  $N \cdot \text{Jm}^3 / (\text{kg m}^2 / \text{s}^2) = N \cdot \text{m}^3$  as for the paramagnetic case. In Cgs units  $e^2$  is in  $\text{erg cm}$  so  $\chi$  has units  $N \text{erg cm}^3 / \text{g} (\text{cm/s})^2 = N \text{cm}^3$ .

3.8) Density =  $1.4 \times 10^3 \text{ g/cm}^3$ . Spin of molecule is  $2/2$  so  $\mu_m^2 = g^2 s(s+1) \mu_m^2 = 8 \mu_m^2$  giving for  $\chi_{\text{mol}} = N \mu_o \mu_m^2 / k_B T$  a value  $1.29 \times 10^{-7}$  (using  $N_A = 6.025 \times 10^{23} \text{ at/mol}$ ). But units of  $\chi_{\text{cgs}}$  are  $\text{cm}^3$ , so our  $\chi_{\text{mol}} = 1.3 \times 10^{-7}$  should be compared to  $4\pi\chi_{\text{mol}} \times 10^{-6} \text{ m}^3 / \text{cm}^3 = 4.3 \times 10^{-8}$  which is  $1/3$  of our value.

3.9) Density of He =  $1.78 \times 10^4 \text{ g/cm}^3$ .  $\chi = [e^2 \mu_o / (6m)] \Sigma \langle r^2 \rangle$  and two electrons in an orbit of radius  $a_o$  gives  $-3.54 \times 10^{-9} \Sigma \langle r^2 \rangle = 1.98 \times 10^{-11}$ . The cgs value should be converted by  $4\pi \times 10^{-6} \chi_m = 2.36 \times 10^{-11}$ .

3.10)  $\nabla \times \mathbf{E} = -d\mathbf{B}/dt$ . Using Stokes' theorem gives

$$\int E \cdot d\ell = -\frac{d\phi}{dt} = -\pi r^2 \frac{dB}{dt}$$

$2\pi r E = \pi r^2 (dB/dt)$ ,  $E = -(r/2)(dB/dt)$ . This field produces a torque  $\mathbf{T} = \mathbf{r} \times \mathbf{F} = rEe$

(where  $e < 0$ ) so  $\mathbf{T} = d\mathbf{L}/dt = +(elr^2/2)(dB/dt)$  giving  $\Delta L = +(elr^2/2)B$

But  $\mu_m = \gamma_L L = eL/(2m)$ , so the field-induced moment is given by

$$\mu_m = -(e^2 r^2 / 4m) \text{ and } \chi = N_v \mu_m / H$$

$$\chi = -(N_v \mu_0 e^2 / 4m) r^2$$

but here  $r^2 = x^2 + y^2 = (2/3)\langle r^2 \rangle$ . So finally  $\chi_d = -(N_v \mu_0 e^2 / 6m)\langle r^2 \rangle$

3.11)  $\chi_p + \chi_d = 0$  implies

$$N\mu_0(\mu_m^2 / 3k_B T - [e^2 / 6m]r^2) = 0$$

The larger the paramagnetic moment squared, the larger the area of the diamagnetic orbit must be to cancel it out at a given  $T$ .

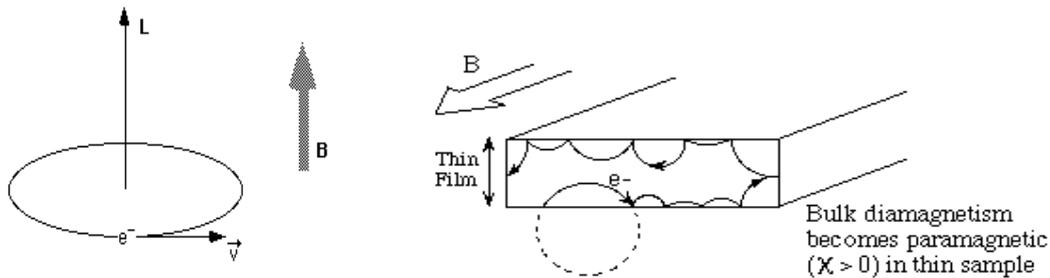
Going further, for a classical orbit,  $\mu_m = gL = (e/2m) mvr = evr/2$ , so the condition becomes

$$v^2 / 12k_B T = e^2 / 6m, \quad (1/2) mv^2 = k_B T$$

This says that the higher the kinetic energy of the classical orbit, the higher the  $T$  at which  $\chi_{\text{tot}} = 0$ . This is because higher  $E_K$  requires a smaller orbit which has smaller area and smaller diamagnetism; the small diamagnetism is only canceled out by the paramagnetism when the temperature is large enough to suppress the Curie paramagnetism. At lower temperatures, the paramagnetism dominates.

3.12)  $\mu_m = IA = e\omega r^2/2$ . For  $\omega = E/\hbar$ ,  $\mu_m = 4.6 \times 10^{-24} \text{ Am}^2$ . However, using  $\omega = v/r_0$ ,  $\mu_m = 9.27 \times 10^{-24} \text{ Am}^2 = \mu_B$ . They differ by a factor of 2. The latter is correct, classically. The former is wrong because it makes use of the Einstein relation for the energy of a photon. This then gives  $\omega = 1/2 \omega$  which is wrong as is the first estimate.

3.13) The diamagnetic contribution of free electrons in metals (between scattering events) is small. But for thin films or wires, the presence of narrow boundaries changes diamagnetism to paramagnetism for  $B$  perpendicular to the thin dimension. This effect is illustrated below.



1. Fig. Above, diamagnetic orbit of electron in  $B$  field. Below, illustration of change of diamagnetism to paramagnetism in a thin film.

$$3.14 \quad \text{Cr}^{3+} = 3d^3. \quad l = 3, s = 3/2, j = l - s = 3/2, \text{ thus } {}^4F_{3/2} g = 0.4, n_{eff} = 3.87$$

$$\text{Fe}^{3+} = 3d^5. \quad l = 0, s = 5/2, j = 5/2, \text{ thus: } {}^6S_{5/2} g = 2 (l = 0) \text{ and } n_{eff} = 5.92$$

$$\text{Co}^{2+} = 3d^7. \quad (3 \text{ holes whereas } \text{Cr}^{3+} \text{ has } 3e\text{'s}) l = 3, s = 3/2, j = 9/2, \text{ thus: } {}^4F_{9/2} g = 1.33, n_{eff} = 6.63$$

3.16) Experiment measures the projection of  $m_J$  on the direction  $J$  about which it precesses:

$\mu_j = \mu_l \cos(LJ) + \mu_s \cos(SJ)$  and from the law of cosines :  $C^2 = A^2 + B^2 - 2AB \cos C$  applied to the vector triangle  $L + S = J$  gives (writing  $\sqrt{l(l+1)}$  as  $L$  for convenience):

$$\cos(LJ) = [L^2 + J^2 - S^2]/[2L J]$$

$$\cos(SJ) = [S^2 + J^2 - L^2]/[2S J]$$

$$\mu_j = \mu_B [(L^2 + J^2 - S^2) + 2(S^2 + J^2 - L^2)]/2J$$

$$\mu_J = \mu_B (3J^2 - L^2 + S^2)/2J = \mu_B J(2J^2 + J^2 + S^2 - L^2)/2J^2$$

$$\mu_J = \mu_B J[1 + (J^2 + S^2 - L^2)/2J^2]$$

Restoring the proper quantum mechanical lengths for the vectors, leads to

$$\mu_j = \mu_B \sqrt{j(j+1)} \{1 - [j(j+1) + s(s+1) - 1(1+1)]/2j(j+1)\}$$

$$\mu_j = g \mu_B \sqrt{j(j+1)}$$

3.17) a) NaCl: diamagnetic (temperature independent) because this ionic crystal satisfies complete octets in each component. No paramagnetism.

b)  $\text{MnSO}_4 \cdot 4\text{H}_2\text{O}$ : Mn is in state  $\text{Mn}^{++} 3d^5$ . It will not order ferromagnetically. Superexchange giving antiferromagnetism is a possibility, but it is not observed, probably because oxygen bonds primarily with the sulfur and so it is not available for mediating exchange between Mn ions. So this is a paramagnetic salt with some diamagnetism supplied by  $\text{H}_2\text{O}$ .  $\chi \sim 1/T$  and  $M$  vs.  $H/T$  follows  $B_{5/2}(x)$ .

c)  $\text{Fe}_3\text{O}_4$ : this is a classic antiferromagnetically coupled two sublattice system:  $\text{Fe}^{3+}$  is on one site (A) and  $\text{Fe}^{3+}$  and  $\text{Fe}^{2+}$  on the other (B).  $\mu^A = -5\mu_B$  and  $\mu^B = +9\mu_B$  so net moment is  $4\mu_B$ .

d) Ne is diamagnetic as is  $\text{H}_2\text{O}$ . Former has complete octet  $l = s = 0$ , latter has covalent bonds with paired spins. In both cases  $\chi < 0$  and temperature independent.

e) Cu is a Pauli paramagnet  $\chi = [\mu_0 N \mu_m^2 / k_B T] (k_B T / E_F)$  which is temperature independent,  $\chi > 0$ .